

**A SYNOPSIS OF  
ELEMENTARY  
RESULTS IN PURE  
MATHEMATICS:  
CONTAINING...**

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George Shoobridge Carr













A SYNOPSIS  
OF  
ELEMENTARY RESULTS  
IN  
PURE MATHEMATICS:

CONTAINING  
PROPOSITIONS, FORMULÆ, AND METHODS OF ANALYSIS,  
WITH  
ABRIDGED DEMONSTRATIONS.

SUPPLEMENTED BY AN INDEX TO THE PAPERS ON PURE MATHEMATICS WHICH ARE TO  
BE FOUND IN THE PRINCIPAL JOURNALS AND TRANSACTIONS OF LEARNED SOCIETIES,  
BOTH ENGLISH AND FOREIGN, OF THE PRESENT CENTURY.

BY  
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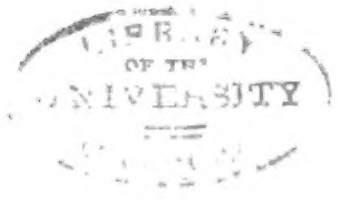
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## PREFACE TO PART I.

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THE work, of which the part now issued is a first instalment, has been compiled from notes made at various periods of the last fourteen years, and chiefly during the engagements of teaching. Many of the abbreviated methods and mnemonic rules are in the form in which I originally wrote them for my pupils.

The general object of the compilation is, as the title indicates, to present within a moderate compass the fundamental theorems, formulæ, and processes in the chief branches of pure and applied mathematics.

The work is intended, in the first place, to follow and supplement the use of the ordinary text-books, and it is arranged with the view of assisting the student in the task of revision of book-work. To this end I have, in many cases, merely indicated the salient points of a demonstration, or merely referred to the theorems by which the proposition is proved. I am convinced that it is more beneficial to the student to recall demonstrations with such aids, than to read and re-read them. Let them be read once, but recalled often. The difference in the effect upon the mind between reading a mathematical demonstration, and originating one wholly or

partly, is very great. It may be compared to the difference between the pleasure experienced, and interest aroused, when in the one case a traveller is passively conducted through the roads of a novel and unexplored country, and in the other case he discovers the roads for himself with the assistance of a map.

In the second place, I venture to hope that the work, when completed, may prove useful to advanced students as an *aide-mémoire* and book of reference. The boundary of mathematical science forms, year by year, an ever widening circle, and the advantage of having at hand some condensed statement of results becomes more and more evident.

To the original investigator occupied with abstruse researches in some one of the many branches of mathematics, a work which gathers together synoptically the leading propositions in all, may not therefore prove unacceptable. Abler hands than mine undoubtedly, might have undertaken the task of making such a digest; but abler hands might also, perhaps, be more usefully employed,—and with this reflection I have the less hesitation in commencing the work myself. The design which I have indicated is somewhat comprehensive, and in relation to it the present essay may be regarded as tentative. The degree of success which it may meet with, and the suggestions or criticisms which it may call forth, will doubtless have their effect on the subsequent portions of the work.

With respect to the abridgment of the demonstrations, I may remark, that while some diffuseness of explanation is not only allowable but very desirable in an initiatory treatise, conciseness is one of the chief requirements in a work intended

for the purposes of revision and reference only. In order, however, not to sacrifice clearness to conciseness, much more labour has been expended upon this part of the subject-matter of the book than will at first sight be at all evident. The only palpable result being a compression of the text, the result is so far a negative one. The amount of compression attained is illustrated in the last section of the present part, in which more than the number of propositions usually given in treatises on Geometrical Conics are contained, together with the figures and demonstrations, in the space of twenty-four pages.

The foregoing remarks have a general application to the work as a whole. With the view, however, of making the earlier sections more acceptable to beginners, it will be found that, in those sections, important principles have sometimes been more fully elucidated and more illustrated by examples, than the plan of the work would admit of in subsequent divisions.

A feature to which attention may be directed is the uniform system of reference adopted throughout all the sections. With the object of facilitating such reference, the articles have been numbered progressively from the commencement in large Clarendon figures; the breaks which will occasionally be found in these numbers having been purposely made, in order to leave room for the insertion of additional matter, if it should be required in a future edition, without disturbing the original numbers and references. With the same object, demonstrations and examples have been made subordinate to enunciations and formulæ, the former being printed in small, the latter in bold

type. By these aids, the interdependence of propositions is more readily shown, and it becomes easy to trace the connexion between theorems in different branches of mathematics, without the loss of time which would be incurred in turning to separate treatises on the subjects. The advantage thus gained will, however, become more apparent as the work proceeds.

The Algebra section was printed some years ago, and does not quite correspond with the succeeding ones in some of the particulars named above. Under the pressure of other occupations, this section moreover was not properly revised before going to press. On that account the table of errata will be found to apply almost exclusively to errors in that section; but I trust that the list is exhaustive. Great pains have been taken to secure the accuracy of the rest of the volume. Any intimation of errors will be gladly received.

I have now to acknowledge some of the sources from which the present part has been compiled. In the Algebra, Theory of Equations, and Trigonometry sections, I am largely indebted to Todhunter's well-known treatises, the accuracy and completeness of which it would be superfluous in me to dwell upon.

In the section entitled Elementary Geometry, I have added to simpler propositions a selection of theorems from Townsend's Modern Geometry and Salmon's Conic Sections.

In Geometrical Conics, the line of demonstration followed agrees, in the main, with that adopted in Drew's treatise on the subject. I am inclined to think that the method of that author cannot be much improved. It is true that some important properties of the ellipse, which are arrived at in

Drew's Conic Sections through certain intermediate propositions, can be deduced at once from the circle by the method of orthogonal projection. But the intermediate propositions cannot on that account be dispensed with, for they are of value in themselves. Moreover, the method of projection applied to the hyperbola is not so successful; because a property which has first to be proved true in the case of the equilateral hyperbola, might as will be proved at once for the general case. I have introduced the method of projection but sparingly, always giving preference to a demonstration which admits of being applied in the same identical form to the ellipse and to the hyperbola. The remarkable analogy subsisting between the two curves is thus kept prominently before the reader.

The account of the C. G. S. system of units given in the preliminary section, has been compiled from a valuable contribution on the subject by Professor Everett, of Belfast, published by the Physical Society of London.\* This abstract, and the tables of physical constants, might perhaps have found a more appropriate place in an after part of the work. I have, however, introduced them at the commencement, from a sense of the great importance of the reform in the selection of units of measurement which is embodied in the C. G. S. system, and from a belief that the student cannot be too early familiarized with the same.

The Factor Table which follows is, to its limited extent, a reprint of Burckhardt's "*Tables des diviseurs*," published in

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\* "Illustrations of the Centimetre-Gramme-Second System of Units." London: Taylor and Francis. 1875.



1814–17, which give the least divisors of all numbers from 1 to 3,036,000. In a certain sense, it may be said that this is the only sort of purely mathematical table which is absolutely indispensable, because the information which it gives cannot be supplied by any process of direct calculation. The logarithm of a number, for instance, may be computed by a formula. Not so its prime factors. These can only be arrived at through the tentative process of successive divisions by the prime numbers, an operation of a most deterrent kind when the subject of it is a high integer.

A table similar to and in continuation of Burckhardt's has recently been constructed for the fourth million by J. W. L. Glaisher, F.R.S., who I believe is also now engaged in completing the fifth and sixth millions. The factors for the seventh, eighth, and ninth millions were calculated previously by Dase and Rosenberg, and published in 1862–65, and the tenth million is said to exist in manuscript. The history of the formation of these tables is both instructive and interesting.\*

As, however, such tables are necessarily expensive to purchase, and not very accessible in any other way to the majority of persons, it seemed to me that a small portion of them would form a useful accompaniment to the present volume. I have, accordingly, introduced the first eleven pages of Burckhardt's tables, which give the least factors of the first 100,000 integers nearly. Each double page of the table here printed is

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\* See "*Factor Table for the Fourth Million.*" By James Glaisher, F.R.S. London: Taylor and Francis. 1880. Also *Camb. Phil. Soc. Proc.*, Vol. III., Pt. IV., and *Nature*, No. 542, p. 462.

an exact reproduction, in all but the type, of a single quarto page of Burckhardt's great work.

It may be noticed here that Prof. Lebesque constructed a table to about this extent, on the plan of omitting the multiples of seven, and thus reducing the size of the table by about one-sixth.\* But a small calculation is required in using the table which counterbalances the advantage so gained.

The values of the Gamma-Function, pages 30 and 31, have been taken from Legendre's table in his "*Exercices de Calcul Intégral*," Tome I. The table belongs to Part II. of this Volume, but it is placed here for the convenience of having all the numerical tables of Volume I. in the same section.

In addition to the authors already named, the following treatises have been consulted—Algebras, by Wood, Bourdon, and Lefebure de Fourcy; Snowball's Trigonometry; Salmon's Higher Algebra; the Geometrical Exercises in Potts's Euclid; and Geometrical Conics by Taylor, Jackson, and Renshaw.

Articles 260, 431, 569, and very nearly all the examples, are original. The latter have been framed with great care, in order that they might illustrate the propositions as completely as possible.

G. S. C.

HADLEY, MIDDLESEX;

May 23, 1880.

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\* "Tables diverses pour la décomposition des nombres en leurs facteurs premiers." Par V. A. Lebesque. Paris. 1864.

# ERRATA.

Art. 13,		for	$-a^2b^2$	read	$+a^2b^2$ .
" 56,	Line 1,	"	3	"	$\frac{2}{3}$ .
" 66,	" 5,	"	$x$	"	$x^2$ .
" 90,	" 4,	"	numerators 1, 1, 1	"	1, $a$ , $a^2$ .
" 99,	" 1,	"	denominator $r-1$	"	$n-1$ .
" 107,	" 1,	"	taken	"	taken $m$ at a time.
" 108,	" 2,	"	(196)	"	(360).
" 131,	" 1, 2,	"	5	"	6.
" "	" 5,	"	$(-1)^5$	"	$(-1)^2$ .
" 133,	" 3, 6, 7,	"	$6x$	"	$3x$ .
" "	" 8,	"	4	"	34.
" "	" 9,	"	204, 459	"	102, 306.
" "	" 10,	"	459	"	$9n$ .
" 138,	" 4,	"	$\frac{10.9.8}{1.2.3}$	"	7.8.9.10.
" 140,		"	$(q+1)^k$	"	$(q+1)^{(k)}$ Notation of (96).
" 182,	" 5,	"	$u_{n-1}$ in numerator	"	$u_{n-1}^2$ .
" 191,	" 4,	"	(163)	"	(164).
" 220,	" 6,	"	$(x+y+z)^3$	"	$2(x+y+z)^2$ .
" 221,	" 4,	"	(1)	"	square of (1).
" 237,	" 11,	"	$x^2 = 1$	"	$x^2 = -1$ .
" 238,	" 5,	"	$(x^2 - 4x + 8)$ on left side	"	$(x^2 - 4x + 8)^2$ .
" 239,	" 11,	"	(234)	"	<i>Dele.</i>
" 248,	" 4,	"	(29)	"	(28).
" 267,	" 4,	"	(267)	"	(266).
" 274,	" 8,	"	$\lfloor 11$	"	$2 \lfloor 11$ .
" 276,	" 13,	"	$p+2$	"	$p+1$ .
" "	" 14,	"	$(p-1)$	"	$\lfloor p-1$ .
" 283,	" 3,	"	$x=1$	"	$a=b$ .
" 288,	" 7,	"	$n-1$	"	$n+1$ .
" 289,	" 4,	"	$H(r, n-1)$	"	$H(n, r-1)$ .
" 290,	" 2,	"	$H(r+1, n-1)$	"	$H(n, r)$ .
" 325,	" 17,	"	$P_2$	"	$P$ .
" "		"	$P_1P_2P_3$ , last line but one	"	$Q_1Q_2Q_3$ .
" 333,	" 3,	"	$\left(\frac{a+b}{2}\right)$	"	$\left(\frac{a+b}{2}\right)^m$ .
" 361,	" 7,	"	3528	"	10284.
" 481,	" 6,	"	$n-3$	"	$n-1$ .
" 514,	" 4,	"	applying Descartes' rule	"	<i>Dele.</i>
" 517,	" 3,	"	$a^3$	"	$x^3$ .
" 544,	" 1,	"		"	Transpose $F$ and $f$ .
" 551,	" 1,	"	$B_1$	"	$B$ .
" "	" 9,	"	$\alpha-n$	"	$\alpha-\kappa$ .
" 704,		"	(11, 12)	"	(9, 10, 1).
" 729,		"	(940)	"	(960).

Article 112 should be as follows:—

$$\frac{1}{1+2\sqrt{3}-\sqrt{2}} = \frac{1+2\sqrt{3}+\sqrt{2}}{(1+2\sqrt{3})^2-2} = \frac{1+2\sqrt{3}+\sqrt{2}}{11+4\sqrt{3}} = \frac{(1+2\sqrt{3}+\sqrt{2})(11-4\sqrt{3})}{73}.$$



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# INDEX TO PROPOSITIONS OF EUCLID

REFERRED TO IN THIS WORK.

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The references to Euclid are made in Roman and Arabic numerals; *e.g.* (VI. 19).

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## BOOK I.

- I. 4.—Triangles are equal and similar if two sides and the included angle of each are equal each to each.
- I. 5.—The angles at the base of an isosceles triangle are equal.
- I. 6.—The converse of 5.
- I. 8.—Triangles are equal and similar if the three sides of each are equal each to each.
- I. 16.—The exterior angle of a triangle is greater than the interior and opposite.
- I. 20.—Two sides of a triangle are greater than the third.
- I. 26.—Triangles are equal and similar if two angles and one corresponding side of each are equal each to each.
- I. 27.—Two straight lines are parallel if they make equal alternate angles with a third line.
- I. 29.—The converse of 27.
- I. 32.—The exterior angle of a triangle is equal to the two interior and opposite; and the three angles of a triangle are equal to two right angles.
  - COR. 1.—The interior angles of a polygon of  $n$  sides  $= (n-2) \pi$ .
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- I. 35 to 38.—Parallelograms or triangles upon the same or equal bases and between the same parallels are equal.
- I. 43.—The complements of the parallelograms about the diameter of a parallelogram are equal.
- I. 47.—The square on the hypotenuse of a right-angled triangle is equal to the squares on the other sides.
- I. 48.—The converse of 47.



## BOOK II.

II. 4.—If  $a, b$  are the two parts of a right line,  $(a+b)^2 = a^2 + 2ab + b^2$ .

If a right line be bisected, and also divided, internally or externally, into two unequal segments, then—

II. 5 and 6.—The rectangle of the unequal segments is equal to the difference of the squares on half the line, and on the line between the points of section; or  $(a+b)(a-b) = a^2 - b^2$ .

II. 9 and 10.—The squares on the same unequal segments are together double the squares on the other parts; or

$$(a+b)^2 + (a-b)^2 = 2a^2 + 2b^2.$$

II. 11.—To divide a right line into two parts so that the rectangle of the whole line and one part may be equal to the square on the other part.

II. 12 and 13.—The square on the base of a triangle is equal to the sum of the squares on the two sides *plus* or *minus* (as the vertical angle is *obtuse* or *acute*), twice the rectangle under either of those sides, and the projection of the other upon it; or  $a^2 = b^2 + c^2 - 2bc \cos A$  (702).

## BOOK III.

III. 3.—If a diameter of a circle bisects a chord, it is perpendicular to it: and conversely.

III. 20.—The angle at the centre of a circle is twice the angle at the circumference on the same arc.

III. 21.—Angles in the same segment of a circle are equal.

III. 22.—The opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles.

III. 31.—The angle in a semicircle is a right angle.

III. 32.—The angle between a tangent and a chord from the point of contact is equal to the angle in the alternate segment.

III. 33 and 34.—To *describe* or to *cut off* a segment of a circle which shall contain a given angle.

III. 35 and 36.—The rectangle of the segments of any chord of a circle drawn through an *internal* or *external* point is equal to the square of the semi-chord perpendicular to the diameter through the internal point, or to the square of the tangent from the external point.

III. 37.—The converse of 36. If the rectangle be equal to the square, the line which meets the circle touches it.



BOOK IV.

- IV. 2.—To inscribe a triangle of given form in a circle.
- IV. 3.—To describe the same about a circle.
- IV. 4.—To inscribe a circle in a triangle.
- IV. 5.—To describe a circle about a triangle.
- IV. 10.—To construct two-fifths of a right angle.
- IV. 11.—To construct a regular pentagon.

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BOOK VI.

- VI. 1.—Triangles and parallelograms of the same altitude are proportional to their bases.
- VI. 2.—A right line parallel to the side of a triangle cuts the other sides proportionally; and conversely.
- VI. 3 and A.—The bisector of the *interior* or *exterior* vertical angle of a triangle divides the base into segments proportional to the sides.
- VI. 4.—Equiangular triangles have their sides proportional homologically.
- VI. 5.—The converse of 4.
- VI. 6.—Two triangles are equiangular if they have two angles equal, and the sides about them proportional.
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- VI. 8.—A right-angled triangle is divided by the perpendicular from the right angle upon the hypotenuse into triangles similar to itself.
- VI. 14 and 15.—Equal *parallelograms*, or *triangles* which have two angles equal, have the sides about those angles reciprocally proportional; and conversely, if the sides are in this proportion, the figures are equal.
- VI. 19.—Similar triangles are in the duplicate ratio of their homologous sides.
- VI. 20.—Likewise similar polygons.
- VI. 23.—Equiangular parallelograms are in the ratio compounded of the ratios of their sides.
- VI. B.—The rectangle of the sides of a triangle is equal to the square of the bisector of the vertical angle *plus* the rectangle of the segments of the base.

- VI. C.—The rectangle of the sides of a triangle is equal to the rectangle under the perpendicular from the vertex on the base and the diameter of the circumscribing circle.
- VI. D.—Ptolemy's Theorem. The rectangle of the diagonals of a quadrilateral inscribed in a circle is equal to both the rectangles under the opposite sides.
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## BOOK XI.

- XI. 4.—A right line perpendicular to two others at their point of intersection is perpendicular to their plane.
- XI. 5.—The converse of 4. If the first line is also perpendicular to a fourth at the same point, that fourth line and the other two are in the same plane.
- XI. 6.—Right lines perpendicular to the same plane are parallel.
- XI. 8.—If one of two parallel lines is perpendicular to a plane, the other is also.
- XI. 20.—Any two of three plane angles containing a solid angle are greater than the third.
- XI. 21.—The plane angles of any solid angle are together less than four right angles.

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## PREFACE TO PART II.

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APOLOGIES for the non-completion of this volume at an earlier period are due to friends and enquirers. The labour involved in its production, and the pressure of other duties, must form the author's excuse.

In the compilation of Sections VIII. to XIV., the following works have been made use of:—

Treatises on the Differential and Integral Calculus, by Bertrand, Hymer, Todhunter, Williamson, and Gregory's Examples on the same subjects; Salmon's Lessons on Higher Algebra.

Treatises on the Calculus of Variations, by Jellett and Todhunter; Boole's Differential Equations and Supplement; Carmichael's Calculus of Operations; Boole's Calculus of Finite Differences, edited by Moulton.

Salmon's Conic Sections; Ferrers's Trilinear Coordinates; Kempe on Linkages (*Proc. of Roy. Soc.*, Vol. 23); Frost and Wolstenholme's Solid Geometry; Salmon's Geometry of Three Dimensions.

Wolstenholme's Problems.

The Index which concludes the work, and which, it is hoped, will supply a felt want, deals with 890 volumes of 32 serial publications: of these publications, thirteen belong to Great Britain, one to New South Wales, two to America, four to France, five to Germany, three to Italy, two to Russia, and two to Sweden.

As the volumes only date from the year 1800, the

important contributions of Euler to the "Transactions of the St. Petersburg Academy," in the last century, are excluded. It was, however, unnecessary to include them, because a very complete classified index to Euler's papers, as well as to those of David Bernoulli, Fuss, and others in the same Transactions, already exists.

The titles of this Index, and of the works of Euler therein referred to, are here appended, for the convenience of those who may wish to refer to the volumes.

Tableau général des publications de l'Académie Impériale de St. Pétersbourg depuis sa fondation. 1872. [B.M.C.\* *R.R.* 2050,*e.*]

- I. Commentarii Academiæ Scientiarum Imperialis Petropolitanæ. 1726–1746; 14 vols. [B. M. C.: 431,*f.*]
  - II. Novi Commentarii A. S. I. P. 1747–75, 1750–77; 21 vols. [B. M. C.: 431,*f.* 15–17, *g.* 1–16, *h.* 1, 2.]
  - III. Acta A. S. I. P. 1778–86; 12 vols. [B. M. C.: 431, *h.* 3–8; or *T.C.* 8, *a.* 11.]
  - IV. Nova Acta A. S. I. P. 1787–1806; 15 vols. [B. M. C.: 431, *h.* 9–15, *i.* 1–8; or *T.C.* 8, *a.* 23.]
  - V. Leonhardi Euler Opera minora collecta, vel Commentationes Arithmeticæ collectæ; 2 vols. 1849. [B. M. C.: 8534, *ee.*]
  - VI. Opera posthuma mathematica et physica; 2 vols. 1862. [B.M.C.: 8534, *f.*]
  - VII. Opuscula analytica; 1783–5; 2 vols. [B. M. C.: 50, *i.* 15.]
- Analysis infinitorum. [B. M. C.: 529, *b.* 11.]

G. S. C.

ENDSLEIGH GARDENS,  
LONDON, N.W., 1886.

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\* British Museum Catalogue.

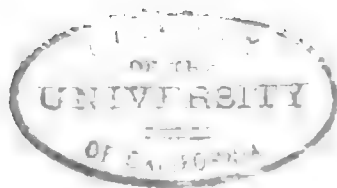
# ERRATA CONTINUED FROM PAGE x.

(Corrections which are important are marked with an asterisk.)

Page	1,	Line 7,	for volume	read weight.
		10,	gramme-million	gramme-six.
• „	6,	5,	1·4971499	·4971499.
• „		6,	·6679358	1·1447299.
* Art.	123,	2,	$2\sqrt{5}$	$2\sqrt{15}$ .
• „	259,	2,	$\alpha^2 + \beta^2$	$(\alpha^2 + \beta^2)^n$ .
• „	276,	6,	$3n^2 + n - 1$	$3n^2 + 3n - 1$ .
„	291,	1,	8	7.
„	292,	3 & 4,	a	a.
• „	322,	9,	45 and 13	35 and 10.
• „	361,	7,	3528	8584.
• „	459,	3 & 9,	-6	-16.
„	470,	1,	$x_m$	$x$ .
• „	489,	9,		$-\frac{4q}{3}$ .
• „	555,	14,	a number of rows	two columns.
„	593,	11 & 12,	$R$ and $R$	$R_1$ and $R_2$ .
„	604,	2,	one-sixtieth	one-ninetieth.
„	713,	2,	II.	III.
• „	897,	5,	$\cos \frac{1}{2}c$	$\sin \frac{1}{2}c$ .
„	922 ii.,		$b^2 + 2c^2$	$2b^2 + c^2$ .
„	949,	last,	$D$	$C$ .
• „	1076,	2,	dele "The projections ... are parallel."	
„	1158,	last,	1201	1217.
„	1178,	4,	$PS$	$PS'$ .
„	1241,	1,	parallel	conjugate.
„	1413,	3,	$-du dv$	$+du dv$ .
„	1491,	3,	-	=
„	1849,	1,	+	-
„	1903,	footnote,	$\int (x)$	$f(x)$ .
„	1925,		supply $dx$ .	
„	1954-6,		supply $x$ .	
„	2030-2,		erroneous, because $l$ in (1427) is necessarily an integer.	
„	2035,	1,	$ax$	$a$ .
• „	2140,	1,	$\sqrt{\quad}$ applies to the whole denominator.	
„	2136,	last,	2294	2293.
„	2354,		-	+
• „	2392,		$x^p$	$x - 1$ .
• „	2465,		$\frac{p}{2}$	$\frac{\pi}{2}$ .
„	3237,	1,	$(n-1)x$	$(n-1)$ .
„	3751,		supply $u_x$ .	
• „	4678,		dele 2 in the second term.	
• „	4680,		supply the factor 4 on the left.	
• „	4692,	5,	dele 2 in the second term.	
• „	4903,	3,	supply the factor 4 on the left.	
„	5154,	4,	3155	5155.
„	5330,	2,	$m$	$b$ .

and refer to Fig. 129, Art. 5332 on the cardioid is wanting.





# MATHEMATICAL TABLES.

## INTRODUCTION.

### *The Centimetre-Gramme-Second system of units.*

NOTATION.—The decimal measures of length are the *kilometre*, *hectometre*, *decametre*, *metre*, *decimetre*, *centimetre*, *millimetre*. The same prefixes are used with the *litre* and *gramme* for measures of capacity and ~~volume~~ <sup>weight</sup>.

Also,  $10^7$  metres is denominated a *metre-seven*;  $10^{-7}$  metres, a *seventh-metre*;  $10^{15}$  grammes, a *gramme-fifteen*; and so on.

A *gramme-million* is also called a *megagramme*; and a *millionth-gramme*, a *microgramme*; and similarly with other measures.

DEFINITIONS.—The C. G. S. system of units refers all physical measurements to the *Centimetre* (cm.), the *Gramme* (gm.), and the *Second* (sec.) as the units of length, mass, and time.

The quadrant of a meridian is approximately a *metre-seven*. More exactly, one metre = 3.2808694 feet = 39.370432 inches.

The *Gramme* is the *Unit of mass*, and the *weight of a gramme* is the *Unit of weight*, being approximately the weight of a cubic centimetre of water; more exactly, 1 gm. = 15.432349 grs.

The *Litre* is a cubic decimetre: but *one cubic centimetre* is the C. G. S. *Unit of volume*.

1 litre = .035317 cubic feet = .2200967 gallons.

The *Dyne* (dn.) is the *Unit of force*, and is the force which, in one second generates in a gramme of matter a velocity of one centimetre per second.

The *Erg* is the *Unit of work and energy*, and is the work done by a dyne in the distance of one centimetre.

The absolute *Unit of atmospheric pressure* is one megadyne per square centimetre = 74.964 cm., or 29.514 in. of mercurial column at 0° at London, where  $g = 981.17$  dynes.

*Elasticity of Volume* =  $k$ , is the pressure per unit area upon a body divided by the cubic dilatation.

*Rigidity*  $= n$ , is the shearing stress divided by the angle of the shear.

*Young's Modulus*  $= M$ , is the longitudinal stress divided by the elongation produced,  $= 9nk \div (3k + n)$ .

*Tenacity* is the tensile strength of the substance in dynes per square centimetre.

The *Gramme-degree* is the *Unit of heat*, and is the amount of heat required to raise by  $1^\circ \text{C.}$  the temperature of 1 gramme of water at or near  $0^\circ$ .

*Thermal capacity* of a body is the increment of heat divided by the increment of temperature. When the increments are small, this is the thermal capacity *at* the given temperature.

*Specific heat* is the thermal capacity of unit mass of the body at the given temperature.

The *Electrostatic unit* is the quantity of electricity which repels an equal quantity at the distance of 1 centimetre with the force of 1 dyne.

The *Electromagnetic unit* of quantity  $= 3 \times 10^{10}$  *electrostatic units* approximately.

The *Unit of potential* is the potential of unit quantity at unit distance.

The *Ohm* is the common *electromagnetic unit* of resistance, and is approximately  $= 10^9$  *C. G. S. units*.

The *Volt* is the *unit of electromotive force*, and is  $= 10^8$  *C. G. S. units of potential*.

The *Weber* is the *unit of current*, being the current due to an electromotive force of 1 Volt, with a resistance of 1 Ohm. It is  $= \frac{1}{10}$  *C. G. S. unit*.

*Resistance of a Wire*  $= \text{Specific resistance} \times \text{Length} \div \text{Section}$ .

### *Physical constants and Formulæ.*

In the latitude of London,  $g = 32.19084$  feet per second.

$= 981.17$  centimetres per second.

In latitude  $\lambda$ , at a height  $h$  above the sea level,

$g = (980.6056 - 2.5028 \cos 2\lambda - .000003 h)$  centimetres per second.

Seconds pendulum  $= (99.3562 - .2536 \cos 2\lambda - .0000003 h)$  centimetres.

*THE EARTH*.—Semi-polar axis,  $20,854,895$  feet\*  $= 6.35411 \times 10^8$  centims.

Mean semi-equatorial diameter,  $20,926,202$  „ \*  $= 6.37824 \times 10^8$  „

Quadrant of meridian,  $39.377786 \times 10^7$  inches\*  $= 1.000196 \times 10^7$  metres.

Volume,  $1.08279$  cubic centimetre-nines.

Mass (with a density  $5\frac{1}{2}$ )  $=$  Six gramme-twenty-sevens nearly.

\* These dimensions are taken from Clarke's "Geodesy," 1880.

Velocity in orbit = 2933000 centims. per sec.    Obliquity,  $23^{\circ} 27' 16''$ .  
 Angular velocity of rotation =  $1 \div 13713$ .  
 Precession,  $50'' \cdot 26$ .<sup>\*</sup>    Progression of Apse,  $11'' \cdot 25$ .    Eccentricity,  $e = \cdot 01679$ .  
 Centrifugal force of rotation at the equator,  $3 \cdot 3912$  dynes per gramme.  
 Force of attraction upon moon,  $\cdot 2701$ .    Force of sun's attraction,  $\cdot 5839$ .  
 Ratio of  $g$  to centrifugal force of rotation,  $g : \omega^2 = 289$ .  
 Sun's horizontal parallax,  $8'' \cdot 7$  to  $9''$ .<sup>\*</sup>    Aberration,  $20'' \cdot 11$  to  $20'' \cdot 79$ .<sup>\*</sup>  
 Semi-diameter at earth's mean distance,  $16' 1'' \cdot 82$ .<sup>\*</sup>  
 Approximate mean distance, 92,000,000 miles, or  $1 \cdot 48$  centimetre-thirteens.<sup>†</sup>  
 Tropical year,    365·242216 days, or 31,556927 seconds.  
 Sidereal year,    365·256374    „    31,558150    „  
 Anomalistic year, 365·259544 days.    Sidereal day, 86164 seconds.

**THE MOON.**—Mass = Earth's mass  $\times \cdot 011364 = 6 \cdot 98 \cdot 10^{25}$  grammes.  
 Horizontal parallax. From  $53' 56''$  to  $61' 24''$ .<sup>\*</sup>  
 Sidereal revolution, 27d. 7h. 43m.  $11 \cdot 46$ s.    Lunar month, 29d. 12h. 44m.  $2 \cdot 87$ s.  
 Greatest distance from the earth, 251700 miles, or  $4 \cdot 05$  centimetre-tens,  
 Least    225600    „    3·63  
 Inclination of Orbit,  $5^{\circ} 9'$ .    Annual regression of Nodes,  $19^{\circ} 20'$ .

**RULE.**—(The Year + 1)  $\div$  19.    The remainder is the Golden Number.  
 (The Golden Number - 1)  $\times$  11  $\div$  30.    The remainder is the Epact.

**GRAVITATION.**—Attraction between masses  $\left. \begin{array}{l} m, m' \text{ at a distance } l \end{array} \right\} = \frac{mm'}{l^2 \times 1 \cdot 543 \times 10^7}$  dynes.

The mass which at unit distance (1 cm.) attracts an equal mass with unit force (1 dn.) is  $= \sqrt{(1 \cdot 543 \times 10^7)} \text{ gm.} = 3928 \text{ gm.}$

**WATER.**—Density at  $0^{\circ} \text{C.}$ , unity; at  $4^{\circ}$ ,  $1 \cdot 000013$  (Kupffer).

Volume elasticity at  $15^{\circ}$ ,  $2 \cdot 22 \times 10^{10}$ .

Compression for 1 megadyne per sq. cm.,  $4 \cdot 51 \times 10^{-6}$  (Amaury and Descamps).

The heat required to raise the temperature of a mass of water from  $0^{\circ}$  to  $t^{\circ}$  is proportional to  $t + \cdot 00002t^2 + 0000003t^3$  (Regnault).

**GASES.**—Expansion for  $1^{\circ} \text{C.}$ ,  $\cdot 003665 = 1 \div 273$ .

$\frac{\text{Specific heat at constant pressure}}{\text{Specific heat at constant volume}} = 1 \cdot 408$ .

Density of dry air at  $0^{\circ}$  with Bar. at 76 cm. =  $\cdot 0012932$  gm. per cb. cm. (Regnault).

At unit pres. (a megadyne) Density =  $\cdot 0012759$ .

Density at press.  $p = p \times 1 \cdot 2759 \times 10^{-9}$ .

Density of saturated steam at  $t^{\circ}$ , with  $p$  taken }  
 from Table II., is approximately }  $= \frac{\cdot 7936098p}{(1 + \cdot 00366t) 10^9}$

**SOUND.**—Velocity  $= \sqrt{(\text{elasticity of medium} \div \text{density})}$ .

Velocity in dry air at  $t^{\circ} = 33240 \sqrt{(1 + \cdot 00366t)}$  centimetres per second.

Velocity in water at  $0^{\circ} = 143000$     „    „

**LIGHT.**—Velocity in a medium of absolute refrangibility  $\mu$   
 $= 3 \cdot 004 \times 10^{10} \div \mu$  (Cornu).

If  $P$  be the pressure in dynes per sq. cm., and  $t$  the temperature,  
 $\mu - 1 = 2903 \times 10^{-13} P \div (1 + \cdot 00366t)$  (Biot & Arago).

\* These data are from the "Nautical Almanack" for 1883.

† Transit of Venus, 1874, "Astrom. Soc. Notices," Vols. 37, 38.

TABLE I.

*Various Measures and their Equivalents in C. G. S. units.*

<i>Dimensions.</i>		<i>Pressure.</i>	
1 inch	= 2·5400 cm.	1 gm. persq. cm.	= 981 dynes persq. cm.
1 foot	= 30·4797 "	1 lb. per sq. foot	= 479 "
1 mile	= 160933 "	1 lb. per sq. in.	= 68971 "
1 nautical do.	= 185230 "	76 centimetres of mercury at 0° C.	= 1,014,000 "
1 sq. inch	= 6·4516 sq. cm.	lbs. per sq. in.	= 70·307 = $\frac{1}{014223}$
1 sq. foot	= 929·01 "	gms. per sq. cm.	
1 sq. yard	= 8361·13 "		
1 sq. mile	= 2·59 × 10 <sup>10</sup> "		
1 cb. inch	= 16·387 cb. cm.	<i>Force of Gravity.</i>	
1 cb. foot	= 28316 "	upon 1 gramme	= 981 dynes
1 cb. yard	= 764535 "	" 1 grain	= 63·56777 "
1 gallon	= 4541 "	" 1 oz.	= 2·7811 × 10 <sup>4</sup> "
	= 277·274 cb. in. or the vo- lume of 10 lbs. of water at 62° Fah., Bar. 30 in.	" 1 lb.	= 4·4497 × 10 <sup>5</sup> "
		" 1 cwt.	= 4·9837 × 10 <sup>7</sup> "
		" 1 ton	= 9·9674 × 10 <sup>8</sup> "
<i>Mass.</i>		<i>Work (g = 981).</i>	
1 grain	= 0·06479895 gm.	1 gramme-centimetre	= 981 ergs.
1 ounce	= 28·3495 "	1 kilogram-metre	= 981 × 10 <sup>5</sup> "
1 pound	= 453·5926 "	1 foot-grain	= 1·937 × 10 <sup>8</sup> "
1 ton	= 1,016047 "	1 foot-pound	= 1·356 × 10 <sup>7</sup> "
1 kilogramme	= 2·20462125 lbs.	1 foot-ton	= 3·04 × 10 <sup>10</sup> "
1 pound Avoir.	= 7000 grains	1 'horse power' p. sec.	= 7·46 × 10 <sup>8</sup> "
1 pound Troy	= 5760 "		
<i>Velocity.</i>		<i>Heat.</i>	
1 mile per hour	= 44·704 cm. per sec.	1 gramme-degree C.	= 42 × 10 <sup>6</sup> ergs.
1 kilometre "	= 27·777 "	1 pound-degree	= 191 × 10 <sup>5</sup> "
		1 pound-degree Fah.	= 106 × 10 <sup>5</sup> "

TABLE II.

*Pressure of Aqueous Vapour in dynes per square centim.*

Temp.	Pressure.	Temp.	Pressure.
-20°	1236	40°	73200
-15°	1866	50°	122600
-10°	2790	60°	198500
-5°	4150	80°	472900
0°	6133	100°	1014000
5°	8710	120°	1988000
10°	12220	140°	3626000
15°	16930	160°	6210000
20°	23190	180°	10060000
25°	31400	200°	15609000
30°	42050		

TABLE III.

*Values for the principal Lines of the Spectrum in air at 160° C. with Bar. 76 cm.*

	Wave-length in centims.	No. of vibrations per second.
A	7·604 × 10 <sup>-5</sup>	3·950 × 10 <sup>14</sup>
B	6·867 "	4·373 "
C	6·56201 "	4·577 "
D (mean)	5·89212 "	5·097 "
E	5·26913 "	5·700 "
F	4·86072 "	6·179 "
G	4·30725 "	6·973 "
H <sub>1</sub>	3·96801 "	7·569 "
H <sub>2</sub>	3·93300 "	7·636 "

TABLE IV. in C. G. S. units.

	Density Water = 1.	Young's Modulus M.	Rigidity n.	Elasticity of volume k.	Tenacity.	Expansion of Volume per degree C.	Linear Expansion between 0 & 100 C.	Specific Heat be- tween 0 & 100 C.	Relative Conduc- tivity.	Rate of Conduction of Sound in cm. per sec.	Elect. Magn. Specific Resistance at 0° C.
Platinum	21	—	—	—	—	.000027	.000875	.0335	381	$2.69 \times 10^5$	9158
Gold	19.26	—	—	—	—	.000045	.001483	—	1000	1.74	2081
Mercury	13.596	—	—	$0.542 \times 10^{12}$	—	.000180	—	.0330	—	—	96190
Lead	11.35	$.059 \times 10^{12}$	—	—	$2.28 \times 10^8$	.000086	.002861	—	180	1.23	19850
Silver	10.47	—	—	—	—	.000061	.00196	.0557	973	2.61	1521
Copper	8.843	1.234	$4.47 \times 10^{11}$	1.684	41.4	.000054	.00175	.0949	898	3.74	1615
Brass, drawn	8.471	1.075	3.66	—	33.8	.000054	.00193	—	357	3.56	—
Iron, cast	7.235	1.349	5.32	0.964	—	.000033	.001111	—	—	4.32	—
Iron, wrought	7.677	1.963	7.69	1.456	58.6	.000040	.001258	.1098	374	5.06	9827
Steel	7.849	2.139	8.19	1.841	79.3	.000037	.001260	—	—	5.22	—
Tin, cast	7.29	—	—	—	3.17	.000063	.00227	.0927	304	—	13360
Zinc, cast	7.19	—	—	—	5.17	.000088	.00294	.1770	363	—	5690
Glass, flint	2.942	0.603	2.40	0.415	—	.000015	.00081	—	—	4.53	—

TABLE V.

	Greatest distance from Sun. Earth's mean distance = 1.	Least distance from Sun.	Sidereal Revolution in Days.	Inclination of Orbit to Ecliptic.	Time of Rotation. h. m. s.	Diameter in Miles.	Mass.	Density.
Sun	—	—	—	0°	600 0 0	888000	354936	0.25
Mercury	0.46669	0.30750	87.969	7°	24 0 5	3000	0.118	2.01
Venus	0.72826	0.71840	224.701	3°	23 21 21	7700	0.883	0.97
Earth	1.01678	0.98322	365.256	—	23 56 4	7926	1.000	1.00
Mars	1.66578	1.38160	686.980	1°	24 37 22	4500	0.132	0.72
Jupiter	5.45378	4.95182	4332.585	1°	9 55 26	92000	338.034	0.24
Saturn	10.07328	9.00442	10759.220	2°	10 29 17	75000	101.064	0.13
Uranus	20.07612	18.28916	30686.821	46°	—	36000	14.789	0.15
Neptune	30.29888	29.77506	60126.722	46°	—	35000	24.648	0.27



TABLE VI.—*Functions of  $\pi$  and  $e$ .*

$\pi = 3.1415926$	$\pi^{-1} = .3183099$	$e = 2.71828183$
$\pi^2 = 9.8696044$	$\pi^{-2} = .1013212$	$e^2 = 7.38905611$
$\pi^3 = 31.0062761$	$\pi^{-3} = .0322515$	$e^{-1} = 0.3678794$
$\sqrt{\pi} = 1.7724539$	$200^\circ \div \pi = 63^\circ.6619772$	$e^{-2} = 0.1353353$
$\log_{10} \pi = 1.4971499$	$180^\circ \div \pi = 57^\circ.2957795$	$\log_{10} e = 0.43429448$
$\log_e \pi = 0.6679358$	$= 206264''.8$	$\log_e 10 = 2.30258509$

TABLE VII.

No.	Square root.	Cube root.
2	1.4142136	1.2599210
3	1.7320508	1.4422496
4	2.0000000	1.5874011
5	2.2360680	1.7099759
6	2.4494897	1.8171206
7	2.6457513	1.9129312
8	2.8284271	2.0000000
9	3.0000000	2.0800837
10	3.1622777	2.1544347
11	3.3166248	2.2239801
12	3.4641016	2.2894286
13	3.6055513	2.3513347
14	3.7416574	2.4101422
15	3.8729833	2.4662121
16	4.0000000	2.5198421
17	4.1231056	2.5712816
18	4.2426407	2.6207414
19	4.3588989	2.6684016
20	4.4721360	2.7144177
21	4.5825757	2.7589243
22	4.6904158	2.8020393
23	4.7958315	2.8438670
24	4.8989795	2.8844991
25	5.0000000	2.9240177
26	5.0990195	2.9624960
27	5.1961524	3.0000000
28	5.2915026	3.0365889
29	5.3851648	3.0723168
30	5.4772256	3.1072325

TABLE VIII.

N.	$\log_{10} N.$	$\log_e N.$
2	.3010300	.69314718
3	.4771213	1.09861229
5	.6989700	1.60943791
7	.8450980	1.94591015
11	1.0413927	2.39789527
13	1.1139434	2.56494936
17	1.2304489	2.83321334
19	1.2787536	2.94443898
23	1.3617278	3.13549422
29	1.4623980	3.36729583
31	1.4913617	3.43398720
37	1.5682017	3.61091791
41	1.6127839	3.71357207
43	1.6334685	3.76120012
47	1.6720979	3.85014760
53	1.7242759	3.97029191
59	1.7708520	4.07753744
61	1.7853298	4.11087386
67	1.8260748	4.20469262
71	1.8512583	4.26267988
73	1.8633229	4.29045944
79	1.8976271	4.36944785
83	1.9190781	4.41884061
89	1.9493900	4.48863637
97	1.9867717	4.57471098
101	2.0043214	4.61512052
103	2.0128372	4.63472899
107	2.0293838	4.67282883
109	2.0374265	4.69134788

NOTE.—The authorities for Table IV. are as follows:—Columns 2, 3, and 4 (Mercury excepted), Everett's experiments (Phil. Trans., 1867);  $g$  is here taken = 981.4. The densities in these cases are those of the specimens employed. Cols. 5 and 7, Rankine. Col. 6, Watt's Dict. of Chemistry. Col. 8, Dulong and Petit. Col. 10, Wertheim. Col. 11, Matthiessen. Table V. is abridged from Loomis's Astronomy. The values in Table III. are Angström's.

## BURCKHARDT'S FACTOR TABLES.

FOR ALL NUMBERS FROM 1 TO 99000.

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EXPLANATION.—The tables give the least divisor of every number from 1 up to 99000: but numbers divisible by 2, 3, or 5 are not printed. All the digits of the number whose divisor is sought, excepting the units and tens, will be found in one of the three rows of larger figures. The two remaining digits will be found in the left-hand column. The least divisor will then be found in the column of the first named digits, and in the row of the units and tens.

If the number be prime, a cipher is printed in the place of its least divisor.

The numbers in the first left-hand column are not consecutive. Those are omitted which have 2, 3, or 5 for a divisor. Since  $2^2 \cdot 3 \cdot 5^2 = 300$ , it follows that this column of numbers will re-appear in the same order after each multiple of 300 is reached.

MODE OF USING THE TABLES.—If the number whose prime factors are required is divisible by 2 or 5, the fact is evident upon inspection, and the division must be effected. The quotient then becomes the number whose factors are required. If this number, being within the range of the tables, is yet not given, *it is divisible by 3*. Dividing by 3, we refer to the tables again for the new quotient and its least factor, and so on.

EXAMPLES.—Required the prime factors of 310155.

Dividing by 5, the quotient is 62031. This number is within the range of the tables. But it is not found printed. Therefore 3 is a divisor of it. Dividing by 3, the quotient is 20677. The table gives 23 for the least factor of 20677. Dividing by 23, the quotient is 899.

The table gives 29 for the least factor of 899. Dividing by 29, the quotient is 31, a prime number. Therefore  $310155 = 3 \cdot 5 \cdot 23 \cdot 29 \cdot 31$ .

Again, required the divisors of 92881. The table gives 293 for the least divisor. Dividing by it, the quotient is 317. Referring to the tables for 317, a cipher is found in the place of the least divisor, and this signifies that 317 is a prime number.

Therefore  $92881 = 293 \times 317$ , the product of two primes.

It may be remarked that, to have resolved 92881 into these factors without the aid of the tables by the method of Art. 360, would have involved fifty-nine fruitless trial divisions by prime numbers.

	00	03	06	09	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63	66	69	72	75	78	81	84	87
01	0	7	0	17	0	19	0	11	7	37	0	0	13	47	0	7	0	0	11	0	17	0	7	67	19	13	29	0	31	7
07	0	0	0	0	17	11	13	7	29	0	31	0	0	0	7	0	11	0	0	13	0	7	0	0	0	0	37	11	7	0
11	0	0	13	0	7	0	0	0	0	0	0	7	23	0	0	13	17	19	7	0	0	0	11	0	0	7	73	0	13	31
13	0	0	0	11	0	17	7	0	19	0	23	0	0	7	11	0	0	0	0	29	7	59	17	31	0	11	13	7	47	0
17	0	0	0	7	0	37	23	29	0	11	7	31	0	0	0	0	0	7	0	0	11	0	13	0	7	0	0	0	19	23
19	0	11	0	0	23	7	17	13	41	0	0	0	7	0	0	0	61	0	0	7	13	71	0	11	0	73	7	23	0	0
23	0	17	7	13	0	0	0	11	0	7	0	0	0	0	41	0	7	47	11	59	19	0	37	7	31	0	0	0	0	11
29	0	7	17	0	0	11	31	0	7	0	13	0	19	0	0	7	11	23	61	17	0	0	7	13	0	0	0	11	0	7
31	0	0	0	0	7	0	0	0	11	0	7	0	0	0	0	23	0	7	0	11	37	13	19	29	7	17	41	47	0	0
37	0	0	7	0	0	29	11	0	0	7	0	47	0	31	19	13	7	11	0	0	0	0	0	7	0	0	17	79	11	0
41	0	11	0	0	17	23	7	0	0	0	0	13	11	7	0	19	47	53	0	0	7	17	29	11	13	0	0	7	23	0
43	0	7	0	23	11	0	19	0	7	13	17	0	0	0	0	7	29	37	0	0	0	0	7	53	0	19	11	17	0	7
47	0	0	0	0	29	7	0	19	0	41	11	0	7	0	31	0	37	0	13	7	0	11	17	0	0	0	7	0	0	0
49	7	0	11	13	0	0	43	7	31	0	0	17	41	11	7	0	13	19	0	0	23	7	61	0	11	47	29	7	13	7
53	0	0	0	0	7	0	0	17	0	0	43	7	13	59	0	29	23	0	7	11	0	0	0	17	0	7	0	31	79	0
59	0	0	0	7	0	0	11	17	0	31	7	0	0	37	0	47	43	7	53	13	73	0	0	0	7	0	29	41	11	19
61	0	19	0	31	13	7	0	0	23	11	0	0	7	17	0	0	0	13	43	7	11	0	0	0	53	0	7	0	0	0
67	0	0	23	0	7	0	0	11	0	0	0	7	19	0	17	0	31	0	7	73	0	59	0	13	7	0	0	0	0	11
71	0	7	11	0	31	0	0	13	7	17	37	0	0	11	0	7	0	0	0	29	13	23	7	0	11	67	17	0	43	7
73	0	0	0	7	19	11	0	41	0	47	7	0	0	29	0	17	11	7	13	23	0	0	0	19	7	0	0	11	37	31
77	7	13	0	0	0	19	0	7	0	0	17	11	0	41	7	23	0	31	0	53	59	7	11	0	19	0	0	13	7	67
79	0	0	7	11	0	0	0	0	37	7	0	31	13	23	11	19	7	0	0	0	0	0	7	7	29	11	0	0	61	0
83	0	0	0	0	0	0	7	37	13	11	0	17	29	7	0	13	19	71	0	0	7	13	41	0	0	0	0	7	17	0
89	0	0	13	23	0	7	0	11	19	0	0	0	7	0	0	13	0	0	11	7	0	0	0	29	37	0	7	19	13	11
91	7	17	0	0	0	37	31	7	47	0	11	0	0	13	7	0	67	29	17	0	0	7	0	0	23	0	13	0	7	59
97	0	0	17	0	0	0	7	13	11	0	19	43	0	7	0	0	59	0	23	11	7	0	37	0	0	71	53	7	29	19

	01	04	07	10	13	16	19	22	25	28	31	34	37	40	43	46	49	52	55	58	61	64	67	70	73	76	79	82	85	88	
01	0	0	0	7	0	0	0	31	41	0	7	19	0	0	11	43	13	7	0	0	0	37	0	0	7	7	11	0	59	0	13
08	0	13	19	17	0	7	11	0	0	0	29	41	7	0	13	0	0	11	0	7	17	19	0	47	67	0	7	13	11	0	0
07	0	11	7	19	0	0	0	0	23	7	13	0	11	0	59	17	7	41	0	0	31	43	19	7	0	0	0	29	47	0	0
09	0	0	0	0	7	0	23	47	13	53	0	7	0	19	31	11	0	0	7	37	41	13	0	43	0	7	11	0	67	23	0
13	0	7	23	0	13	0	0	0	7	29	11	0	47	0	19	7	17	13	37	0	0	11	7	0	71	23	41	43	0	7	0
19	7	0	0	0	0	0	19	7	11	0	0	13	0	0	7	31	0	17	0	11	29	7	0	0	13	19	0	0	7	0	0
21	11	0	7	0	0	0	17	0	0	7	0	11	61	0	29	0	7	23	0	0	0	0	11	7	0	89	0	0	0	0	0
27	0	7	0	13	0	0	41	17	7	11	53	23	0	0	0	7	13	0	0	0	11	0	7	0	17	29	0	19	0	7	0
31	0	0	17	0	11	7	0	23	0	19	31	47	7	29	61	11	0	0	0	7	0	59	53	79	0	13	7	0	19	0	0
33	7	0	0	0	31	23	0	7	17	0	13	0	0	37	7	41	0	0	11	19	0	7	0	13	0	17	0	0	7	11	0
37	0	19	11	17	7	0	13	0	43	0	0	7	37	11	0	0	0	0	7	13	17	41	0	31	11	7	0	0	0	0	0
39	0	0	0	0	13	11	7	0	0	17	43	19	0	7	0	0	11	13	29	0	7	47	23	0	41	0	17	7	0	0	0
43	11	0	0	7	17	31	29	0	0	0	7	11	19	13	43	0	0	7	23	0	0	17	11	0	7	0	13	0	0	37	0

	03	05	08	11	14	17	20	23	26	29	32	35	38	41	44	47	50	53	56	59	62	65	68	71	74	77	80	83	86	89			
49	0	0	7	0	0	19	17	0	13	0	7	47	0	23	0	0	0	7	29	31	0	11	0	17	7	0	0	0	73	83	0		
51	0	11	0	0	0	7	13	0	0	0	23	0	7	11	0	19	0	0	69	7	0	0	0	43	11	0	7	0	37	17	63		
57	0	0	0	7	23	0	0	19	37	0	7	0	0	13	0	0	0	0	7	0	0	47	11	29	0	7	13	73	23	43	17		
61	7	0	0	0	0	0	11	37	7	13	0	29	0	0	31	7	59	11	0	67	0	61	7	0	23	17	47	19	11	7	0		
63	0	0	7	0	0	29	0	13	31	11	7	0	0	53	17	0	0	7	19	0	11	0	23	0	7	37	79	0	0	0	0		
67	0	0	13	11	0	0	0	7	0	17	47	0	0	0	7	11	13	0	23	19	0	7	29	67	37	53	11	31	7	13	0		
69	13	7	0	0	0	37	0	11	0	7	19	0	0	0	13	17	7	0	11	0	31	0	0	7	0	0	0	0	0	11	7	0	
73	0	11	0	29	0	0	7	0	31	13	19	23	7	0	0	0	0	0	0	0	7	0	0	13	11	73	0	7	0	0	19	0	
79	0	0	19	13	7	7	21	0	43	0	11	7	0	0	29	0	13	0	0	7	37	11	0	0	0	47	7	79	17	23	13		
81	0	13	11	23	0	41	0	7	0	29	43	0	59	19	7	13	31	17	0	0	7	0	0	0	73	11	0	23	7	0	83		
87	11	0	0	0	0	19	7	0	0	13	0	0	11	7	61	41	43	0	17	37	7	23	13	11	19	83	0	7	0	31	0		
91	0	0	7	0	0	13	19	11	29	0	7	0	0	17	0	0	7	11	0	0	43	41	0	0	7	19	0	0	0	11	17	0	
93	0	17	13	0	0	7	0	0	0	11	31	7	0	0	23	13	0	0	67	7	71	11	43	0	41	0	7	0	0	13	0	7	
97	0	7	0	0	11	0	0	0	0	23	0	23	13	0	17	0	7	19	0	29	0	0	73	7	47	13	43	11	0	0	0	7	
99	0	0	17	7	0	0	0	11	23	13	7	7	0	29	0	53	37	0	7	11	17	0	67	13	31	7	0	19	43	0	11	0	
03	7	0	11	0	0	23	13	0	7	19	0	0	31	0	11	7	0	0	0	13	0	0	7	0	0	11	0	53	19	7	29	0	
09	11	0	0	0	0	0	0	7	0	0	0	0	11	7	0	17	0	0	0	71	19	7	11	0	31	13	13	0	7	0	59	7	
11	0	7	0	11	17	0	29	0	0	41	13	0	0	37	0	11	7	0	47	31	23	0	17	7	13	0	11	0	0	79	7	0	
17	7	11	19	0	13	17	0	0	7	0	0	0	11	23	7	53	29	13	13	41	61	7	17	11	0	0	0	0	0	7	37	0	
21	13	0	0	19	7	0	43	11	0	23	0	0	7	13	0	0	0	0	17	7	31	0	19	0	41	7	13	53	37	11	11	0	
23	0	0	0	0	0	0	7	23	43	37	11	13	0	7	0	0	0	0	0	0	7	11	0	17	13	0	71	7	0	0	0	0	
27	0	17	0	7	0	11	0	13	37	0	7	0	43	0	19	29	11	7	7	17	13	61	0	0	7	0	23	11	0	0	79	0	
29	0	23	0	0	0	7	0	17	11	29	0	0	7	0	43	0	47	73	13	13	7	0	0	0	17	59	7	0	0	0	0	0	
33	0	13	7	11	0	0	19	0	0	7	53	0	0	0	11	0	7	0	43	17	23	47	0	7	0	11	29	13	89	0	0	0	
39	0	7	0	17	0	37	0	0	0	0	41	0	11	0	23	7	0	19	0	0	17	13	7	11	43	71	0	31	53	7	0	0	
41	0	0	29	7	11	0	13	0	19	17	7	0	23	41	0	0	11	71	7	0	79	31	0	37	7	0	11	19	0	0	0	0	
47	13	0	7	31	0	0	23	0	0	7	17	0	0	11	0	47	7	0	0	0	0	41	0	7	11	61	13	17	0	0	23	0	
51	0	19	23	0	0	0	7	0	13	13	0	53	0	7	0	0	0	0	0	0	7	0	13	0	0	23	83	7	41	0	0	0	
53	11	7	0	0	0	0	0	13	7	0	0	11	0	0	61	7	31	53	0	0	13	0	7	23	29	0	0	0	17	7	7	0	
57	0	0	0	13	31	7	11	0	0	0	0	0	7	0	0	67	13	11	11	0	7	79	0	17	0	0	7	61	11	13	13	0	
59	7	13	0	19	0	0	29	7	0	11	0	0	17	0	7	0	0	23	0	59	11	7	19	0	0	0	0	0	13	7	17	0	
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[illegible]



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1.32	16485	15850	15220	14595	13975	13359	12748	12142	11540	10944
1.33	10353	09766	09184	08606	08034	07466	06903	06344	05791	05242
1.34	04698	04158	03624	03094	02568	02048	01532	01021	00514	00012
1.35	9.9499515	99023	98535	98052	97573	97100	96630	96166	95706	95251
1.36	94800	94355	93913	93477	93044	92617	92194	91776	91362	90953
1.37	90549	90149	89754	89363	88977	88595	88218	87846	87478	87115
1.38	86756	86402	86052	85707	85366	85030	84698	84371	84049	83731
1.39	83417	83108	82803	82503	82208	81916	81630	81348	81070	80797
1.40	80528	80263	80003	79748	79497	79250	79008	78770	78537	78308
1.41	78084	77864	77648	77437	77230	77027	76829	76636	76446	76261
1.42	76081	75905	75733	75565	75402	75243	75089	74939	74793	74652
1.43	74515	74382	74254	74130	74010	73894	73783	73676	73574	73476
1.44	73382	73292	73207	73125	73049	72976	72908	72844	72784	72728
1.45	72677	72630	72587	72549	72514	72484	72459	72437	72419	72406
1.46	72397	72393	72392	72396	72404	72416	72432	72452	72477	72506
1.47	72539	72576	72617	72662	72712	72766	72824	72886	72952	73022
1.48	73097	73175	73258	73345	73436	73531	73630	73734	73841	73953
1.49	74068	74188	74312	74440	74572	74708	74848	74992	75141	75293

NOTE.—This table is taken from Vol. II. of Legendre's work, and not from Vol. I., as stated in the Preface: the numbers given in Vol. I. being inaccurate in the seventh decimal place. In Vol. II. the values are given to twelve places of decimals. The figure here printed in the seventh place is



$n$	0	1	2	3	4	5	6	7	8	9
1.50	9.9475449	75610	75774	75943	76116	76292	76473	76658	76847	77040
1.51	77237	77438	77642	77851	78064	78281	78502	78727	78956	79189
1.52	79426	79667	79912	80161	80414	80671	80932	81196	81465	81738
1.53	82015	82295	82580	82868	83161	83457	83758	84062	84370	84682
1.54	84998	85318	85642	85970	86302	86638	86977	87321	87668	88019
1.55	88374	88733	89096	89463	89834	90208	90587	90969	91355	91745
1.56	92139	92537	92938	93344	93753	94166	94583	95004	95429	95857
1.57	96289	96725	97165	97609	98056	98508	98963	99422	99885	00351
1.58	9.9500822	01296	01774	02255	02741	03230	03723	04220	04720	05225
1.59	05733	06245	06760	07280	07803	08330	08860	09395	09933	10475
1.60	11020	11569	12122	12679	13240	13804	14372	14943	15519	16098
1.61	16680	17267	17857	18451	19048	19650	20254	20862	21475	22091
1.62	22710	23333	23960	24591	25225	25863	26504	27149	27798	28451
1.63	29107	29767	30430	31097	31767	32442	33120	33801	34486	35175
1.64	35867	36563	37263	37966	38673	39383	40097	40815	41536	42260
1.65	42989	43721	44456	45195	45938	46684	47434	48187	48944	49704
1.66	50468	51236	52007	52782	53560	54342	55127	55916	56708	57504
1.67	58303	59106	59913	60723	61536	62353	63174	63998	64826	65656
1.68	66491	67329	68170	69015	69864	70716	71571	72430	73293	74159
1.69	75028	75901	76777	77657	78540	79427	80317	81211	82108	83008
1.70	83912	84820	85731	86645	87563	88484	89409	90337	91268	92203
1.71	93141	94083	95028	95977	96929	97884	98843	99805	00771	01740
1.72	9.9602712	03688	04667	05650	06636	07625	08618	09614	10613	11616
1.73	12622	13632	14645	15661	16681	17704	18730	19760	20793	21830
1.74	22869	23912	24959	26009	27062	28118	29178	30241	31308	32377
1.75	33451	34527	35607	36690	37776	38866	39959	41055	42155	43258
1.76	44364	45473	46586	47702	48821	49944	51070	52200	53331	54467
1.77	55606	56749	57894	59043	60195	61350	62509	63671	64836	66004
1.78	67176	68351	69529	70710	71895	73082	74274	75468	76665	77866
1.79	79070	80277	81488	82701	83918	85138	86361	87588	88818	90051
1.80	91287	92526	93768	95014	96263	97515	98770	00029	01291	02555
1.81	9.9703823	05095	06369	07646	08927	10211	11498	12788	14082	15378
1.82	16678	17981	19287	20596	21908	23224	24542	25864	27189	28517
1.83	29848	31182	32520	33860	35204	36551	37900	39254	40610	41969
1.84	43331	44697	46065	47437	48812	50190	51571	52955	54342	55733
1.85	57126	58522	59922	61325	62730	64140	65551	66966	68384	69805
1.86	71230	72657	74087	75521	76957	78397	79839	81285	82734	84186
1.87	85640	87098	88559	90023	91490	92960	94433	95910	97389	98871
1.88	9.9800356	01844	03335	04830	06327	07827	09331	10837	12346	13859
1.89	15374	16893	18414	19939	21466	22996	24530	26066	27606	29148
1.90	30693	32242	33793	35348	36905	38465	40028	41595	43164	44736
1.91	46311	47890	49471	51055	52642	54232	55825	57421	59020	60622
1.92	62226	63834	65445	67058	68675	70294	71917	73542	75170	76802
1.93	78436	80073	81713	83356	85002	86651	88302	89957	91614	93275
1.94	94938	96605	98274	99946	01621	03299	04980	06663	08350	10039
1.95	9.9911732	13427	15125	16826	18530	20237	21947	23659	25375	27093
1.96	28815	30539	32266	33995	35728	37464	39202	40943	42688	44435
1.97	46185	47937	49693	51451	53213	54977	56744	58513	60286	62062
1.98	63840	65621	67405	69192	70982	72774	74570	76368	78169	79972
1.99	81779	83588	85401	87216	89034	90854	92678	94504	96333	98165

the one nearest to the true value whether in excess or defect. This table, and the table of Least Factors, have each been subjected to two complete and independent revisions before finally printing off.



# ALGEBRA.

## FACTORS.

$$1 \quad a^2 - b^2 = (a - b)(a + b).$$

$$2 \quad a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

$$3 \quad a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

And generally,

$$4 \quad a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \dots + b^{n-1}) \text{ always.}$$

$$5 \quad a^n - b^n = (a + b)(a^{n-1} - a^{n-2}b + \dots - b^{n-1}) \text{ if } n \text{ be even.}$$

$$6 \quad a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + \dots + b^{n-1}) \text{ if } n \text{ be odd.}$$

$$7 \quad (x + a)(x + b) = x^2 + (a + b)x + ab.$$

$$8 \quad (x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2$$

$$9 \quad (a + b)^2 = a^2 + 2ab + b^2. \quad + (bc + ca + ab)x + abc.$$

$$10 \quad (a - b)^2 = a^2 - 2ab + b^2.$$

$$11 \quad (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a + b).$$

$$12 \quad (a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - b^3 - 3ab(a - b).$$

Generally,

$$(a \pm b)^7 = a^7 \pm 7a^6b + 21a^5b^2 \pm 35a^4b^3 + 35a^3b^4 \pm 21a^2b^5 + 7ab^6 \pm b^7.$$

Newton's Rule for forming the coefficients: *Multiply any coefficient by the index of the leading quantity, and divide by the number of terms to that place to obtain the coefficient of the term next following.* Thus  $21 \times 5 \div 3$  gives 35, the following coefficient in the example given above. See also (125).

To square a polynomial: *Add to the square of each term twice the product of that term and every term that follows it.*

Thus,

$$(a + b + c + d)^2 \\ = a^2 + 2a(b + c + d) + b^2 + 2b(c + d) + c^2 + 2cd + d^2.$$

F

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$$13 \quad a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2).$$

$$14 \quad a^4 + b^4 = (a^2 + ab\sqrt{2} + b^2)(a^2 - ab\sqrt{2} + b^2).$$

$$15 \quad \left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2, \quad \left(x + \frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right).$$

$$16 \quad (a + b + c)^2 = a^2 + b^2 + c^2 + 2bc + 2ca + 2ab.$$

$$17 \quad (a + b + c)^3 = a^3 + b^3 + c^3 + 3(b^2c + bc^2 + c^2a + ca^2 + a^2b + ab^2) + 6abc.$$


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Observe that in an algebraical equation *the sign of any letter may be changed throughout*, and thus a new formula obtained, it being borne in mind that an *even* power of a negative quantity is positive. For example, by changing the sign of  $c$  in (16), we obtain

$$(a + b - c)^2 = a^2 + b^2 + c^2 - 2bc - 2ca + 2ab.$$


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$$18 \quad a^3 + b^3 - c^3 + 2ab = (a + b)^2 - c^3 = (a + b + c)(a + b - c) \text{ by (1).}$$

$$19 \quad a^2 - b^2 - c^2 + 2bc = a^2 - (b - c)^2 = (a + b - c)(a - b + c).$$

$$20 \quad a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab).$$

$$21 \quad bc^2 + b^2c + ca^2 + c^2a + ab^2 + a^2b + a^3 + b^3 + c^3 \\ = (a + b + c)(a^2 + b^2 + c^2).$$

$$22 \quad bc^2 + b^2c + ca^2 + c^2a + ab^2 + a^2b + 3abc \\ = (a + b + c)(bc + ca + ab).$$

$$23 \quad bc^2 + b^2c + ca^2 + c^2a + ab^3 + a^2b + 2abc = (b + c)(c + a)(a + b)$$

$$24 \quad bc^2 + b^2c + ca^2 + c^2a + ab^2 + a^2b - 2abc - a^3 - b^3 - c^3 \\ = (b + c - a)(c + a - b)(a + b - c).$$

$$25 \quad bc^2 - b^2c + ca^2 - c^2a + ab^2 - a^2b = (b - c)(c - a)(a - b).$$

$$26 \quad 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4 \\ = (a + b + c)(b + c - a)(c + a - b)(a + b - c).$$

$$27 \quad x^3 + 2x^2y + 2xy^2 + y^3 = (x + y)(x^2 + xy + y^2).$$

Generally for the division of  $(x + y)^n - (x^n + y^n)$  by  $x^2 + xy + y^2$  see (545).

## MULTIPLICATION AND DIVISION,

BY THE METHOD OF DETACHED COEFFICIENTS.

**28** Ex. 1:  $(a^4 - 3a^3b^3 + 2ab^3 + b^4) \times (a^3 - 2ab^2 - 2b^3).$

$$\begin{array}{r}
 1+0-3+2+1 \\
 1+0-2-2 \\
 \hline
 1+0-3+2+1 \\
 \quad -2-0+6-4-2 \\
 \quad \quad -2-0+6-4-2 \\
 \hline
 1+0-5+0+7+2-6-2
 \end{array}$$

Result  $a^7 - 5a^5b^3 + 7a^3b^4 + 2a^3b^6 - 6ab^6 - 2b^7.$

Ex. 2:  $(x^7 - 5x^5 + 7x^3 + 2x^2 - 6x - 2) \div (x^4 - 3x^2 + 2x + 1).$

$$\begin{array}{r}
 1+0-3+2+1) \quad 1+0-5+0+7+2-6-2 \quad (1+0-2-2 \\
 \quad \quad \quad -1-0+3-2-1 \\
 \hline
 \quad \quad \quad 0-2-2+6+2-6 \\
 \quad \quad \quad \quad +2+0-6+4+2 \\
 \hline
 \quad \quad \quad \quad -2+0+6-4-2 \\
 \quad \quad \quad \quad \quad +2+0-6+4+2 \\
 \hline
 \end{array}$$

Result  $x^3 - 2x - 2.$

*Synthetic Division.*

Ex. 3: Employing the last example, the work stands thus,

$$\begin{array}{r|l}
 & 1+0-5+0+7+2-6-2 \\
 -0 & 0+0+0+0 \\
 +3 & +3+0-6-6 \\
 -2 & -2+0+4+4 \\
 -1 & -1+0+2+2 \\
 \hline
 & 1+0-2-2
 \end{array}$$

Result  $x^3 - 2x - 2.$

[See also (248).]

Note that, in all operations with detached coefficients, the result must be written out in successive powers of the quantity which stood in its successive powers in the original expression.



## INDICES.

**29** Multiplication:  $a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3} + \frac{1}{3}} = a^{\frac{2}{3}}$ , or  $\sqrt[3]{a^2}$ ;

$$a^{\frac{1}{m}} \times a^{\frac{1}{n}} = a^{\frac{1}{m} + \frac{1}{n}} = a^{\frac{m+n}{mn}}, \text{ or } \sqrt[mn]{a^{m+n}}.$$

Division:  $a^{\frac{1}{3}} \div a^{\frac{1}{3}} = a^{\frac{1}{3} - \frac{1}{3}} = a^0$ , or  $\sqrt[3]{a}$ ;

$$a^{\frac{1}{n}} \div a^{\frac{1}{m}} = a^{\frac{1}{n} - \frac{1}{m}} = a^{\frac{m-n}{mn}}, \text{ or } \sqrt[mn]{a^{m-n}}.$$

Involution:  $(a^{\frac{1}{3}})^{\frac{1}{2}} = a^{\frac{1}{3} \times \frac{1}{2}} = a^{\frac{1}{6}}$ , or  $\sqrt[6]{a}$ .

Evolution:  $\sqrt[7]{a^{\frac{1}{3}}} = a^{\frac{1}{3} \times \frac{1}{7}} = a^{\frac{1}{21}}$ , or  $\sqrt[21]{a^{\frac{1}{3}}}$ .

$$a^{-n} = \frac{1}{a^n}, \quad a^0 = 1.$$

## HIGHEST COMMON FACTOR.

**30** RULE.—To find the highest common factor of two expressions: *Divide the one which is of the highest dimension by the other, rejecting first any factor of either expression which is not also a factor of the other. Operate in the same manner upon the remainder and the divisor, and continue the process until there is no remainder. The last divisor will be the highest common factor required.*

**31** EXAMPLE.—To find the H. C. F. of

$$3x^5 - 10x^4 + 15x^3 + 8 \quad \text{and} \quad x^5 - 2x^4 - 6x^3 + 4x^2 + 13x + 6.$$

1—	2—	6+	4+	13+	6		3+0—	10+	0+	15+	8	3
							—3+	6+	18—	12—	39—	18
							2)	6+	8—	12—	24—	10
								3+	4—	6—	12—	5
							—3—	9—	9—	3		
								—	5—	15—	15—	5
								+	5+	15+	15+	5
1	3—	6—	18+	12+	39+	18						
	—3—	4+	6+	12+	5							
	2)	—10—	12+	24+	44+	18						
		—	5—	6+	12+	22+	9					
						3						
5	—15—	18+	36+	66+	27							
	+15+	20—	30—	60—	25							
	2)	2+	6+	6+	2							
		1+	3+	3+	1							

Result H. C. F. =  $x^3 + 3x^2 + 3x + 1$ .

**32** Otherwise.—To form the H. C. F. of two or more algebraical expressions: *Separate the expressions into their simplest factors. The H. C. F. will be the product of the factors common to all the expressions, taken in the lowest powers that occur.*

### LOWEST COMMON MULTIPLE.

**33** The L. C. M. of two quantities is equal to their product divided by the H. C. F.

**34** Otherwise.—To form the L. C. M. of two or more algebraical expressions: *Separate them into their simplest factors. The L. C. M. will be the product of all the factors that occur, taken in the highest powers that occur.*

EXAMPLE.—The H. C. F. of  $a^3(b-x)^5c^2d$  and  $a^3(b-x)^3c^4e$  is  $a^3(b-x)^3c^4$ ; and the L. C. M. is  $a^3(b-x)^5c^4de$ .

### EVOLUTION.

To extract the Square Root of

$$a^3 - \frac{3a\sqrt{a}}{2} - \frac{3\sqrt{a}}{2} + \frac{41a}{16} + 1.$$

Arranging according to powers of  $a$ , and reducing to one denominator, the

expression becomes  $\frac{16a^3 - 24a^{\frac{3}{2}} + 41a - 24a^{\frac{1}{2}} + 16}{16}.$

**35** Detaching the coefficients, the work is as follows:—

$$\begin{array}{r} 16 - 24 + 41 - 24 + 16 \quad (4 - 3 + 4 \\ 16 \\ \hline 8 - 3 \quad \boxed{-24 + 41} \\ -3 \quad \boxed{24 - 9} \\ \hline 8 - 6 + 4 \quad \boxed{32 - 24 + 16} \\ \quad \boxed{-32 + 24 - 16} \end{array}$$

Result

$$\frac{4a - 3a^{\frac{1}{2}} + 4}{4} = a - \frac{3}{4}\sqrt{a} + 1$$

To extract the Cube Root of

$$37 \quad 8x^6 - 36x^5\sqrt{y} + 66x^4y - 63x^3y\sqrt{y} + 33x^2y^2 - 9x^2\sqrt{y} + y^3.$$

The terms here contain the successive powers of  $x$  and  $\sqrt{y}$ ; therefore, detaching the coefficients, the work will be as follows:—

I.	II.	III.
$\begin{array}{r} 6-3 \} \\ -6 \} \\ \hline 6-9+1 \end{array}$	$\begin{array}{r} 12 \\ -18+9 \} \\ \hline 12-18+9 \\ +9 \} \\ \hline 12-36+27 \\ 6-9+1 \\ \hline 12-36+33-9+1 \end{array}$	$\begin{array}{r} 8-36+66-63+33-9+1 (2-3+1 \\ -8 \\ \hline -36+66-63+33-9+1 \\ +36-54+27 \\ \hline 12-36+33-9+1 \\ -12+36-33+9-1 \\ \hline \end{array}$
<b>Result</b> $2x^2-3x\sqrt{y}+y.$		

EXPLANATION.—The cube root of 8 is 2, the first term of the result.

Place  $3 \times 2 = 6$  in the first column I.,  $3 \times 2^2 = 12$  in column II., and  $2^3 = 8$  in III., changing its sign for subtraction.

$-36 \div 12 = -3$ , the second term of the result.

Put  $-3$  in I.;  $(6-3) \times (-3)$  gives  $-18+9$  for II.

$(12-18+9) \times 3$  (changing sign) gives  $36-54+27$  for III. Then add.

Put twice  $(-3)$ , the term last found, in I., and the square of it in II.

Add the two last rows in I., and the three last in II.

$12 \div 12$  gives 1, the third term of the result.

Put 1 in col. I.,  $(6-9+1) \times 1$  gives  $6-9+1$  for col. II.

$(12-36+33-9+1) \times 1$  gives the same for III. Change the signs, and add, and the work is finished.

The foregoing process is but a slight variation of Horner's rule for solving an equation of any degree. See (533).

Transformations frequently required.

$$38 \quad \text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a+b}{a-b} = \frac{c+d}{c-d} \dots\dots\dots [68.]$$

$$39 \quad \text{If } \left. \begin{array}{l} x+y=a \\ \text{and } x-y=b \end{array} \right\}, \text{ then } \left\{ \begin{array}{l} x = \frac{1}{2}(a+b). \\ x = \frac{1}{2}(a-b). \end{array} \right.$$

$$40 \quad (x+y)^2 + (x-y)^2 = 2(x^2 + y^2).$$

$$41 \quad (x+y)^2 - (x-y)^2 = 4xy.$$

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42  $(x+y)^2 = (x-y)^2 + 4xy.$

43  $(x-y)^2 = (x+y)^2 - 4xy.$

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44

## EXAMPLES.

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$$\frac{2\sqrt{a^2-b^2} + \sqrt{c^2-x^2}}{2\sqrt{a^2-b^2} - \sqrt{c^2-x^2}} = \frac{3\sqrt{a^2-b^2} + \sqrt{c^2-d^2}}{3\sqrt{a^2-b^2} - \sqrt{c^2-d^2}},$$

$$\frac{\sqrt{c^2-x^2}}{2\sqrt{a^2-b^2}} = \frac{\sqrt{c^2-d^2}}{3\sqrt{a^2-b^2}} \dots\dots\dots [38.]$$

$$9(c^2-x^2) = 4(c^2-d^2),$$

$$x = \sqrt{5c^2+4d^2}.$$


---

To simplify a compound fraction, as

$$\frac{\frac{1}{a^2-ab+b^2} + \frac{1}{a^2+ab+b^2}}{\frac{1}{a^2-ab+b^2} - \frac{1}{a^2+ab+b^2}},$$

multiply the numerator and denominator by the L. C. M. of all the smaller denominators.

Result  $\frac{(a^2+ab+b^2) + (a^2-ab+b^2)}{(a^2+ab+b^2) - (a^2-ab+b^2)} = \frac{a^2+b^2}{ab}.$

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## QUADRATIC EQUATIONS.

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45 If  $ax^2+bx+c=0$ ,  $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$

46 If  $ax^2+2bx+c=0$ ; that is, if the coefficient of  $x$  be an even number,  $x = \frac{-b \pm \sqrt{b^2-ac}}{a}.$

---

47 *Method of solution without the formula.*

Ex.:  $2x^2-7x+3=0.$

Divide by 2,  $x^2 - \frac{7}{2}x + \frac{3}{2} = 0.$

Complete the square,  $x^2 - \frac{7}{2}x + \left(\frac{7}{4}\right)^2 = \frac{49}{16} - \frac{3}{2} = \frac{25}{16}$ .

Take square root,  $x - \frac{7}{4} = \pm \frac{5}{4},$

$$x = \frac{7 \pm 5}{4} = 3 \quad \text{or} \quad \frac{1}{2}.$$

**48** Rule for “completing the square” of an expression like  $x^2 - \frac{7}{2}x$ : *Add the square of half the coefficient of  $x$ .*

**49** The solution of the foregoing equation, employing formula (45), is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{7 \pm \sqrt{49 - 24}}{4} = \frac{7 \pm 5}{4} = 3 \quad \text{or} \quad \frac{1}{2}.$$

## THEORY OF QUADRATIC EXPRESSIONS.

If  $\alpha, \beta$  be the roots of the equation  $ax^2 + bx + c = 0$ , then

**50**  $ax^2 + bx + c = a(x - \alpha)(x - \beta).$

**51** Sum of roots  $\alpha + \beta = -\frac{b}{a}.$

**52** Product of roots  $\alpha\beta = \frac{c}{a}.$

Condition for the existence of equal roots—

**53**  $b^2 - 4ac$  must vanish.

**54** The solution of equations in one unknown quantity may sometimes be simplified by changing the quantity sought.

Ex. (1):  $2x + \frac{3x-1}{3x+1} + \frac{18x+6}{6x^2+5x-1} = 14 \dots\dots\dots (1).$

$$\frac{6x^2+5x-1}{3x+1} + \frac{6(3x+1)}{6x^2+5x-1} = 14.$$

Put  $y = \frac{6x^2+5x-1}{3x+1} \dots\dots\dots (2).$



thus

$$y + \frac{6}{y} = 14.$$

$$y^2 - 14y + 6 = 0.$$

$y$  having been determined from this quadratic,  $x$  is afterwards found from (2).

55 Ex. 2:

$$x^2 + \frac{1}{x^2} + x + \frac{1}{x} = 4.$$

$$\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) = 6.$$

Put  $x + \frac{1}{x} = y$ , and solve the quadratic in  $y$ .

56 Ex. 3:

$$x^2 + x + \frac{3}{2}\sqrt{2x^2 + x + 2} = \frac{x}{2} + 1.$$

$$2x^2 + x + 3\sqrt{2x^2 + x + 2} = 2,$$

$$2x^2 + x + 2 + 3\sqrt{2x^2 + x + 2} = 4.$$

Put  $\sqrt{2x^2 + x + 2} = y$ , and solve the quadratic

$$y^2 + 3y = 4.$$

57 Ex. 4:

$$\sqrt[3]{x^n} + \frac{2}{3\sqrt[3]{x^n}} = \frac{16}{3}x^{-n}.$$

$$x^{\frac{n}{3}} + \frac{2}{3}x^{\frac{2n}{3}} = \frac{16}{3}.$$

A quadratic in

$$y = x^{\frac{2n}{3}}.$$

### 58 To find Maxima and Minima values by means of a Quadratic Equation.

Ex.—Given

$$y = 3x^2 + 6x + 7,$$

to find what value of  $x$  will make  $y$  a maximum or minimum.

Solve the quadratic equation

$$3x^2 + 6x + 7 - y = 0.$$

Thus

$$x = \frac{-3 \pm \sqrt{3y - 12}}{3}. \quad [46.]$$

In order that  $x$  may be a real quantity, we must have  $3y$  not less than 12; therefore 4 is a minimum value of  $y$ , and the value of  $x$  which makes  $y$  a minimum is  $-1$ .

G

## SIMULTANEOUS EQUATIONS.

*General solution with two unknown quantities.*

Given

$$59 \quad \left. \begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned} \right\}, \quad x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \quad y = \frac{c_1a_2 - c_2a_1}{b_1a_2 - b_2a_1}.$$

*General solution with three unknown quantities.*

$$60 \quad \text{Given} \quad \left. \begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \right\},$$

$$x = \frac{d_1(b_2c_3 - b_3c_2) + d_2(b_3c_1 - b_1c_3) + d_3(b_1c_2 - b_2c_1)}{a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)},$$

and symmetrical forms for  $y$  and  $z$ .*Methods of solving simultaneous equations between two unknown quantities  $x$  and  $y$ .*

**61** I. *By substitution.*—Find one unknown in terms of the other from one of the two equations, and substitute this value in the remaining equation. Then solve the resulting equation.

$$\text{Ex.:} \quad \left. \begin{aligned} x + 5y &= 23 \dots\dots (1) \\ 7y &= 28 \dots\dots (2) \end{aligned} \right\}.$$

From (2),  $y = 4$ . Substitute in (1); thus

$$x + 20 = 23, \quad x = 3.$$

**62** II. *By the method of Multipliers.*

$$\text{Ex.:} \quad \left. \begin{aligned} 3x + 5y &= 36 \dots\dots (1) \\ 2x - 3y &= 5 \dots\dots (2) \end{aligned} \right\}.$$

Eliminate  $x$  by multiplying eq. (1) by 2, and (2) by 3; thus

$$6x + 10y = 72,$$

$$6x - 9y = 15,$$

$$19y = 57, \text{ by subtraction,}$$

$$y = 3;$$

$$\therefore x = 7, \text{ by substitution in eq. (2).}$$

**63**      *III. By changing the quantities sought.*

Ex. 1: 
$$\left. \begin{aligned} x-y &= 2 \dots\dots (1) \\ x^2-y^2+x+y &= 30 \dots\dots (2) \end{aligned} \right\}.$$

Let 
$$x+y = u, \quad x-y = v.$$

Substitute these values in (1) and (2),

$$\left. \begin{aligned} v &= 2 \\ uv+u &= 30 \end{aligned} \right\};$$

$$\therefore 2u+u = 30,$$

$$u = 10;$$

$$\therefore x+y = 10,$$

$$x-y = 2.$$

From which 
$$x = 6 \quad \text{and} \quad y = 4.$$

**64**    Ex. 2: 
$$\left. \begin{aligned} 2\frac{x+y}{x-y} + 10\frac{x-y}{x+y} &= 9 \dots\dots (1) \\ x^2+7y^2 &= 64 \dots\dots (2) \end{aligned} \right\}.$$

Substitute  $z$  for  $\frac{x+y}{x-y}$  in (1);

$$\therefore 2z + \frac{10}{z} = 9;$$

$$2z^2 - 9z + 10 = 0.$$

From which 
$$z = \frac{5}{2} \quad \text{or} \quad 2,$$

$$\frac{x+y}{x-y} = 2 \quad \text{or} \quad \frac{5}{2}.$$

From which 
$$x = 3y \quad \text{or} \quad \frac{7}{3}y.$$

Substitute in (2); thus 
$$y = 2 \quad \text{and} \quad x = 6,$$

or 
$$y = \frac{6}{\sqrt{7}} \quad \text{and} \quad x = 2\sqrt{7}.$$

**65**    Ex. 3: 
$$\left. \begin{aligned} 3x+5y &= xy \dots\dots (1) \\ 2x+7y &= 3xy \dots\dots (2) \end{aligned} \right\}.$$

Divide each quantity by  $xy$ ;

$$\left. \begin{aligned} \frac{3}{y} + \frac{5}{x} &= 1 \dots\dots (3) \\ \frac{2}{y} + \frac{7}{x} &= 3 \dots\dots (4) \end{aligned} \right\}.$$

Multiply (3) by 2, and (4) by 3, and by subtraction  $y$  is eliminated.

**66** IV. *By substituting  $y = tx$ , when the equations are homogeneous in the terms which contain  $x$  and  $y$ .*

Ex. 1: 
$$\begin{aligned} 52x^2 + 7xy &= 5y^2 \dots\dots (1) \\ 5x - 3y &= 17 \dots\dots (2) \end{aligned} \left. \vphantom{\begin{aligned} 52x^2 + 7xy &= 5y^2 \dots\dots (1) \\ 5x - 3y &= 17 \dots\dots (2) \end{aligned}} \right\}.$$

From (1),  $52x^2 + 7tx^2 = 5t^2x^2 \dots\dots (3)$   
 and, from (2),  $5x - 3tx = 17 \dots\dots (4)$   
 (3) gives  $52 + 7t = 5t^2$ ,  
 a quadratic equation from which  $t$  must be found, and its value substituted in (4).  
 $x$  is thus determined; and then  $y$  from  $y = tx$ .

**67** Ex. 2: 
$$\begin{aligned} 2x^2 + xy + 3y^2 &= 16 \dots\dots (1) \\ 3y - 2x &= 4 \dots\dots (2) \end{aligned} \left. \vphantom{\begin{aligned} 2x^2 + xy + 3y^2 &= 16 \dots\dots (1) \\ 3y - 2x &= 4 \dots\dots (2) \end{aligned}} \right\}.$$

From (1), by putting  $y = tx$ ,  
 $x^2(2 + t + 3t^2) = 16 \dots\dots (3)$   
 from (2),  $x(3t - 2) = 4 \dots\dots (4)$   
 squaring,  $x^2(9t^2 - 12t + 4) = 16$ ;  
 $\therefore 9t^2 - 12t + 4 = 2 + t + 3t^2$ ,

a quadratic equation for  $t$ .

$t$  being found from this, equation (4) will determine  $x$ ; and finally  $y = tx$ .

## RATIO AND PROPORTION.

**68** If  $a : b :: c : d$ ; then  $ad = bc$ , and  $\frac{a}{b} = \frac{c}{d}$ ;

$$\frac{a+b}{b} = \frac{c+d}{d}; \quad \frac{a-b}{b} = \frac{c-d}{d}; \quad \frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

**69** If  $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \&c.$ ; then  $\frac{a}{b} = \frac{a+c+e+\&c.}{b+d+f+\&c.}$

*General theorem.*

**70** If  $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \&c. = k$  say, then

$$k = \left\{ \frac{pa^n + qc^n + re^n + \&c.}{pb^n + qd^n + rf^n + \&c.} \right\}^{\frac{1}{n}}$$

where  $p, q, r, \&c.$  are any quantities whatever. Proved as in (71).

**71** RULE.—To verify any equation between such proportional quantities: *Substitute for  $a, c, e$ , &c., their equivalents  $kb, kd, kf$ , &c. respectively, in the given equation.*

Ex.—If  $a : b :: c : d$ , to show that

$$\frac{\sqrt{a-b}}{\sqrt{c-d}} = \frac{\sqrt{a}-\sqrt{b}}{\sqrt{c}-\sqrt{d}}.$$

Put  $kb$  for  $a$ , and  $kd$  for  $c$ ; thus

$$\frac{\sqrt{a-b}}{\sqrt{c-d}} = \frac{\sqrt{kb-b}}{\sqrt{kd-d}} = \frac{\sqrt{b}\sqrt{k-1}}{\sqrt{d}\sqrt{k-1}} = \frac{\sqrt{b}}{\sqrt{d}};$$

also 
$$\frac{\sqrt{a}-\sqrt{b}}{\sqrt{c}-\sqrt{d}} = \frac{\sqrt{kb}-\sqrt{b}}{\sqrt{kd}-\sqrt{d}} = \frac{\sqrt{b}(\sqrt{k}-1)}{\sqrt{d}(\sqrt{k}-1)} = \frac{\sqrt{b}}{\sqrt{d}}.$$

Identical results being obtained, the proposed equation must be true.

**72** If  $a : b : c : d : e$  &c., forming a continued proportion, then  $a : c :: a^2 : b^2$ , the duplicate ratio of  $a : b$ ,

$a : d :: a^3 : b^3$ , the triplicate ratio of  $a : b$ , and so on.

Also  $\sqrt{a} : \sqrt{b}$  is the subduplicate ratio of  $a : b$ ,

$a^{\frac{1}{2}} : b^{\frac{1}{2}}$  is the sesquiplicate ratio of  $a : b$ .

**73** The fraction  $\frac{a}{b}$  is made to approach nearer to unity in value, by adding the same quantity to the numerator and denominator. Thus

$$\frac{a+x}{b+x} \text{ is nearer to } 1 \text{ than } \frac{a}{b} \text{ is.}$$

**74** DEF.—The ratio compounded of the ratios  $a : b$  and  $c : d$  is the ratio  $ac : bd$ .

**75** If  $a : b :: c : d$ , and  $a' : b' :: c' : d'$ ; then, by compounding ratios,  $aa' : bb' :: cc' : dd'$ .

### VARIATION.

**76** If  $a \propto c$  and  $b \propto c$ , then  $(a \pm b) \propto c$  and  $\sqrt{ab} \propto c$ .

**77** If  $a \propto b$  }  
and  $c \propto d$  }, then  $ac \propto bd$  and  $\frac{a}{c} \propto \frac{b}{d}$ .

**78** If  $a \propto b$ , we may assume  $a = mb$ , where  $m$  is some constant.



## ARITHMETICAL PROGRESSION.

*General form of a series in A. P.*

$$79 \quad a, a+d, a+2d, a+3d, \dots a+(n-1)d.$$

$a$  = first term,

$d$  = common difference,

$l$  = last of  $n$  terms,

$s$  = sum of  $n$  terms; then

$$80 \quad l = a + (n-1)d.$$

$$81 \quad s = (a+l) \frac{n}{2}.$$

$$82 \quad s = \{2a + (n-1)d\} \frac{n}{2}.$$

PROOF.—By writing (79) in reversed order, and adding both series together.

## GEOMETRICAL PROGRESSION.

*General form of a series in G. P.*

$$83 \quad a, ar, ar^2, ar^3, \dots ar^{n-1}.$$

$a$  = first term,

$r$  = common ratio,

$l$  = last of  $n$  terms,

$s$  = sum of  $n$  terms; then

$$84 \quad l = ar^{n-1}.$$

$$85 \quad s = a \frac{r^n - 1}{r - 1} \quad \text{or} \quad a \frac{1 - r^n}{1 - r}.$$

If  $r$  be less than 1, and  $n$  be infinite,

$$86 \quad s = \frac{a}{1-r}, \text{ since } r^n = 0.$$

PROOF.—(85) is obtained by multiplying (83) by  $r$ , and subtracting one series from the other.

HARMONICAL PROGRESSION.

**87**  $a, b, c, d$ , &c. are in Harm. Prog. when the reciprocals  $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$ , &c. are in Arith. Prog.,

**88** Or when  $a : b :: a - b : b - c$  is the relation subsisting between any three consecutive terms.

**89**  $n^{\text{th}}$  term of the series  $= \frac{ab}{(n-1)a - (n-2)b}$ . [87, 80.]

**90** Approximate sum of  $n$  terms of the Harm. Prog.  $\frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}$ , &c., when  $d$  is small compared with  $a$ ,  

$$= \frac{(a+d)^n - a^n}{d(a+d)^n}.$$

*Evrate*

PROOF.—By taking instead the G.P.  $\frac{1}{a+d} + \frac{a}{(a+d)^2} + \frac{a^2}{(a+d)^3} + \dots$

**91** Arithmetic mean between  $a$  and  $b = \frac{a+b}{2}$ .

**92** Geometric do.  $= \sqrt{ab}$ .

**93** Harmonic do.  $= \frac{2ab}{a+b}$ .

The three means are in continued proportion.

PERMUTATIONS AND COMBINATIONS.

**94** The number of permutations of  $n$  things taken *all* at a time  $= n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1 \equiv n!$  or  $n^{(n)}$ .

PROOF BY INDUCTION.—Assume the formula to be true for  $n$  things. Now take  $n+1$  things. After each of these the remaining  $n$  things may be arranged in  $n!$  ways, making in all  $n \times n!$  [that is  $(n+1)!$ ] permutations of  $n+1$  things; therefore, &c. See also (233) for the mode of proof by Induction.

**95** The number of permutations of  $n$  things taken  $r$  at a time is denoted by  $P(n, r)$ .

$$P(n, r) = n(n-1)(n-2) \dots (n-r+1) \equiv n^{(r)}.$$

PROOF.—By (94); for  $(n-r)$  things are left out of each permutation; therefore  $P(n, r) = n! \div (n-r)!$ .

Observe that  $r$  = the number of factors.

**96** The number of combinations of  $n$  things taken  $r$  at a time is denoted by  $C(n, r)$ .

$$\begin{aligned} C(n, r) &= \frac{n(n-1)(n-2) \dots (n-r+1)}{1 \cdot 2 \cdot 3 \dots r} \equiv \frac{n^{(r)}}{r!} \\ &= \frac{n!}{r!(n-r)!} = C(n, n-r). \end{aligned}$$

For every combination of  $r$  things admits of  $r!$  permutations; therefore  $C(n, r) = P(n, r) \div r!$

**97**  $C(n, r)$  is greatest when  $r = \frac{1}{2}n$  or  $\frac{1}{2}(n \pm 1)$ , according as  $n$  is even or odd.

**98** The number of homogeneous products of  $r$  dimensions of  $n$  things is denoted by  $H(n, r)$ .

$$H(n, r) = \frac{n(n+1)(n+2) \dots (n+r-1)}{1 \cdot 2 \dots r} \equiv \frac{(n+r-1)^{(r)}}{r!}.$$

When  $r$  is  $> n$ , this reduces to

$$99 \quad \frac{(r+1)(r+2) \dots (n+r-1)}{(r-n)!}.$$

PROOF.— $H(n, r)$  is equal to the number of terms in the product of the expansions by the Bin. Th. of the  $n$  expressions  $(1-ax)^{-1}$ ,  $(1-bx)^{-1}$ ,  $(1-cx)^{-1}$ , &c.

Put  $a = b = c = \&c. = 1$ . The number will be the coefficient of  $x^r$  in  $(1-x)^{-n}$ . (128, 129.)

errata

**100** The number of permutations of  $n$  things taken all together, when  $a$  of them are alike,  $b$  of them alike,  $c$  alike, &c.

$$= \frac{n!}{a! b! c! \dots \&c.}$$

For, if the  $a$  things were all different, they would form  $a!$  permutations where there is now but one. So of  $b$ ,  $c$ , &c.

**101** The number of combinations of  $n$  things  $r$  at a time, in which any  $p$  of them will always be found, is

$$= C(n-p, r-p).$$

For, if the  $p$  things be set on one side, we have to add to them  $r-p$  things taken from the remaining  $n-p$  things in every possible way.

**102** THEOREM:  $C(n-1, r-1) + C(n-1, r) = C(n, r).$

PROOF BY INDUCTION; or as follows: Put one out of  $n$  letters aside; there are  $C(n-1, r)$  combinations of the remaining  $n-1$  letters  $r$  at a time. To complete the total  $C(n, r)$ , we must place with the excluded letter all the combinations of the remaining  $n-1$  letters  $r-1$  at a time.

**103** If there be one set of  $P$  things, another of  $Q$  things, another of  $R$  things, and so on; the number of combinations formed by taking one out of each set is  $= PQR \dots \&c.$ , the product of the numbers in the several sets.

For one of the  $P$  things will form  $Q$  combinations with the  $Q$  things. A second of the  $P$  things will form  $Q$  more combinations; and so on. In all,  $PQ$  combinations of two things. Similarly there will be  $PQR$  combinations of three things; and so on. This principle is very important.

**104** On the same principle, if  $p$ ,  $q$ ,  $r$ , &c. things be taken out of each set respectively, the number of combinations will be the product of the numbers of the separate combinations;

that is,  $= C(Pp) \cdot C(Qq) \cdot C(Rr) \dots \&c.$

**105** The number of combinations of  $n$  things taken  $m$  at a time, when  $p$  of the  $n$  things are alike,  $q$  of them alike,  $r$  of them alike, &c., will be the sum of all the combinations of each possible form of  $m$  dimensions, and this is equal to the coefficient of  $x^m$  in the expansion of

$$(1+x+x^2+\dots+x^p)(1+x+x^2+\dots+x^q)(1+x+x^2+\dots+x^r)\dots$$

**106** The total number of possible combinations under the same circumstances, when the  $n$  things are taken in all ways, 1, 2, 3 ...  $n$  at a time,

$$= (p+1)(q+1)(r+1)\dots - 1.$$

**107** The number of permutations when they are taken  $m$  at a time in all possible ways will be equal to the product of  $m!$  and the coefficient of  $x^m$  in the expansion of

$$\left\{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^p}{p!}\right\}\left\{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^q}{q!}\right\}\dots$$

.....&c.

## SURDS.

**108** To reduce  $\sqrt[3]{2808}$ . Decompose the number into its prime factors by (360); thus,

$$\sqrt[3]{2808} = \sqrt[3]{2^3 \cdot 3^3 \cdot 13} = 6 \sqrt[3]{13},$$

$$\sqrt[3]{a^{15}b^{10}c^8} = a^{5\frac{1}{3}}b^{3\frac{1}{3}}c^2 = a^5b^3c^2b^{\frac{1}{3}}c^{\frac{2}{3}} = a^5b^3c^2\sqrt[3]{bc^2}.$$

**109** To bring  $5\sqrt[4]{3}$  to an entire surd.

$$5\sqrt[4]{3} = \sqrt[4]{5^4 \cdot 3} = \sqrt[4]{1875},$$

$$x^{\frac{1}{3}}y^{\frac{1}{4}}z^{\frac{1}{5}} = x^{\frac{20}{60}}y^{\frac{15}{60}}z^{\frac{12}{60}} = \sqrt[60]{x^{20}y^{15}z^{12}}.$$

**110** To rationalise fractions having surds in their denominators.

$$\frac{1}{\sqrt{7}} = \frac{\sqrt{7}}{7}; \quad \frac{1}{\sqrt[3]{7}} = \frac{\sqrt[3]{49}}{\sqrt[3]{7 \times 49}} = \frac{\sqrt[3]{49}}{7}.$$



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**111**  $\frac{3}{9-\sqrt{80}} = \frac{3(9+\sqrt{80})}{81-80} = 3(9+\sqrt{80}),$   
 since  $(9-\sqrt{80})(9+\sqrt{80}) = 81-80$ , by (1).

---

**112**  $\frac{1}{1+2\sqrt{3}-\sqrt{2}} = \frac{1+2\sqrt{3}+\sqrt{2}}{(1+2\sqrt{3})^2-2} = \frac{1+2\sqrt{3}+\sqrt{2}}{11+4\sqrt{3}}$   
 $= \frac{(1+2\sqrt{3}+\sqrt{2})(11-4\sqrt{3})}{73}.$

---

**113**  $\frac{1}{\sqrt[3]{3}-\sqrt{2}} = \frac{1}{3^{\frac{1}{3}}-2^{\frac{1}{3}}}.$

Put  $3^{\frac{1}{3}} = x$ ,  $2^{\frac{1}{3}} = y$ , and take 6 the L.C.M. of the denominators 2 and 3, then

$$\frac{1}{x-y} = \frac{x^6+x^4y+x^2y^2+x^2y^3+xy^4+y^6}{x^6-y^6}, \text{ by (4);}$$

therefore  $\frac{1}{3^{\frac{1}{3}}-2^{\frac{1}{3}}} = \frac{3^{\frac{1}{3}}+3^{\frac{1}{3}}2^{\frac{1}{3}}+3^{\frac{1}{3}}2^{\frac{2}{3}}+3^{\frac{1}{3}}2^{\frac{1}{3}}+3^{\frac{1}{3}}2^{\frac{2}{3}}+2^{\frac{1}{3}}}{3^{\frac{1}{3}}-2^{\frac{1}{3}}}$   
 $= 3\sqrt[3]{9}+3\sqrt[3]{72}+6+2\sqrt[3]{648}+4\sqrt[3]{3}+4\sqrt[3]{2}.$

---

**114**  $\frac{1}{\sqrt[3]{3}+\sqrt{2}}.$  Here the result will be the same as in the last example if the signs of the even terms be changed. [Sec 5.]

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**115** A surd cannot be partly rational; that is,  $\sqrt{a}$  cannot be equal to  $\sqrt{b} \pm c$ . Proved by squaring.

**116** The product of two unlike squares is irrational;  
 $\sqrt{7} \times \sqrt{3} = \sqrt{21}$ , an irrational quantity.

**117** The sum or difference of two unlike surds cannot produce a single surd; that is,  $\sqrt{a} + \sqrt{b}$  cannot be equal to  $\sqrt{c}$ . By squaring.

**118** If  $a + \sqrt{m} = b + \sqrt{n}$ ; then  $a = b$  and  $m = n$ .  
 Theorems (115) to (118) are proved indirectly.

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**119** If  $\sqrt{a+\sqrt{b}} = \sqrt{x} + \sqrt{y},$   
 then  $\sqrt{a-\sqrt{b}} = \sqrt{x} - \sqrt{y}.$   
 By squaring and by (118).

**120** To express in two terms  $\sqrt{7+2\sqrt{6}}$ .

Let  $\sqrt{7+2\sqrt{6}} = \sqrt{x} + \sqrt{y};$

then  $x+y = 7$  ..... by squaring and by (118),

and  $x-y = \sqrt{7^2 - (2\sqrt{6})^2} = \sqrt{49-24} = 5$ , by (119);

$\therefore x = 6$  and  $y = 1$ .

Result  $\sqrt{6}+1$ .

General formula for the same—

**121**  $\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{1}{2}(a + \sqrt{a^2 - b})} \pm \sqrt{\frac{1}{2}(a - \sqrt{a^2 - b})}.$

Observe that no simplification is effected unless  $a^2 - b$  is a perfect square.

**122** To simplify  $\sqrt[3]{a + \sqrt{b}}$ .

Assume  $\sqrt[3]{a + \sqrt{b}} = x + \sqrt{y}.$

Let  $c = \sqrt[3]{a^2 - b}.$

Then  $x$  must be found by trial from the cubic equation

$$4x^3 - 3cx = a,$$

and  $y = x^2 - c.$

No simplification is effected unless  $a^2 - b$  is a perfect cube.

Ex. 1:  $\sqrt[3]{7+5\sqrt{2}} = x + \sqrt{y}.$

$$c = \sqrt[3]{49-50} = -1.$$

$$4x^3 + 3x = 7; \therefore x = 1, y = 2.$$

Result  $1 + \sqrt{2}.$

Ex. 2:  $\sqrt[3]{9\sqrt{3}-11\sqrt{2}} = \sqrt{x} + \sqrt{y}$ , two different surds.

Cubing,  $9\sqrt{3}-11\sqrt{2} = x\sqrt{x} + 3x\sqrt{y} + 3y\sqrt{x} + y\sqrt{y};$

$$\therefore \begin{cases} 9\sqrt{3} = (x+3y)\sqrt{x} \\ 11\sqrt{2} = (3x+y)\sqrt{y} \end{cases}; \quad (118)$$

$$\therefore x = 3 \text{ and } y = 2.$$

**123** To simplify  $\sqrt{(12+4\sqrt{3}+4\sqrt{5}+2\sqrt{15})}$ .

Assume  $\sqrt{(12+4\sqrt{3}+4\sqrt{5}+2\sqrt{15})} = \sqrt{x} + \sqrt{y} + \sqrt{z}$ .

Square, and equate corresponding surds.

Result  $\sqrt{3} + \sqrt{4} + \sqrt{5}$ .

**124** To express  $\sqrt[n]{A \pm B}$  in the form of two surds, where  $A$  and  $B$  are one or both quadratic surds and  $n$  is odd. Take  $q$  such that  $q(A^2 - B^2)$  may be a perfect  $n^{\text{th}}$  power, say  $p^n$ , by (361). Take  $s$  and  $t$  the nearest integers to  $\sqrt[n]{q(A+B)^2}$  and  $\sqrt[n]{q(A-B)^2}$ , then

$$\sqrt[n]{A+B} = \frac{1}{2^{\frac{2n}{n}}q} \{ \sqrt{s+t+2p} \pm \sqrt{s+t-2p} \}.$$

EXAMPLE: To reduce  $\sqrt[5]{89\sqrt{3}+109\sqrt{2}}$ .

Here  $A = 89\sqrt{3}$ ,  $B = 109\sqrt{2}$ ,

$$A^2 - B^2 = 1; \quad \therefore p = 1 \text{ and } q = 1.$$

$$\left. \begin{aligned} \sqrt[5]{q(A+B)^2} &= 9+f \\ \sqrt[5]{q(A-B)^2} &= 1-f \end{aligned} \right\} \quad \begin{aligned} &f \text{ being a proper fraction;} \\ &\therefore s = 9, \quad t = 1. \end{aligned}$$

$$\text{Result} \quad \frac{1}{2} (\sqrt{9+1+2} \pm \sqrt{9+1-2}) = \sqrt{3} + \sqrt{2}.$$

## BINOMIAL THEOREM.

**125**  $(a+b)^n =$

$$a^n + na^{n-1}b + \frac{n(n-1)}{2!} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!} a^{n-3}b^3 + \&c.$$

**126** General or  $(r+1)^{\text{th}}$  term,

$$\frac{n(n-1)(n-2) \dots (n-r+1)}{r!} a^{n-r} b^r$$

**127** or  $\frac{n!}{(n-r)! r!} a^{n-r} b^r$

if  $n$  be a positive integer.

If  $b$  be negative, the signs of the even terms will be changed.

If  $n$  be negative the expansion reduces to

$$128 \quad (a+b)^{-n} = a^{-n} - na^{-n-1}b + \frac{n(n+1)}{2!}a^{-n-2}b^2 - \frac{n(n+1)(n+2)}{3!}a^{-n-3}b^3 + \&c.$$

129 General term,

$$(-1)^r \frac{n(n+1)(n+2) \dots (n+r-1)}{r!} a^{-n-r} b^r. \quad [\text{See 98.}]$$

*Euler's proof.*—Let the expansion of  $(1+x)^n$ , as in (125), be called  $f(n)$ . Then it may be proved by Induction that the equation  $f(m) \times f(n) = f(m+n)$  ..... (1) is true when  $m$  and  $n$  are integers, and therefore universally true; because the *form* of an algebraical product is not altered by changing the letters involved into fractional or negative quantities. Hence

$$f(m+n+p+\&c.) = f(m) \times f(n) \times f(p), \&c.$$

Put  $m = n = p = \&c.$  to  $k$  terms, each equal  $\frac{h}{k}$ , and the theorem is proved for a fractional index.

Again, put  $-n$  for  $m$  in (1); thus, whatever  $n$  may be,

$$f(-n) \times f(n) = f(0) = 1,$$

which proves the theorem for a negative index.

130 For the greatest term in the expansion of  $(a+b)^n$ , take  $r =$  the integral part of  $\frac{(n+1)b}{a+b}$  or  $\frac{(n-1)b}{a-b}$ , according as  $n$  is positive or negative.

But if  $b$  be greater than  $a$ , and  $n$  negative or fractional, the terms increase without limit.

#### EXAMPLES.

Required the 40th term of  $\left(1 - \frac{2x}{3}\right)^{42}$ .

Here  $r = 39$ ;  $a = 1$ ;  $b = -\frac{2x}{3}$ ;  $n = 42$ .

By (127), the term will be

$$\frac{42!}{3!39!} \left(-\frac{2x}{3}\right)^{39} = -\frac{42 \cdot 41 \cdot 40}{1 \cdot 2 \cdot 3} \left(\frac{2x}{3}\right)^{39} \text{ by (96).}$$



**134** To write the coefficient of  $x^{3m+1}$  in the expansion of  $\left(x^2 - \frac{1}{x^2}\right)^{2n+1}$ .

The general term is

$$\frac{(2n+1)!}{(2n+1-r)! r!} x^{2(2n-r+1)} \cdot \frac{1}{x^{2r}} = \frac{(2n+1)!}{(2n+1-r)! r!} x^{4n-4r+2}.$$

Equate  $4n-4r+2$  to  $3m+1$ , thus

$$r = \frac{4n-3m+1}{4}.$$

Substitute this value of  $r$  in the general term; the required coefficient becomes

$$\frac{(2n+1)!}{\left[\frac{1}{4}(4n+3m+3)\right]! \left[\frac{1}{4}(4n-3m+1)\right]!}.$$

The value of  $r$  shows that there is no term in  $x^{3m+1}$  unless  $\frac{4n-3m+1}{4}$  is an integer.

**135** An approximate value of  $(1+x)^n$ , when  $x$  is small, is  $1+nx$ , by (125), neglecting  $x^2$  and higher powers of  $x$ .

**136** Ex.—An approximation to  $\sqrt[3]{999}$  by Bin. Th. (125) is obtained from the first two or three terms of the expansion of

$$(1000-1)^{\frac{1}{3}} = 10 - \frac{1}{3} \cdot 1000^{-\frac{2}{3}} = 10 - \frac{1}{300} = 9\frac{299}{300} \text{ nearly.}$$

## MULTINOMIAL THEOREM.

The general term in the expansion of  $(a+bx+cx^2+\&c.)^n$  is

$$\mathbf{137} \quad \frac{n(n-1)(n-2)\dots(p+1)}{q! r! s! \dots} a^p b^q c^r d^s \dots x^{q+2r+3s+\dots},$$

where  $p+q+r+s+\&c. = n$ ,

and the number of terms  $p, q, r, \&c.$  corresponds to the number of terms in the given multinomial.

$p$  is integral, fractional, or negative, according as  $n$  is one or the other.

If  $n$  be an integer, (137) may be written

$$\mathbf{138} \quad \frac{n!}{p! q! r! s!} a^p b^q c^r d^s \dots x^{q+2r+3s}.$$

[Deduced from the Bin. Theor.]



Ex. 1.—To write the coefficient of  $a^3bc^5$  in the expansion of  $(a+b+c+d)^{10}$ . Here put  $n=10, x=1, p=3, q=1, r=5, s=0$  in (138).

Result 
$$\frac{10!}{3! 5!} = 7.8.9.10.$$

Ex. 2.—To obtain the coefficient of  $x^8$  in the expansion of  $(1-2x+3x^2-4x^3)^4$ .

Here, comparing with (137), we have  $a=1, b=-2, c=3, d=-4$ ,

$$p+q+r+s=4,$$

$$q+2r+3s=8,$$

1	0	1	2
0	2	0	2
0	1	2	1
0	0	4	0

The numbers 1, 0, 1, 2 are particular values of  $p, q, r, s$  respectively, which satisfy the two equations given above.

0, 2, 0, 2 are another set of values which also satisfy those equations; and the four rows of numbers constitute all the solutions. In forming these rows always try the highest possible numbers on the right first.

Now substitute each set of values of  $p, q, r, s$  in formula (138) successively, as under:

$$\begin{aligned} \frac{4!}{2!} 1^1 (-2)^0 3^1 (-4)^2 &= 576 \\ \frac{4!}{2! 2!} 1^0 (-2)^2 3^0 (-4)^2 &= 384 \\ \frac{4!}{2!} 1^0 (-2)^1 3^2 (-4)^1 &= 864 \\ \frac{4!}{4!} 1^0 (-2)^0 3^4 (-4)^0 &= 81 \\ \text{Result} & \quad 1905 \end{aligned}$$

Ex. 3.—Required the coefficient of  $x^4$  in  $(1+2x-4x^2-2x^3)^{-\frac{1}{2}}$ .

Here  $a=1, b=2, c=-4, d=-2, n=-\frac{1}{2}$ ; and the two equations are

$$p+q+r+s=-\frac{1}{2},$$

$$q+2r+3s=4,$$

$-\frac{5}{2}$	1	0	1
$-\frac{5}{2}$	0	2	0
$-\frac{7}{2}$	2	1	0
$-\frac{9}{2}$	4	0	0

Employing formula (137), the remainder of the work stands as follows :

$$\begin{array}{rcl}
 \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)1^{-\frac{1}{2}}2^1(-4)^0(-2)^1 & = & -3 \\
 \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)1^{-\frac{1}{2}}2^0(-4)^2(-2)^0 & = & 6 \\
 \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)1^{-\frac{1}{2}}2^3(-4)^1(-2)^0 & = & 15 \\
 \frac{1}{4!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)1^{-\frac{1}{2}}2^4(-4)^0(-2)^0 & = & \frac{35}{8} \\
 \text{Result} & & \underline{22\frac{3}{8}}
 \end{array}$$

**139** The number of terms in the expansion of the multinomial  $(a+b+c+\text{ to } n \text{ terms})^r$  is the same as the number of homogeneous products of  $n$  things of  $r$  dimensions. See (97) and (98).

The greatest coefficient in the expansion of  $(a+b+c+\text{ to } m \text{ terms})^n$ ,  $n$  being an integer, is

**140**  $\frac{n!}{(q!)^m (q+1)^{(k)}}, \text{ where } qm+k=n.$

PROOF.—By making the denominator in (138) as small as possible. The notation is explained in (96).

## LOGARITHMS.

**142**  $\log_a N = x$  signifies that  $a^x = N$ , or

DEF.—The logarithm of a number is the power to which the base must be raised to produce that number.

**143**  $\log_a a = 1, \log 1 = 0.$

**144**  $\log MN = \log M + \log N.$

$$\log \frac{M}{N} = \log M - \log N.$$

$$\log (M)^n = n \log M.$$

$$\log \sqrt[n]{M} = \frac{1}{n} \log M. \quad [142]$$

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145  $\log_b a = \frac{\log_c a}{\log_c b}.$

That is—*The logarithm of a number to any base is equal to the logarithm of the number divided by the logarithm of the base, the two last named logarithms being taken to any the same base at pleasure.*

PROOF.—Let  $\log_c a = x$  and  $\log_c b = y$ ; then  $a = c^x$ ,  $b = c^y$ . Eliminate  $c$ .

$$c = a^{\frac{1}{x}} = b^{\frac{1}{y}}; \quad \therefore a = b^{\frac{x}{y}}, \quad \text{that is, } \log_b a = \frac{x}{y}. \quad \text{Q. e. d.}$$


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146  $\log_b a = \frac{1}{\log_a b}.$  Put  $c = a$  in (145).

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147  $\log_{10} N = \frac{\log_e N}{\log_e 10}$  by (145).

148  $\frac{1}{\log_e 10} = .43429448 \dots$

is called the modulus of the common system of logarithms; that is, the factor which will convert logarithms of numbers calculated to the base  $e$  into the corresponding logarithms to the base 10. See (154).

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## EXPONENTIAL THEOREM.

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149  $a^x = 1 + cx + \frac{c^2 x^2}{2!} + \frac{c^3 x^3}{3!} + \&c.,$

where  $c = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.$

PROOF.  $a^x = \{1 + (a-1)\}^x$ . Expand this by Binomial Theorem, and collect the coefficients of  $x$ ; thus  $c$  is obtained. Assume  $c_2, c_3, \&c.$ , as the coefficients of the succeeding powers of  $x$ , and with this assumption write out the expansions of  $a^x, a^y$ , and  $a^{x+y}$ . Form the product of the first two series, which product must be equivalent to the third. Therefore equate the coefficient of  $x$  in this product with that in the expansion of  $a^{x+y}$ . In the identity so obtained, equate the coefficients of the successive powers of  $y$  to determine  $c_2, c_3, \&c.$

Let  $e$  be that value of  $a$  which makes  $c = 1$ , then

$$150 \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \&c.$$

$$151 \quad e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \&c.$$

$$= 2.718281828 \dots$$

[See (295).]

PROOF.—By making  $x = 1$  in (150).

152 By making  $x = 1$  in (149) and  $x = c$  in (150), we obtain

$$a = e^c; \text{ that is, } c = \log_e a. \text{ Therefore by (149)}$$

$$154 \quad \log_e a = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.$$

$$155 \quad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.$$

$$156 \quad \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \&c. \quad [154]$$

$$157 \quad \therefore \log \frac{1+x}{1-x} = 2 \left\{ x + \frac{x^3}{3} + \frac{x^5}{5} + \&c. \right\}.$$

Put  $\frac{m-1}{m+1}$  for  $x$  in (157); thus,

$$158 \quad \log m = 2 \left\{ \frac{m-1}{m+1} + \frac{1}{3} \left( \frac{m-1}{m+1} \right)^3 + \frac{1}{5} \left( \frac{m-1}{m+1} \right)^5 + \&c. \right\}.$$

Put  $\frac{1}{2n+1}$  for  $x$  in (157); thus,

$$159 \quad \log(n+1) - \log n \\ = 2 \left\{ \frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \&c. \right\}.$$

## CONTINUED FRACTIONS AND CONVERGENTS.

**160** To find convergents to  $3.14159 = \frac{314159}{100000}$ . Proceed as in the rule for H. C. F.

7	100000	314159	3
	99113	300000	
1	887	14159	15
	854	887	
1	33	5289	
	29	4435	
4	4	854	25
	4	66	
		194	
		165	
		29	7
		28	
		1	

The continued fraction is

$$3 + \frac{1}{7 + \frac{1}{15 + \&c.}}$$

or, as it is more conveniently written,

$$3 + \frac{1}{7 + \frac{1}{15 + \&c.}}$$

The convergents are formed as follows:—

3	7	15	1	25	1	7	4
3	22	333	355	9208	9563	76149	314159
1	7	106	113	2931	3044	24239	100000

**161** RULE.—Write the quotients in a row, and the first two convergents at sight (in the example 3 and  $3 + \frac{1}{7}$ ). Multiply the numerator of any convergent by the next quotient, and add the previous numerator. The result is the numerator of the next convergent. Proceed in the same way to determine the denominator. The last convergent should be the original fraction in its lowest terms.

**162** *Formula for forming the convergents.*

If  $\frac{p_{n-2}}{q_{n-2}}, \frac{p_{n-1}}{q_{n-1}}, \frac{p_n}{q_n}$  are any consecutive convergents, and  $a_{n-2}, a_{n-1}, a_n$  the corresponding quotients; then

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}.$$

The  $n^{\text{th}}$  convergent is therefore

$$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} \equiv F_n.$$

The true value of the continued fraction will be expressed by

$$163 \quad F = \frac{a'_n p_{n-1} + p_{n-2}}{a'_n q_{n-1} + q_{n-2}},$$

in which  $a'_n$  is the complete quotient or value of the continued fraction commencing with  $a_n$ .

$$164 \quad p_n q_{n-1} - p_{n-1} q_n = \pm 1 \text{ alternately, by (162).}$$

The convergents are alternately greater and less than the original fraction, and are always in their lowest terms.

165 The difference between  $F_n$  and the true value of the continued fraction is

$$< \frac{1}{q_n q_{n+1}} \quad \text{and} \quad > \frac{1}{q_n (q_n + q_{n+1})}$$

and this difference therefore diminishes as  $n$  increases.

$$\text{PROOF.}—\text{By taking the difference, } \frac{p_n}{q_n} - \frac{a'_n p_{n+1} + p_n}{a'_n q_{n+1} + q_n}. \quad (163)$$

Also  $F$  is nearer the true value than any other fraction with a less denominator.

166  $F_n F_{n+1}$  is greater or less than  $F^2$  according as  $F_n$  is greater or less than  $F_{n+1}$ .

### *General Theory of Continued Fractions.*

167 First class of continued fraction.

$$F = \frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \&c.$$

Second class of continued fraction.

$$V = \frac{b_1}{a_1 -} \frac{b_2}{a_2 -} \frac{b_3}{a_3 -} \&c.$$

$a_1, b_1, \&c.$  are taken as positive quantities.



$\frac{b_1}{a_1}, \frac{b_2}{a_2}, \&c.$  are termed *components* of the continued fraction. If the components be infinite in number, the continued fraction is said to be infinite.

Let the successive convergents be denoted by

$$\frac{p_1}{q_1} = \frac{b_1}{a_1}; \quad \frac{p_2}{q_2} = \frac{b_1}{a_1 + \frac{b_2}{a_2}}; \quad \frac{p_3}{q_3} = \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3}}}; \text{ and so on.}$$

**168** The law of formation of the convergents is

<p>For <math>F</math>,</p> $\begin{cases} p_n = a_n p_{n-1} + b_n p_{n-2} \\ q_n = a_n q_{n-1} + b_n q_{n-2} \end{cases}$		<p>For <math>V</math>,</p> $\begin{cases} p_n = a_n p_{n-1} - b_n p_{n-2} \\ q_n = a_n q_{n-1} - b_n q_{n-2} \end{cases}$ <p style="text-align: right;">[Proved by Induction.]</p>
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The relation between the successive differences of the convergents is, by (168),

$$\mathbf{169} \quad \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \mp \frac{b_{n+1} q_{n-1}}{q_{n+1}} \left( \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} \right).$$

Take the  $-$  sign for  $F$ , and the  $+$  for  $V$ .

$$\mathbf{170} \quad p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} b_1 b_2 b_3 \dots b_n. \quad (168)$$

**171** The odd convergents for  $F$ ,  $\frac{p_1}{q_1}, \frac{p_3}{q_3}, \&c.$ , continually decrease, and the even convergents,  $\frac{p_2}{q_2}, \frac{p_4}{q_4}, \&c.$ , continually increase. (167)

Every odd convergent is greater, and every even convergent is less, than all following convergents. (169)

**172** DEF.—If the difference between consecutive convergents diminishes without limit, the infinite continued fraction is said to be *definite*. If the same difference tends to a fixed value greater than zero, the infinite continued fraction is *indefinite*; the odd convergents tending to one value, and the even convergents to another.

**173**  $F$  is definite if the ratio of every quotient to the next component is greater than a fixed quantity.

PROOF.—Apply (169) successively.

**174**  $F$  is incommensurable when the components are all proper fractions and infinite in number.

PROOF.—Indirectly, and by (168).

**175** If  $a$  be never less than  $b+1$ , the convergents of  $V$  are all positive proper fractions, increasing in magnitude,  $p_n$  and  $q_n$  also increasing with  $n$ . By (167) and (168).

**176** If, in this case,  $V$  be infinite, it is also definite, being  $=1$ , if  $a$  always  $=b+1$  while  $b$  is less than 1, (175); and being less than 1, if  $a$  is ever greater than  $b+1$ . By (180).

**177**  $V$  is incommensurable when it is less than 1, and the components are all proper fractions and infinite in number.

**180** If in the continued fraction  $V$  (167), we have  $a_n = b_n + 1$  always; then, by (168),

$$p_n = b_1 + b_1 b_2 + b_1 b_2 b_3 + \dots \text{ to } n \text{ terms, and } q_n = p_n + 1.$$

**181** If, in the continued fraction  $F$ ,  $a_n$  and  $b_n$  are constant and equal, say, to  $a$  and  $b$  respectively; then  $p_n$  and  $q_n$  are respectively equal to the coefficients of  $x^{n-1}$  in the expansions

of 
$$\frac{b}{1-ax-bx^2} \quad \text{and} \quad \frac{a+bx}{1-ax-bx^2}.$$

PROOF.— $p_n$  and  $q_n$  are the  $n^{\text{th}}$  terms of two recurring series. See (168) and (251).

**182** *To convert a Series into a Continued Fraction.*

The series 
$$\frac{1}{u} + \frac{x}{u_1} + \frac{x^2}{u_2} + \dots + \frac{x^n}{u_n}$$

is equal to a continued fraction  $V$  (167), with  $n+1$  components; the first, second, and  $n+1^{\text{th}}$  components being

$$\frac{1}{u}, \quad \frac{u^2 x}{u_1 + ux}, \quad \dots \quad \frac{u_{n-1}^2 x}{u_n + u_{n-1} x}.$$

[Proved by Induction.

**183** The series

$$\frac{1}{v} + \frac{x}{vv_1} + \frac{x^2}{vv_1v_2} + \dots + \frac{x^n}{vv_1v_2\dots v_n}$$

is equal to a continued fraction  $V$  (167), with  $n+1$  components, the first, second, and  $n+1^{\text{th}}$  components being

$$\frac{1}{v}, \quad \frac{vx}{v_1+x}, \quad \dots, \quad \frac{v_{n-1}x}{v_n+x}. \quad [\text{Proved by Induction.}]$$

**184** The sign of  $x$  may be changed in either of the statements in (182) or (183).

**185** Also, if any of these series are convergent and infinite, the continued fractions become infinite.

**186** *To find the value of a continued fraction with recurring quotients.*

Let the continued fraction be

$$x = \frac{b_1}{a_1 + \dots + \frac{b_n}{a_n + y}} \quad \text{where} \quad y = \frac{b_{n+1}}{a_{n+1} + \dots + \frac{b_{n+m}}{a_{n+m} + y}}$$

so that there are  $m$  recurring quotients. Form the  $n^{\text{th}}$  convergent for  $x$ , and the  $m^{\text{th}}$  for  $y$ . Then, by substituting the complete quotients  $a_n + y$  for  $a_n$ , and  $a_{n+m} + y$  for  $a_{n+m}$  in (168), two equations are obtained of the forms

$$x = \frac{Ay + B}{Cy + D} \quad \text{and} \quad y = \frac{Ey + F}{Gy + H},$$

from which, by eliminating  $y$ , a quadratic equation for determining  $x$  is obtained.

**187** If  $\frac{b_1}{a_1 + \dots + \frac{b_n}{a_n +}}$

be a continued fraction, and

$$\frac{p_1}{q_1}, \quad \dots, \quad \frac{p_n}{q_n}$$

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the corresponding first  $n$  convergents; then  $\frac{q_{n-1}}{q_n}$ , developed by (168), produces the continued fraction

$$\frac{1}{a_n + \frac{b_n}{a_{n-1} + \frac{b_{n-1}}{a_{n-2} + \dots + \frac{b_3}{a_2 + \frac{b_2}{a_1}}}}$$

the quotients being the same but in reversed order.

### INDETERMINATE EQUATIONS.

**188** Given  $ax + by = c$

free from fractions, and  $\alpha, \beta$  integral values of  $x$  and  $y$  which satisfy the equation, the complete integral solution is given by

$$x = \alpha - bt$$

$$y = \beta + at$$

where  $t$  is any integer.

**EXAMPLE.**—Given  $5x + 3y = 112$ .

Then  $x = 20, y = 4$  are values;

$$\therefore \begin{cases} x = 20 - 3t \\ y = 4 + 5t \end{cases}$$

The values of  $x$  and  $y$  may be exhibited as under:

$t =$	-2	-1	0	1	2	3	4	5	6	7
$x =$	26	23	20	17	14	11	8	5	2	-1
$y =$	-6	-1	4	9	14	19	24	29	34	39

For solutions in positive integers  $t$  must lie between  $\frac{20}{3} = 6\frac{2}{3}$  and  $-\frac{4}{5}$ ; that is,  $t$  must be 0, 1, 2, 3, 4, 5, or 6, giving 7 positive integral solutions.

**189** If the equation be

$$ax - by = c$$

the solutions are given by

$$x = \alpha + bt$$

$$y = \beta + at.$$

EXAMPLE :  $4x - 3y = 19$ .

Here  $x = 10$ ,  $y = 7$  satisfy the equation ;

$$\therefore \begin{cases} x = 10 + 3t \\ y = 7 + 4t \end{cases} \text{ furnish all the solutions.}$$

The simultaneous values of  $t$ ,  $x$ , and  $y$  will be as follows :—

$t = -5$	$-4$	$-3$	$-2$	$-1$	$0$	$1$	$2$	$3$
$x = -5$	$-2$	$1$	$4$	$7$	$10$	$13$	$16$	$19$
$y = -13$	$-9$	$-5$	$-1$	$3$	$7$	$11$	$15$	$19$

The number of positive integral solutions is infinite, and the least positive integral values of  $x$  and  $y$  are given by the limiting value of  $t$ , viz.,

$$t > -\frac{10}{3} \quad \text{and} \quad t > -\frac{7}{4};$$

that is,  $t$  must be  $-1, 0, 1, 2, 3$ , or greater.

**190** If two values,  $\alpha$  and  $\beta$ , cannot readily be found by inspection, as, for example, in the equation

$$17x + 13y = 14900,$$

*divide by the least coefficient, and equate the remaining fractions to  $t$ , an integer ; thus*

$$y + x + \frac{4x}{13} = 1146 + \frac{2}{13} \dots\dots\dots(1);$$

$$\therefore 4x - 2 = 13t.$$

Repeat the process ; thus

$$x - \frac{2}{4} = 3t + \frac{t}{4},$$

$$\therefore t + 2 = 4u.$$

Put  $u = 1,$

$$\therefore t = 2,$$

$$x = \frac{13t + 2}{4} = 7 = \alpha;$$

and

$$y + x + t = 1146, \text{ by (1),}$$

$$\therefore y = 1146 - 7 - 2 = 1137 = \beta.$$

The general solution will be

$$x = 7 - 13t,$$

$$y = 1137 + 17t,$$

Or, changing the sign of  $t$  for convenience,

$$x = 7 + 13t,$$

$$y = 1137 - 17t.$$

Here the number of solutions in positive integers is equal to the number of integers lying between  $-\frac{7}{13}$  and  $\frac{1137}{17}$ ;

or  $-\frac{7}{13}$  and  $66\frac{11}{17}$ ; that is, 67.

**191** Otherwise.—Two values of  $x$  and  $y$  may be found in the following manner:—

Find the nearest converging fraction to  $\frac{17}{13}$ . [By (160).

This is  $\frac{4}{3}$ . By (164) we have

$$17 \times 3 - 13 \times 4 = -1.$$

Multiply by 14900, and change the signs;

$$\therefore 17(-44700) + 13(59600) = 14900;$$

which shows that we may take  $\begin{cases} \alpha = -44700 \\ \beta = 59600 \end{cases}$

and the general solution may be written

$$x = -44700 + 13t,$$

$$y = 59600 - 17t.$$

This method has the disadvantage of producing high values of  $\alpha$  and  $\beta$ .

**192** The values of  $x$  and  $y$ , in positive integers, which satisfy the equation  $ax \pm by = c$ , form two Arithmetic Progressions, of which  $b$  and  $a$  are respectively the common differences. See examples (188) and (189).

**193** Abbreviation of the method in (169).

EXAMPLE:  $11x - 18y = 63.$

Put  $x = 9z$ , and divide by 9; then proceed as before.

**194** To obtain integral solutions of  $ax + by + cz = d.$

Write the equation thus

$$ax + by = d - cz.$$

Put successive integers for  $z$ , and solve for  $x, y$  in each case.



TO REDUCE A QUADRATIC SURD TO A  
CONTINUED FRACTION.

**195** EXAMPLE :

$$\begin{aligned}\sqrt{29} &= 5 + \sqrt{29-5} = 5 + \frac{4}{\sqrt{29+5}}, \\ \frac{\sqrt{29+5}}{4} &= 2 + \frac{\sqrt{29-3}}{4} = 2 + \frac{5}{\sqrt{29+3}}, \\ \frac{\sqrt{29+3}}{5} &= 1 + \frac{\sqrt{29-2}}{5} = 1 + \frac{5}{\sqrt{29+2}}, \\ \frac{\sqrt{29+2}}{5} &= 1 + \frac{\sqrt{29-3}}{5} = 1 + \frac{4}{\sqrt{29+3}}, \\ \frac{\sqrt{29+3}}{4} &= 2 + \frac{\sqrt{29-5}}{4} = 2 + \frac{1}{\sqrt{29+5}}, \\ \sqrt{29+5} &= 10 + \sqrt{29-5} = 10 + \frac{4}{\sqrt{29+5}}.\end{aligned}$$

The quotients 5, 2, 1, 1, 2, 10 are the greatest integers contained in the quantities in the first column. The quotients now recur, and the surd  $\sqrt{29}$  is equivalent to the continued fraction

$$5 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{10 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2 + \&c.}}}}}}}}}$$

The convergents to  $\sqrt{29}$ , formed as in (160), will be

$$\frac{5}{1}, \frac{11}{2}, \frac{16}{3}, \frac{27}{5}, \frac{70}{13}, \frac{727}{135}, \frac{1524}{283}, \frac{2251}{418}, \frac{3775}{701}, \frac{9801}{1820}.$$

**196** Note that the last quotient 10 is the greatest and twice the first, that the *second* is the first of the recurring ones, and that the recurring quotients, excluding the last, consist of pairs of equal terms, quotients equi-distant from the first and last being equal. These properties are universal. (See 204-210).

*To form high convergents rapidly.*

**197** Suppose  $m$  the number of recurring quotients, or any

multiple of that number, and let the  $m^{\text{th}}$  convergent to  $\sqrt{Q}$  be represented by  $F_m$ ; then the  $2m^{\text{th}}$  convergent is given by the

formula 
$$F_{2m} = \frac{1}{2} \left\{ F_m + \frac{Q}{F_m} \right\} \text{ by (203) and (210).}$$

**198** For example, in approximating to  $\sqrt{29}$  above, there are five recurring quotients. Take  $m = 2 \times 5 = 10$ ; therefore, by

$$F_{20} = \frac{1}{2} \left\{ F_{10} + \frac{29}{F_{10}} \right\},$$

$$F_{10} = \frac{9801}{1820}, \text{ the } 10^{\text{th}} \text{ convergent.}$$

Therefore 
$$F_{20} = \left\{ \frac{9801}{1820} + 29 \times \frac{1820}{9801} \right\} = \frac{192119201}{35675640}$$

the  $20^{\text{th}}$  convergent to  $\sqrt{29}$ ; and the labour of calculating the intervening convergents is saved.

## GENERAL THEORY.

**199** The process of (174) may be exhibited as follows:—

$$\begin{aligned} \frac{\sqrt{Q+c_1}}{r_1} &= a_1 + \frac{r_2}{\sqrt{Q+c_2}} \\ \frac{\sqrt{Q+c_2}}{r_2} &= a_2 + \frac{r_3}{\sqrt{Q+c_3}} \\ \dots & \dots \dots \dots \\ \dots & \dots \dots \dots \\ \frac{\sqrt{Q+c_n}}{r_n} &= a_n + \frac{r_{n+1}}{\sqrt{Q+c_{n+1}}}. \end{aligned}$$

**200** Then 
$$\sqrt{Q} = a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \&c.$$

The quotients  $a_1, a_2, a_3, \&c.$  are the integral parts of the fractions on the left.

**201** The equations connecting the remaining quantities are

$$\begin{array}{ll}
 c_1 = 0 & r_1 = 1 \\
 c_2 = a_1 r_1 - c_1 & r_2 = \frac{Q - c_2^2}{r_1} \\
 c_3 = a_2 r_2 - c_2 & r_3 = \frac{Q - c_3^2}{r_2} \\
 \dots \quad \dots \quad \dots & \dots \quad \dots \quad \dots \\
 c_n = a_{n-1} r_{n-1} - c_{n-1} & r_n = \frac{Q - c_n^2}{r_{n-1}}
 \end{array}$$

The  $n^{\text{th}}$  convergent to  $\sqrt{Q}$  will be

**202** 
$$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}. \quad [\text{By Induction.}]$$

The true value of  $\sqrt{Q}$  is what this becomes when we substitute for  $a_n$  the complete quotient  $\frac{\sqrt{Q} + c_n}{r_n}$ , of which  $a_n$  is only the integral part. This gives

**203** 
$$\sqrt{Q} = \frac{(\sqrt{Q} + c_n) p_{n-1} + r_n p_{n-2}}{(\sqrt{Q} + c_n) q_{n-1} + r_n q_{n-2}}.$$

By the relations (199) to (203) the following theorems are demonstrated:—

**204** All the quantities  $a$ ,  $r$ , and  $c$  are positive integers.

**205** The greatest  $c$  is  $c_2$ , and  $c_2 = a_1$ .

**206** No  $a$  or  $r$  can be greater than  $2a_1$ .

**207** If  $r_n = 1$ , then  $c_n = a_1$ .

**208** For all values of  $n$  greater than 1,  $a - c_n$  is  $< r_n$ .

**209** The number of quotients cannot be greater than  $2a_1^2$ . The last quotient is  $2a_1$ , and after that the terms repeat. The first complete quotient that is repeated is  $\frac{\sqrt{Q} + c_2}{r_2}$ , and  $a_2$ ,  $r_2$ ,  $c_2$  commence each cycle of repeated terms.

**210** Let  $a_m, r_m, c_m$  be the last terms of the first cycle ; then  $a_{m-1}, r_{m-1}, c_{m-1}$  are respectively equal to  $a_2, r_2, c_2$  ;  $a_{m-2}, r_{m-2}, c_{m-2}$  are equal to  $a_3, r_3, c_3$ , and so on. [By (187).]

## EQUATIONS.

### *Special Cases in the Solution of Simultaneous Equations.*

**211** First, with two unknown quantities.

$$\left. \begin{array}{l} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{array} \right\} \cdot \quad x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \quad y = \frac{c_1a_2 - c_2a_1}{b_1a_2 - b_2a_1}.$$

If the denominators vanish, we have

$$\frac{a_1}{a_2} = \frac{b_1}{b_2}, \text{ and } x = \infty, y = \infty ;$$

unless at the same time the numerators vanish, for then

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} ; \quad x = \frac{0}{0} ; \quad y = \frac{0}{0} ;$$

and the equations are not *independent*, one being produced by multiplying the other by some constant.

**212** Next, with three unknown quantities. See (60) for the equations.

If  $d_1, d_2, d_3$  all vanish, divide each equation by  $z$ , and we have three equations for finding the two ratios  $\frac{x}{z}$  and  $\frac{y}{z}$ , two only of which equations are necessary, any one being deducible from the other two if the three be consistent.

### **213** *To solve simultaneous equations by Indeterminate Multipliers.*

Ex.—Take the equations

$$\begin{array}{rcl} x + 2y + 3z + 4w & = & 27, \\ 3x + 5y + 7z + w & = & 48, \\ 5x + 8y + 10z - 2w & = & 65, \\ 7x + 6y + 5z + 4w & = & 53. \end{array}$$

Multiply the first by  $A$ , the second by  $B$ , the third by  $C$ , leaving one equation unmultiplied; and then add the results.

$$\begin{aligned}\text{Thus} \quad & (A+3B+5C+7)x + (2A+5B+8C+6)y \\ & + (3A+7B+10C+5)z + (4A+B-2C+4)w \\ & = 27A+48B+65C+53.\end{aligned}$$

To determine either of the unknowns, for instance  $x$ , equate the coefficients of the other three separately to zero, and from the three equations find  $A, B, C$ . Then

$$x = \frac{27A+48B+65C+53}{A+3B+5C+7}.$$

MISCELLANEOUS EQUATIONS AND SOLUTIONS.

**214**  $x^6 \pm 1 = 0.$

Divide by  $x^3$ , and throw into factors, by (2) or (3). See also (480).

**215**  $x^3 - 7x - 6 = 0.$

$x = -1$  is a root, by inspection; therefore  $x+1$  is a factor. Divide by  $x+1$ , and solve the resulting quadratic.

**216**  $x^3 + 16x = 455.$

$$\begin{aligned}x^4 + 16x^2 &= 455x = 65 \times 7x, \\ x^4 + 65x^2 + \left(\frac{65}{2}\right)^2 &= 49x^2 + 65 \times 7x + \left(\frac{65}{2}\right)^2, \\ x^2 + \frac{65}{2} &= 7x + \frac{65}{2}, \\ x^2 = 7x; \quad \therefore x &= 7.\end{aligned}$$

**RULE.**—Divide the absolute term (here 455) into two factors, if possible, such that one of them, minus the square of the other, equals the coefficient of  $x$ . See (483) for general solution of a cubic equation.

**217**  $x^4 - y^4 = 14560, \quad x - y = 8.$

Put  $x = z + v$  and  $y = z - v.$

Eliminate  $v$ , and obtain a cubic in  $z$ , which solve as in (216).

**218**  $x^5 - y^5 = 3093, \quad x - y = 3.$

Divide the first equation by the second, and subtract from the result the fourth power of  $x - y$ . Eliminate  $(x^2 + y^2)$ , and obtain a quadratic in  $xy$ .

**219** *On forming Symmetrical Expressions.*

Take, for example, the equation

$$(y - c)(z - b) = a^2.$$

To form the remaining equations symmetrical with this, write the corresponding letters in *vertical* columns, observing the circular order in which  $a$  is followed by  $b$ ,  $b$  by  $c$ , and  $c$  by  $a$ . So with  $x$ ,  $y$ , and  $z$ . Thus the equations become

$$(y - c)(z - b) = a^2,$$

$$(z - a)(x - c) = b^2,$$

$$(x - b)(y - a) = c^2.$$

To solve these equations, substitute

$$x = b + c + x', \quad y = c + a + y', \quad z = a + b + z';$$

and, multiplying out, and eliminating  $y$  and  $z$ , we obtain

$$x = \frac{bc(b+c) - a(b^2+c^2)}{bc - ca - ab},$$

and therefore, by symmetry, the values of  $y$  and  $z$ , by the rule just given.

**220**  $y^2 + z^2 + yz = a^2 \dots\dots\dots (1),$

$$z^2 + x^2 + zx = b^2 \dots\dots\dots (2),$$

$$x^2 + y^2 + xy = c^2 \dots\dots\dots (3);$$

$$\therefore 3(yz + zx + xy)^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4 \dots\dots (4).$$



Now add (1), (2), and (3), and we obtain

$$2(x+y+z)^2 - 3(yz+zx+xy) = a^2 + b^2 + c^2 \dots\dots (5).$$

From (4) and (5),  $(x+y+z)$  is obtained, and then (1), (2), and (3) are readily solved.

$$\begin{aligned} 221 \quad x^2 - yz &= a^2 \dots\dots\dots (1), \\ y^2 - zx &= b^2 \dots\dots\dots (2), \\ z^2 - xy &= c^2 \dots\dots\dots (3). \end{aligned}$$

Multiply (2) by (3), and subtract the square of (1).

Result  $x(3xyz - x^3 - y^3 - z^3) = b^2c^2 - a^4,$

$$\therefore \frac{x}{b^2c^2 - a^4} = \frac{y}{c^2a^2 - b^4} = \frac{z}{a^2b^2 - c^4} = \lambda \dots\dots\dots (4).$$

Obtain  $\lambda^2$  by proportion as a fraction with numerator

$$= x^2 - yz = a^2.$$

$$\begin{aligned} 222 \quad x &= cy + bz \dots\dots\dots (1), \\ y &= az + cx \dots\dots\dots (2), \\ z &= bx + ay \dots\dots\dots (3). \end{aligned}$$

Eliminate  $a$  between (2) and (3), and substitute the value of  $x$  from equation (1).

Result  $\frac{y^2}{1-b^2} = \frac{z^2}{1-c^2} = \frac{x^2}{1-a^2}.$

## IMAGINARY EXPRESSIONS.

**223** The following are conventions:—

That  $\sqrt{(-a^2)}$  is equivalent to  $a\sqrt{(-1)}$ ; that  $a\sqrt{(-1)}$  vanishes when  $a$  vanishes; that the symbol  $a\sqrt{(-1)}$  is subject to the ordinary rules of Algebra.  $\sqrt{(-1)}$  is denoted by  $i$ .

**224** If  $a + i\beta = \gamma + i\delta$ ; then  $a = \gamma$  and  $\beta = \delta$ .

**225**  $a + i\beta$  and  $a - i\beta$  are conjugate expressions; their product  $= a^2 + \beta^2$ .

**226** The sum and product of two conjugate expressions are both real, but their difference is imaginary.

**227** The modulus is  $+\sqrt{a^2 + \beta^2}$ .

**228** If the modulus vanishes,  $a$  and  $\beta$  must vanish.

**229** If two imaginary expressions are equal, their moduli are equal, by (224).

**230** The modulus of the product of two imaginary expressions is equal to the product of their moduli.

**231** Also the modulus of the quotient is equal to the quotient of their moduli.

## METHOD OF INDETERMINATE COEFFICIENTS.

**232** If  $A + Bx + Cx^2 + \dots = A' + B'x + C'x^2 + \dots$  be an equation which holds for all values of  $x$ , the coefficients  $A, B$ , &c. not involving  $x$ , then  $A = A', B = B', C = C'$ , &c.; that is, the coefficients of like powers of  $x$  must be equal. Proved by putting  $x = 0$ , and dividing by  $x$  alternately. See (234) for an example.

## 233 METHOD OF PROOF BY INDUCTION.

Ex.—To prove that

$$1 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Assume

$$1 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6};$$

$$\begin{aligned}\therefore 1+2^2+3^2+\dots+n^2+(n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)+6(n+1)^2}{6} = \frac{(n+1)\{n(2n+1)+6(n+1)\}}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6} = \frac{n'(n'+1)(2n'+1)}{6},\end{aligned}$$

where  $n'$  is written for  $n+1$ ;

$$\therefore 1+2^2+3^2+\dots+n^2 = \frac{n'(n'+1)(2n'+1)}{6}.$$

It is thus proved that if the formula be true for  $n$  it is also true for  $n+1$ .

But the formula is true when  $n=2$  or  $3$ , as may be shewn by actual trial; therefore it is true when  $n=4$ ; therefore also when  $n=5$ , and so on; therefore universally true.

**234** Ex.—The same theorem proved by the method of Indeterminate coefficients.

Assume

$$1+2^2+3^2+\dots+n^2 = A+Bn + Cn^2 + Dn^3 + \&c.;$$

$$\therefore 1+2^2+3^2+\dots+n^2+(n+1)^2 = A+B(n+1)+C(n+1)^2+D(n+1)^3+\&c.;$$

therefore, by subtraction,

$$n^2+2n+1 = B+C(2n+1)+D(3n^2+3n+1),$$

writing no terms in this equation which contain higher powers of  $n$  than the highest which occurs on the left-hand side, for the coefficients of such terms may be shewn to be separately equal to zero.

Now equate the coefficients of like powers of  $n$ ; thus

$$3D = 1, \quad \therefore D = \frac{1}{3};$$

$$2C+3D = 2, \quad \therefore C = \frac{1}{2}, \quad \text{and} \quad A = 0;$$

$$B+C+D = 1, \quad \therefore B = \frac{1}{6};$$

therefore the sum of the series is equal to

$$\frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3} = \frac{n(n+1)(2n+1)}{6}.$$

## PARTIAL FRACTIONS.

In the resolution of a fraction into partial fractions four cases present themselves, which are illustrated in the following examples.

**235** First.—When there are no repeated factors in the denominator of the given fraction.

Ex.—To resolve  $\frac{3x-2}{(x-1)(x-2)(x-3)}$  into partial fractions.

$$\text{Assume } \frac{3x-2}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3};$$

$$\therefore 3x-2 = A(x-2)(x-3) + B(x-3)(x-1) + C(x-1)(x-2).$$

Since  $A$ ,  $B$ , and  $C$  do not contain  $x$ , and this equation is true for all values of  $x$ , put  $x = 1$ ; then

$$3-2 = A(1-2)(1-3), \text{ from which } A = \frac{1}{2}.$$

Similarly, if  $x$  be put  $= 2$ , we have

$$6-2 = B(2-3)(2-1); \quad \therefore B = -4;$$

and, putting  $x = 3$ ,

$$9-2 = C(3-1)(3-2); \quad \therefore C = \frac{7}{2}.$$

$$\text{Hence } \frac{3x-2}{(x-1)(x-2)(x-3)} = \frac{1}{2(x-1)} - \frac{4}{x-2} + \frac{7}{2(x-3)}.$$

**236** Secondly.—When there is a repeated factor.

Ex.—Resolve into partial fractions  $\frac{7x^3-10x^2+6x}{(x-1)^3(x+2)}$ .

$$\text{Assume } \frac{7x^3-10x^2+6x}{(x-1)^3(x+2)} = \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{x-1} + \frac{D}{x+2}.$$

These forms are necessary and sufficient. Multiplying up, we have

$$7x^3-10x^2+6x = A(x+2) + B(x-1)(x+2) + C(x-1)^2(x+2) + D(x-1)^3 \dots\dots\dots (1).$$

$$\text{Make } x = 1; \quad \therefore 7-10+6 = A(1+2); \quad \therefore A = 1.$$

Substitute this value of  $A$  in (1); thus

$$7x^3-10x^2+5x-2 = B(x-1)(x+2) + C(x-1)^2(x+2) + D(x-1)^3.$$

Divide by  $x-1$ ; thus

$$7x^2-3x+2 = B(x+2) + C(x-1)(x+2) + D(x-1)^2 \dots\dots\dots (2).$$

$$\text{Make } x = 1 \text{ again, } 7-3+2 = B(1+2); \quad \therefore B = 2.$$

Substitute this value of  $B$  in (2), and we have

$$7x^2-5x-2 = C(x-1)(x+2) + D(x-1)^2.$$

$$\text{Divide by } x-1, \quad 7x+2 = C(x+2) + D(x-1) \dots\dots\dots (3).$$

$$\text{Put } x = 1 \text{ a third time, } 7+2 = C(1+2); \quad \therefore C = 3.$$

Lastly, make  $x = -2$  in (3),

$$-14 + 2 = D(-2 - 1); \quad \therefore D = 4.$$

Result 
$$\frac{1}{(x-1)^3} + \frac{2}{(x-1)^2} + \frac{3}{x-1} + \frac{4}{x+2}.$$

**237** Thirdly.—When there is a quadratic factor of imaginary roots not repeated.

Ex.—Resolve  $\frac{1}{(x^2+1)(x^2+x+1)}$  into partial fractions.

Here we must assume

$$\frac{1}{(x^2+1)(x^2+x+1)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+x+1};$$

$x^2+1$  and  $x^2+x+1$  have no real factors, and are therefore retained as denominators. The requisite form of the numerators is seen by adding together two simple fractions, such as  $\frac{a}{x+b} + \frac{c}{x+d}$ .

Multiplying up, we have the equation

$$1 = (Ax+B)(x^2+x+1) + (Cx+D)(x^2+1) \dots\dots\dots (1).$$

Let  $x^2+1 = 0; \quad \therefore x^2 = -1.$

Substitute this value of  $x^2$  in (1) repeatedly; thus

$$1 = (Ax+B)x = Ax^2+Bx = -A+Bx;$$

or  $Bx - A - 1 = 0.$

Equate coefficients to zero;  $\therefore B = 0,$

$$A = -1.$$

Again, let  $x^2+x+1 = 0;$

$$\therefore x^2 = -x-1.$$

Substitute this value of  $x^2$  repeatedly in (1); thus

$$1 = (Cx+D)(-x) = -Cx^2-Dx = Cx+C-Dx;$$

or  $(C-D)x + C - 1 = 0.$

Equate coefficients to zero; thus  $C = 1,$

$$D = 1.$$

Hence 
$$\frac{1}{(x^2+1)(x^2+x+1)} = \frac{x+1}{x^2+x+1} - \frac{x}{x^2+1}.$$

**238** Fourthly.—When there is a repeated quadratic factor of imaginary roots.

Ex.—Resolve  $\frac{40x-103}{(x+1)^2(x^2-4x+8)^3}$  into partial fractions.

Assume

$$\frac{40x-103}{(x+1)^2(x^2-4x+8)^3} = \frac{Ax+B}{(x^2-4x+8)^3} + \frac{Cx+D}{(x^2-4x+8)^2} + \frac{Ex+F}{x^2-4x+8} + \frac{G}{(x+1)^2} + \frac{H}{x+1};$$

∴

$$40x-103 = \{(Ax+B) + (Cx+D)(x^2-4x+8) + (Ex+F)(x^2-4x+8)^2\}(x+1)^2 + \{G+H(x+1)\}(x^2-4x+8)^3 \dots\dots\dots (1).$$

In the first place, to determine  $A$  and  $B$ , equate  $x^2-4x+8$  to zero; thus  $x^2 = 4x-8$ .

Substitute this value of  $x^2$  repeatedly in (1), as in the previous example, until the first power of  $x$  alone remains. The resulting equation is

$$40x-103 = (17A+6B)x - 48A - 7B.$$

Equating coefficients, we obtain two equations

$$\left. \begin{aligned} 17A+6B &= 40 \\ 48A+7B &= 103 \end{aligned} \right\}, \text{ from which } \begin{aligned} A &= 2 \\ B &= 1. \end{aligned}$$

Next, to determine  $C$  and  $D$ , substitute these values of  $A$  and  $B$  in (1); the equation will then be divisible by  $x^2-4x+8$ . Divide, and the resulting equation is

$$0 = 2x+13 + \{Cx+D + (Ex+F)(x^2-4x+8)\}(x+1)^2 + \{G+H(x+1)\}(x^2-4x+8)^2 \dots\dots\dots (2).$$

Equate  $x^2-4x+8$  again to zero, and proceed exactly as before, when finding  $A$  and  $B$ .

Next, to determine  $E$  and  $F$ , substitute the values of  $C$  and  $D$ , last found in equation (2); divide, and proceed as before.

Lastly,  $G$  and  $H$  are determined by equating  $x+1$  to zero successively, as in Example 2.

## CONVERGENCY AND DIVERGENCY OF SERIES.

**239** Let  $a_1 + a_2 + a_3 + \&c.$  be a series, and  $a_n, a_{n+1}$  any two consecutive terms. The following tests of convergency may be applied. The series will converge, if, after any fixed term—

(i.) The terms decrease and are alternately positive and negative.

(ii.) Or if  $\frac{a_n}{a_{n+1}}$  is always *greater* than some quantity greater than unity.



(iii.) Or if  $\frac{a_n}{a_{n+1}}$  is never less than the corresponding ratio in a known converging series.

(iv.) Or if  $\left(\frac{na_n}{a_{n+1}} - n\right)$  is always *greater* than some quantity greater than unity. [By 244 and iii.]

(v.) Or if  $\left(\frac{na_n}{a_{n+1}} - n - 1\right) \log n$  is always *greater* than some quantity greater than unity.

**240** The conditions of divergency are obviously the converse of rules (i.) to (v.).

**241** The series  $a_1 + a_2x + a_3x^2 + \&c.$  converges, if  $\frac{a_{n+1}}{a_n}$  is always less than some quantity  $p$ , and  $x$  less than  $\frac{1}{p}$ .  
[By 239 (ii.)]

**242** To make the sum of the last series less than an assigned quantity  $p$ , make  $x$  less than  $\frac{p}{p+k}$ ,  $k$  being the greatest coefficient.

### General Theorem.

**243** If  $\phi(x)$  be positive for all positive integral values of  $x$ , and continually diminish as  $x$  increases, and if  $m$  be any positive integer, then the two series

$$\phi(1) + \phi(2) + \phi(3) + \phi(4) + \dots$$

$$\phi(1) + m\phi(m) + m^2\phi(m^2) + m^3\phi(m^3) + \dots$$

are either both convergent or divergent.

**244** Application of this theorem. To ascertain whether the series

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$$

is divergent or convergent when  $p$  is greater than unity.

Taking  $m = 2$ , the second series in (243) becomes

$$1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \&c.....,$$

a geometrical progression which converges; therefore the given series converges.

**245** The series of which  $\frac{1}{n(\log n)^p}$  is the general term is convergent if  $p$  be greater than unity, and divergent if  $p$  be not greater than unity. [By (243), (244).]

**246** The series of which the general term is

$$\frac{1}{n\lambda(n)\lambda^2(n)\dots\dots\lambda^r(n)\{\lambda^{r+1}(n)\}^p},$$

where  $\lambda(n)$  signifies  $\log n$ ,  $\lambda^2(n)$  signifies  $\log\{\log(n)\}$ , and so on, is convergent if  $p$  be greater than unity, and divergent if  $p$  be not greater than unity. [By Induction, and by (243).]

**247** The series  $a_1 + a_2 + \&c.$  is convergent if

$$na_n \log(n) \log^2(n) \dots\dots \log^r(n) \{\log_{r+1}(n)\}^p$$

is always finite for a value of  $p$  greater than unity;  $\log^2(n)$  here signifying  $\log(\log n)$ , and so on.

[See Todhunter's *Algebra*, or Boole's *Finite Differences*.]

## EXPANSION OF A FRACTION.

**248** A fractional expression such as  $\frac{4x - 10x^2}{1 - 6x + 11x^2 - 6x^3}$  may be expanded in ascending powers of  $x$  in three different ways.

First, by dividing the numerator by the denominator in the ordinary way, or by Synthetic Division, as shewn in (28).

Secondly, by the method of Indeterminate Coefficients (232).

Thirdly, by Partial Fractions and the Binomial Theorem.

To expand by the method of Indeterminate Coefficients, proceed as follows:—

$$\text{Assume } \frac{4x-10x^2}{1-6x+11x^2-6x^3} = A+Bx+Cx^2+Dx^3+Ex^4+\&c.$$

$$\begin{aligned} \therefore 4x-10x^2 &= A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+\dots \\ &\quad -6Ax-6Bx^2-6Cx^3-6Dx^4-6Ex^5-\dots \\ &\quad +11Ax^2+11Bx^3+11Cx^4+11Dx^5+\dots \\ &\quad -6Ax^3-6Bx^4-6Cx^5-\dots \end{aligned}$$

Equate coefficients of like powers of  $x$ , thus

$$\begin{aligned} A &= 0, \\ B-6A &= 4, \quad \therefore B = 4; \\ C-6B+11A &= -10, \quad \therefore C = 14; \\ D-6C+11B-6A &= 0, \quad \therefore D = 40; \\ E-6D+11C-6B &= 0, \quad \therefore E = 110; \\ F-6E+11D-6C &= 0, \quad \therefore F = 304; \\ \dots &\quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \end{aligned}$$

The formation of the same coefficients by synthetic division is now exhibited, in order that the connexion between the two processes may be clearly seen.

The division of  $4x-10x^2$  by  $1-6x+11x^2-6x^3$  is as follows:—

$$\begin{array}{r|l} & 0+4-10 \\ +6 & 24+84+240+660 \\ -11 & -44-154-440-1210 \\ +6 & +24+84+240+660 \\ \hline & 0+4+14+40+110+304+\dots \end{array}$$

$A \quad B \quad C \quad D \quad E \quad F$

If we stop at the term  $110x^4$ , then the undivided remainder will be  $304x^5-970x^6+660x^7$ , and the complete result will be

$$4x+14x^2+40x^3+110x^4+\frac{304x^5-970x^6+660x^7}{1-6x+11x^2-6x^3}.$$

**249** Here the concluding fraction may be regarded as the sum to infinity after four terms of the series, just as the original expression is considered to be the sum to infinity of the whole series.

**250** If the general term be required, the method of expansion by partial fractions must be adopted. See (257), where the general term of the foregoing series is obtained.

## RECURRING SERIES.

$a_0 + a_1x + a_2x^2 + a_3x^3 + \&c.$  is a recurring series if the coefficients are connected by the relation

$$251 \quad a_n = p_1 a_{n-1} + p_2 a_{n-2} + \dots + p_m a_{n-m}.$$

The Scale of Relation is

$$252 \quad 1 - p_1x - p_2x^2 - \dots - p_mx^m.$$

The sum of  $n$  terms of the series is equal to

253 [The first  $m$  terms

$-p_1x$  (first  $m-1$  terms + the last term)

$-p_2x^2$  (first  $m-2$  terms + the last 2 terms)

$-p_3x^3$  (first  $m-3$  terms + the last 3 terms)

... ..

$-p_{m-1}x^{m-1}$  (first term + the last  $m-1$  terms)

$-p_mx^m$  (the last  $m$  terms)]  $\div [1 - p_1x - p_2x^2 - \dots - p_mx^m]$ .

254 If the series converges, and the sum to infinity is required, omit all "the last terms" from the formula.

255 EXAMPLE.—Required the Scale of Relation, the general term, and the apparent sum to infinity, of the series

$$4x + 14x^2 + 40x^3 + 110x^4 + 304x^5 - 854x^6 + \dots$$

Observe that six arbitrary terms given are sufficient to determine a Scale of Relation of the form  $1 - px - qx^2 - rx^3$ , involving three constants  $p, q, r$ , for, by (251), we can write three equations to determine these constants;

$$\text{namely, } \left. \begin{array}{l} 110 = 40p + 14q + 4r \\ 304 = 110p + 40q + 14r \\ 854 = 304p + 110q + 40r \end{array} \right\} \quad \begin{array}{l} \text{The solution gives} \\ p = 6, \quad q = -11, \quad r = 6. \end{array}$$

Hence the Scale of Relation is  $1 - 6x + 11x^2 - 6x^3$ .

The sum of the series without limit will be found from (254), by putting  $p_1 = 6, p_2 = -11, p_3 = 6, m = 3$ .

$$\begin{array}{rcl} \text{The first three terms} & = & 4x + 14x^2 + 40x^3 \\ -6 \times \text{the first two terms} & = & -24x^2 - 84x^3 \\ +11x \times \text{the first term} & = & \phantom{-24x^2 - 84x^3} + 44x^3 \\ \hline & & 4x - 10x^2 \end{array}$$

$$\therefore S = \frac{4x-10x^3}{1-6x+11x^2-6x^3};$$

the meaning of which is that, if this fraction be expanded in ascending powers of  $x$ , the first six terms will be those given in the question.

**256** To obtain more terms of the series, we may use the Scale of Relation; thus the 7th term will be

$$(6 \times 854 - 11 \times 304 + 6 \times 110) x^7 = 2440x^7.$$

**257** To find the general term,  $S$  must be decomposed into partial fractions; thus, by the method of (235),

$$\frac{4x-10x^3}{1-6x+11x^2-6x^3} = \frac{1}{1-3x} + \frac{2}{1-2x} - \frac{3}{1-x}.$$

By the Binomial Theorem (128),

$$\begin{aligned} \frac{1}{1-3x} &= 1 + 3x + 3^2x^2 + \dots + 3^n x^n, \\ \frac{2}{1-2x} &= 2 + 2^2x + 2^3x^2 + \dots + 2^{n+1}x^n, \\ -\frac{3}{1-x} &= -3 - 3x - 3x^2 - \dots - 3x^n. \end{aligned}$$

Hence the general term involving  $x^n$  is

$$(3^n + 2^{n+1} - 3) x^n.$$

And by this formula we can write the "last terms" required in (253), and so obtain the sum of any finite number of terms of the given series. Also, by the same formula we can calculate the successive terms at the beginning of the series. In the present case this mode will be more expeditious than that of employing the Scale of Relation.

**258** If, in decomposing  $S$  into partial fractions for the sake of obtaining the general term, a quadratic factor with imaginary roots should occur as a denominator, the same method must be pursued, for the imaginary quantities will disappear in the final result. In this case, however, it is more convenient to employ a general formula. Suppose the fraction which gives rise to the imaginary roots to be

$$\frac{L+Mx}{a+bx+x^2} = \frac{L+Mx}{(p-x)(q-x)},$$

$p$  and  $q$  being the imaginary roots of  $a+bx+x^2=0$ .

Suppose

$$p = a + i\beta,$$

$$q = a - i\beta, \text{ where } i = \sqrt{-1}.$$

If, now, the above fraction be resolved into two partial fractions in the ordinary way, and if these fractions be expanded separately by the Binomial Theorem, and that part of the general term furnished by these two expansions written out, still retaining  $p$  and  $q$ , and if the imaginary values of  $p$  and  $q$  be then substituted, it will be found that the factor will disappear, and that the result may be enunciated as follows.

**259** The coefficient of  $x^{n-1}$  in the expansion of

$$\frac{L + Mx}{(a^2 + \beta^2)^n - 2ax + x^2}$$

will be

$$\begin{aligned} & \frac{L}{\beta(a^2 + \beta^2)} \{ na^{n-1}\beta - C(n, 3) a^{n-3}\beta^3 + C(n, 5) a^{n-5}\beta^5 - \dots \} \\ & + \frac{M}{\beta(a^2 + \beta^2)^{n-1}} \{ (n-1) a^{n-2}\beta - C(n-1, 3) a^{n-4}\beta^3 \\ & \qquad \qquad \qquad + C(n-1, 5) a^{n-6}\beta^5 - \dots \}. \end{aligned}$$

**260** With the aid of the known expansion of  $\sin n\theta$  in Trigonometry, this formula for the  $n^{\text{th}}$  term may be reduced to

$$\sqrt{\frac{(L + Ma)^2 + M^2\beta^2}{\beta^2(a^2 + \beta^2)^n}} \cdot \sin(n\theta - \phi),$$

in which  $\theta = \tan^{-1} \frac{\beta}{a}, \quad \phi = \tan^{-1} \frac{M\beta}{L + Ma}.$

If  $n$  be not greater than 100,  $\sin(n\theta - \phi)$  may be obtained from the tables correct to about six places of decimals, and accordingly the  $n^{\text{th}}$  term of the expansion may be found with corresponding accuracy. As an example, the 100<sup>th</sup> term in the expansion of  $\frac{1+x}{5-2x+x^2}$  is readily found by this method to be  $\frac{41824}{10^{41}} x^{99}$ .

*To determine whether a given Series is recurring or not.*

**261** If certain first terms only of the series be given, a scale of relation may be found which shall produce a recurring



series whose first terms are those given. The method is exemplified in (255). The number of unknown coefficients  $p, q, r, \&c.$  to be assumed for the scale of relation must be equal to half the number of the given terms of the series, if that number be even. If the number of given terms be odd, it may be made even by prefixing zero for the first term of the series.

**262** Since this method may, however, produce zero values for one or more of the last coefficients in the scale of relation, it may be advisable in practice to determine a scale from the first two terms of the series, and if that scale does not produce the following terms, we may try a scale determined from the first four terms, and so on until the true scale is arrived at.

If an indefinite number of terms of the series be given, we may find whether it is recurring or not by a rule of Lagrange's.

**263** Let the series be

$$S = A + Bx + Cx^2 + Dx^3 + \&c.$$

Divide unity by  $S$  as far as two terms of the quotient, which will be of the form  $p + qx$ , and write the remainder in the form  $S'x^2$ ,  $S'$  being another indefinite series of the same form as  $S$ .

Next, divide  $S$  by  $S'$  as far as two terms of the quotient, and write the remainder in the form  $S''x^2$ .

Again, divide  $S'$  by  $S''$ , and proceed as before, and repeat this process until there is no remainder after one of the divisions. The series will then be proved to be a recurring series, and the order of the series, that is, the degree of the scale of relation, will be the same as the number of divisions which have been effected in the process.

EXAMPLE.—To determine whether the series 1, 3, 6, 10, 15, 21, 28, 36, 45, ... is recurring or not.

Introducing  $x$ , we may write

$$S = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + 28x^6 + 36x^7 + 45x^8 \dots$$

Then we shall have  $\frac{S}{1} = 1 - 3x + \dots$  with a remainder

$$3x^2 + 8x^3 + 15x^4 + 24x^5 + 35x^6 + \&c.$$

Therefore

$$S' = 3 + 8x + 15x^2 + 24x^3 + 35x^4 + \&c.,$$

$$\frac{S}{S'} = \frac{1}{3} + \frac{x}{9}$$

with a remainder  $\frac{1}{9} (x^2 + 3x^3 + 6x^4 + 10x^5 + \&c. \dots)$ .

Therefore we may take  $S'' = 1 + 3x + 6x^2 + 10x^3 + \&c.$

Lastly  $\frac{S'}{S''} = 3 - x$  without any remainder.

Consequently the series is a recurring series of the third order. It is, in fact, the expansion of  $\frac{1}{1 - 3x + 3x^2 - x^3}$ .

## SUMMATION OF SERIES BY THE METHOD OF DIFFERENCES.

**264** RULE.—*Form successive series of differences until a series of equal differences is obtained. Let  $a, b, c, d, \&c.$  be the first terms of the several series; then the  $n^{\text{th}}$  term of the given series is*

$$\mathbf{265} \quad a + (n-1)b + \frac{(n-1)(n-2)}{1.2}c + \frac{(n-1)(n-2)(n-3)}{1.2.3}d +$$

The sum of  $n$  terms

$$\mathbf{266} \quad = na + \frac{n(n-1)}{1.2}b + \frac{n(n-1)(n-2)}{1.2.3}c + \&c.$$

Proved by Induction.

**EXAMPLE:**

$a \dots$	$1 + 5 + 15 + 35 + 70 + 126 + \dots$
$b \dots$	$4 + 10 + 20 + 35 + 56 + \dots$
$c \dots$	$6 + 10 + 15 + 21 + \dots$
$d \dots$	$4 + 5 + 6 + \dots$
$e \dots$	$1 + 1 + \dots$

The 100<sup>th</sup> term of the first series

$$= 1 + 99.4 + \frac{99.98}{1.2}6 + \frac{99.98.97}{1.2.3}4 + \frac{99.98.97.96}{1.2.3.4}.$$

The sum of 100 terms

$$= 100 + \frac{100.99}{1.2}4 + \frac{100.99.98}{1.2.3}6 + \frac{100.99.98.97}{1.2.3.4}4 + \frac{100.99.98.97.96}{1.2.3.4.5}.$$

**267** To interpolate a term between two terms of a series by the method of differences.

Ex.—Given  $\log 71$ ,  $\log 72$ ,  $\log 73$ ,  $\log 74$ , it is required to find  $\log 72.54$ .  
Form the series of differences from the given logarithms, as in (266),

	$\log 71$	$\log 72$	$\log 73$	$\log 74$
$a \dots$	1.8512583	1.8573325	1.8633229	1.8692317
$b \dots$	.0060742	.0059904	.0059088	
$c \dots$	— .0000838	— .0000816		
$d \dots$	— .0000022	considered to vanish.		

$\log 72.54$  must be regarded as an interpolated term, the number of its place being 2.54.

Therefore put 2.54 for  $n$  in formula (265).

Result  $\log 72.54 = 1.8605777$ .

## DIRECT FACTORIAL SERIES.

**268** Ex.:  $5.7.9 + 7.9.11 + 9.11.13 + 11.13.15 + \dots$

$d$  = common difference of factors,

$m$  = number of factors in each term,

$n$  = number of terms,

$a$  = first factor of first term  $-d$ .

$$n^{\text{th}} \text{ term} = (a+nd) (a+\overline{n+1}d) \dots (a+\overline{n+m-1}d).$$

**269** To find the sum of  $n$  terms.

RULE.—From the last term with the next highest factor take the first term with the next lowest factor, and divide by  $(m+1)d$ .

PROOF.—By Induction.

Thus the sum of 4 terms of the above series will be, putting  $d=2$ ,  $m=3$ ,

$$n=4, a=3, \quad S = \frac{11.13.15.17 - 3.5.7.9}{(3+1)2}.$$

Proved either by Induction, or by the method of Indeterminate Coefficients.

### INVERSE FACTORIAL SERIES.

**270** Ex.:  $\frac{1}{5.7.9} + \frac{1}{7.9.11} + \frac{1}{9.11.13} + \frac{1}{11.13.15} + \dots$

Defining  $d, m, n, a$  as before, the

$$n^{\text{th}} \text{ term} = \frac{1}{(a+nd)(a+n+1d) \dots (a+n+m-1d)}.$$

**271** To find the sum of  $n$  terms. **RULE.**—From the first term wanting its last factor take the last term wanting its first factor, and divide by  $(m-1)d$ .

Thus the sum of 4 terms of the above series will be, putting  $d=2, m=3,$

$$n=4, a=3, \quad \frac{\frac{1}{5.7} - \frac{1}{13.15}}{(3-1)2}.$$

**PROOF.**—By Induction, or by decomposing the terms, as in the following example.

**272** Ex.: To sum the same series by decomposing the terms into partial fractions. Take the general term in the simple form

$$\frac{2}{(r-2)r(r+2)}.$$

Resolve this into the three fractions

$$\frac{1}{8(r-2)} - \frac{1}{4r} + \frac{1}{8(r+2)} \text{ by (235).}$$

Substitute 7, 9, 11, &c. successively for  $r$ , and the given series has for its equivalent the three series

$$\begin{aligned} & \frac{1}{8} \left\{ \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} \dots \dots \dots + \frac{1}{2n+3} \right\} \\ & + \frac{1}{8} \left\{ -\frac{2}{7} - \frac{2}{9} - \frac{2}{11} - \frac{2}{13} - \dots \dots \dots - \frac{2}{2n+3} - \frac{2}{2n+5} \right\} \\ & + \frac{1}{8} \left\{ \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \dots \dots + \frac{1}{2n+3} + \frac{1}{2n+5} + \frac{1}{2n+7} \right\}, \end{aligned}$$

and the sum of  $n$  terms is seen, by inspection, to be

$$\frac{1}{8} \left\{ \frac{1}{5} - \frac{1}{7} - \frac{1}{2n+5} + \frac{1}{2n+7} \right\} = \frac{1}{4} \left\{ \frac{1}{5.7} - \frac{1}{(2n+5)(2n+7)} \right\},$$

a result obtained at once by the rule in (271), taking  $\frac{1}{5.7.9}$  for the first

term, and  $\frac{1}{(2n+3)(2n+5)(2n+7)}$  for the  $n^{\text{th}}$  or last term.

**273** Analogous series may be reduced to the types in (268) and (270), or else the terms may be decomposed in the manner shewn in (272).

$$\text{Ex.:} \quad \frac{1}{1.2.3} + \frac{4}{2.3.4} + \frac{7}{3.4.5} + \frac{10}{4.5.6} + \dots$$

has for its general term

$$\frac{3n-2}{n(n+1)(n+2)} = -\frac{1}{n} + \frac{5}{n+1} - \frac{4}{n+2} \text{ by (235),}$$

and we may proceed as in (272) to find the sum of  $n$  terms.

The method of (272) includes the method known as "Summation by Subtraction," but it has the advantage of being more general and easier of application to complex series.

## COMPOSITE FACTORIAL SERIES.

**274** If the two series

$$(1-x)^{-5} = 1 + 5x + \frac{5.6}{1.2}x^2 + \frac{5.6.7}{1.2.3}x^3 + \frac{5.6.7.8}{1.2.3.4}x^4 + \dots$$

$$(1-x)^{-3} = 1 + 3x + \frac{3.4}{1.2}x^2 + \frac{3.4.5}{1.2.3}x^3 + \frac{3.4.5.6}{1.2.3.4}x^4 + \dots$$

be multiplied together, and the coefficient of  $x^4$  in the product be equated to the coefficient of  $x^4$  in the expansion of  $(1-x)^{-8}$ , we obtain as the result the sum of the composite series

$$\begin{aligned} & 5.6.7.8 \times 1.2 + 4.5.6.7 \times 2.3 + 3.4.5.6 \times 3.4 \\ & + 2.3.4.5 \times 4.5 + 1.2.3.4 \times 5.6 = \frac{4! \cdot 2.11!}{7! \cdot 4!}. \end{aligned}$$

**275** Generally, if the given series be

$$P_1Q_1 + P_2Q_2 + \dots + P_{n-1}Q_{n-1} \dots \dots \dots (1),$$

where  $Q_r = r(r+1)(r+2) \dots (r+q-1)$ ,

and  $P_r = (n-r)(n-r+1) \dots (n-r+p-1)$ ;

the sum of  $n-1$  terms will be

$$\frac{p! \cdot q!}{(p+q+1)!} \cdot \frac{(n+p+q-1)!}{(n-2)!}.$$

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 MISCELLANEOUS SERIES.
 

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**276** *Sum of the powers of the terms of an Arithmetical Progression.*

$$1+2+3+\dots+n = \frac{n(n+1)}{2} = S_1$$

$$1+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6} = S_2$$

$$1+2^3+3^3+\dots+n^3 = \left\{ \frac{n(n+1)}{2} \right\}^2 = S_3$$

$$1+2^4+3^4+\dots+n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = S_4$$

[By the method of Indeterminate Coefficients (234).

A general formula for the sum of the  $r^{\text{th}}$  powers of  $1.2.3\dots n$ , obtained in the same way is

$$S_r = \frac{1}{r+1} n^{r+1} + \frac{1}{2} n^r + A_1 n^{r-1} + \dots + A_{r-1} n,$$

where  $A_1, A_2, \&c.$ , are determined by putting  $p = 1, 2, 3, \&c.$  successively in the equation

$$\begin{aligned} & \frac{1}{2(p+1)!} \\ &= \frac{1}{(p+2)!} + \frac{A_1}{r(p)!} + \frac{A_2}{r(r-1)(p-1)!} + \dots + \frac{A_p}{r(r-1)\dots(r-p+1)!} \end{aligned}$$


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**277**  $a^m + (a+d)^m + (a+2d)^m + \dots + (a+nd)^m$   
 $= (n+1)a^m + S_1 m a^{m-1} d + S_2 C(m, 2) a^{m-2} d^2$   
 $+ S_3 C(m, 3) a^{m-3} d^3 + \&c.$

PROOF.—By Binomial Theorem and (276).

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**278** *Summation of a series partly Arithmetical and partly Geometrical.*

EXAMPLE.—To find the sum of the series  $1+3x+5x^2+\dots$  to  $n$  terms.



$$\begin{aligned}\text{Let } s &= 1 + 3x + 5x^2 + 7x^3 + \dots + (2n-1)x^{n-1}, \\ sx &= x + 3x^2 + 5x^3 + \dots + (2n-3)x^{n-1} + (2n-1)x^n,\end{aligned}$$

∴ by subtraction,

$$\begin{aligned}s(1-x) &= 1 + 2x + 2x^2 + 2x^3 + \dots + 2x^{n-1} - (2n-1)x^n \\ &= 1 + 2x \frac{1-x^{n-1}}{1-x} - (2n-1)x^n,\end{aligned}$$

$$\therefore s = \frac{1-(2n-1)x^n}{1-x} + \frac{2x(1-x^{n-1})}{(1-x)^2}.$$

**279** A general formula for the sum of  $n$  terms of

$$a + (a+d)r + (a+2d)r^2 + (a+3d)r^3 + \&c.$$

is 
$$S = \frac{a - (a + \overline{n-1}d)r^n}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2}.$$

Obtained as in (278).

**RULE.**—*Multiply by the ratio and subtract the resulting series.*

**280** 
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^{n-1} + \frac{x^n}{1-x}.$$

**281** 
$$\begin{aligned}\frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + 4x^3 + \dots \\ &\quad \dots + nx^{n-1} + \frac{(n+1)x^n - nx^{n+1}}{(1-x)^2}.\end{aligned}$$

**282** 
$$\begin{aligned}(n-1)x + (n-2)x^2 + (n-3)x^3 + \dots + 2x^{n-2} + x^{n-1} \\ = \frac{(n-1)x - nx^2 + x^{n+1}}{(1-x)^2}.\end{aligned}$$
 By (253).

**283** 
$$1 + n + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \&c. = 2^n,$$

$$1 - n + \frac{n(n-1)}{2!} - \frac{n(n-1)(n-2)}{3!} + \&c. = 0.$$

By making  $a=b$  in (125).  
 $x=1$

errata

**284** The series

$$1 - \frac{n-3}{2} + \frac{(n-4)(n-5)}{3!} - \frac{(n-5)(n-6)(n-7)}{4!} + \dots$$

$$\dots + (-1)^{r-1} \frac{(n-r-1)(n-r-2) \dots (n-2r+1)}{r!}$$

consists of  $\frac{n}{2}$  or  $\frac{n-1}{2}$  terms, and the sum is given by

$$S = \frac{3}{n} \text{ if } n \text{ be of the form } 6m+3,$$

$$S = 0 \text{ if } n \text{ be of the form } 6m \pm 1,$$

$$S = -\frac{1}{n} \text{ if } n \text{ be of the form } 6m,$$

$$S = \frac{2}{n} \text{ if } n \text{ be of the form } 6m \pm 2.$$

PROOF.—By (545), putting  $p = x + y$ ,  $q = xy$ , and applying (546).

**285** The series  $n^r - n(n-1)^r + \frac{n(n-1)}{2!}(n-2)^r$   
 $-\frac{n(n-1)(n-2)}{3!}(n-3)^r + \&c. \dots$

takes the values  $0, n!, \frac{1}{2}n(n+1)!$

according as  $r$  is  $< n$ ,  $= n$ , or  $= n+1$ .

PROOF.—By expanding  $(e^x - 1)^n$ , in two ways: first, by the Exponential Theorem and Multinomial; secondly, by the Bin. Th., and each term of the expansion by the Exponential. Equate the coefficients of  $x^r$  in the two results.

Other results are obtained by putting  $r = n+2, n+3, \&c.$

The series (285), when divided by  $r!$ , is, in fact, equal to the coefficient of  $x^r$  in the expansion of

$$\left\{ x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right\}^n.$$

**286** By exactly the same process we may deduce from the function  $\{e^x - e^{-x}\}^n$  the result that the series

$$n^r - n(n-2)^r + \frac{n(n-1)}{2!}(n-4)^r - \&c.$$

takes the values 0 or  $2^n.n!$ , according as  $r$  is  $< n$  or  $= n$ ; this series, divided by  $r!$ , being equal to the coefficient of  $x^r$  in the expansion of

$$2^n \left\{ x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right\}^n.$$

POLYGONAL NUMBERS.

**287** The  $n^{\text{th}}$  term of the  $r^{\text{th}}$  order of polygonal numbers is equal to the sum of  $n$  terms of an Arith. Prog. whose first term is unity and common difference  $r-2$ ; that is

$$= \frac{n}{2} \{ 2 + (n-1)(r-2) \} = n + \frac{1}{2}n(n-1)(r-2).$$

**288** The sum of  $n$  terms

$$= \frac{n(n+1)}{2} + \frac{n(n-1)(n+1)(r-2)}{6}.$$

By resolving into two series.

Order.	$n^{\text{th}}$ term.	
1	1	1 1 1 1 1 1 1
2	$n$	1 2 3 4 5 6 7
3	$\frac{1}{2}n(n+1)$	1 3 6 10 15 21 28
4	$n^2$	1 4 9 16 25 36 49
5	$\frac{1}{3}n(3n-1)$	1 5 12 22 35 51 70
6	$(2n-1)n$	1 6 15 28 45 66 91
...	...	...
$r$	$n + \frac{n(n-1)}{2}(r-2)$	1, $r$ , $3+3(r-2)$ , $4+6(r-2)$ , $5+10(r-2)$ , $6+15(r-2)$ , &c.

In practice—to form, for instance, the 6<sup>th</sup> order of polygonal numbers—write the first three terms by the formula, and form the rest by the method of differences.

Ex.:	1	6	15	28	45	66	91	120	...
		5	9	13	17	21	25	29	...
[ $r-2=4$ ]		4	4	4	4	4	4	4	...

FIGURATE NUMBERS.

**289** The  $n^{\text{th}}$  term of any order is the sum of  $n$  terms of the preceding order.

The  $n^{\text{th}}$  term of the  $r^{\text{th}}$  order is

$$\frac{n(n+1) \dots (n+r-2)}{(r-1)!} = H(n, r-1). \quad [\text{By 98.}]$$

**290** The sum of  $n$  terms is

$$\frac{n(n+1) \dots (n+r-1)}{r!} = H(n, r).$$

Order.	Figurate Numbers.	$n^{\text{th}}$ term.
1	1, 1, 1, 1, 1, 1	1
2	1, 2, 3, 4, 5, 6	$n$
3	1, 3, 6, 10, 15, 21	$\frac{n(n+1)}{1.2}$
4	1, 4, 10, 20, 35, 56	$\frac{n(n+1)(n+2)}{1.2.3}$
5	1, 5, 15, 35, 70, 126	$\frac{n(n+1)(n+2)(n+3)}{1.2.3.4}$
6	1, 6, 21, 56, 126, 252	$\frac{n(n+1)(n+2)(n+3)(n+4)}{1.2.3.4.5}$

## HYPERGEOMETRICAL SERIES.

$$291 \quad 1 + \frac{a \cdot \beta}{1 \cdot \gamma} x + \frac{a(a+1) \beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 \\ + \frac{a(a+1)(a+2) \beta(\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma+1)(\gamma+2)} x^3 + \&c.$$

is convergent if  $x$  is  $< 1$ ,

and divergent if  $x$  is  $> 1$ ; (239 ii.)

and if  $x = 1$ , the series is

convergent if  $\gamma - a - \beta$  is positive,

divergent if  $\gamma - a - \beta$  is negative, (239 iv.)

and divergent if  $\gamma - a - \beta$  is zero. (239 v.)

Let the hypergeometrical series (291) be denoted by  $F(a, \beta, \gamma)$ ; then, the series being convergent, it is shewn by induction that

$$292 \quad \frac{F(a, \beta+1, \gamma+1)}{F(a, \beta, \gamma)} = \frac{1}{1-k_1} \frac{1-k_2}{1-\&c. \dots} \frac{1-k_{2r-1}}{1-k_{2r} z_{2r}} \quad \text{concluding with}$$

where  $k_1, k_2, k_3, \&c.$ , with  $z_{2r}$ , are given by the formulæ

$$k_{2r-1} = \frac{(a+r-1)(\gamma+r-1-\beta)x}{(\gamma+2r-2)(\gamma+2r-1)}$$

$$k_{2r} = \frac{(\beta+r)(\gamma+r-a)x}{(\gamma+2r-1)(\gamma+2r)}$$

$$z_{2r} = \frac{F(a+r, \beta+r+1, \gamma+2r+1)}{F(a+r, \beta+r, \gamma+2r)}.$$

The continued fraction may be concluded at any point with  $k_{2r} z_{2r}$ . When  $r$  is infinite,  $z_{2r} = 1$  and the continued fraction is infinite.

**293** Let

$$f(\gamma) \equiv 1 + \frac{x^2}{1 \cdot \gamma} + \frac{x^4}{1 \cdot 2 \cdot \gamma(\gamma+1)} + \frac{x^6}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma+1)(\gamma+2)} + \&c.$$

the result of substituting  $\frac{x^2}{a\beta}$  for  $x$  in (291), and making  $\beta = a = \infty$ . Then, by last, or independently by induction,

$$\frac{f(\gamma+1)}{f(\gamma)} = \frac{1}{1+} \frac{p_1}{1+} \frac{p_2}{1+} \dots + \frac{p_m}{1+} \&c.$$

$$\text{with } p_m = \frac{x^2}{(\gamma+m-1)(\gamma+m)}.$$

**294** In this result put  $\gamma = \frac{1}{2}$  and  $\frac{y}{2}$  for  $x$ , and we obtain by Exp. Th. (150),

$$\frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{y}{1+} \frac{y^2}{3+} \frac{y^2}{5+} \&c. \quad \text{the } r^{\text{th}} \text{ component being } \frac{y^2}{2r-1}.$$

Or the continued fraction may be formed by ordinary division of one series by the other.

**295**  $e^{\frac{m}{n}}$  is incommensurable,  $m$  and  $n$  being integers. From the last and (174), by putting  $x = \frac{m}{n}$ .

## INTEREST.

If  $r$  be the Interest on £1 for 1 year,

$n$  the number of years,

$P$  the Principal,

$A$  the amount in  $n$  years. Then

**296** At Simple Interest  $A = P(1+nr)$ .

**297** At Compound Interest  $A = P(1+r)^n$ . By (84).



$$298 \quad \left. \begin{array}{l} \text{But if the payments of} \\ \text{Interest be made } q \\ \text{times a year ... } \end{array} \right\} A = P \left( 1 + \frac{r}{q} \right)^{nq}.$$

If  $A$  be an amount due in  $n$  years' time, and  $P$  the present worth of  $A$ . Then

$$299 \quad \text{At Simple Interest} \quad P = \frac{A}{1+nr}. \quad \text{By (296).}$$

$$300 \quad \text{At Compound Interest} \quad P = \frac{A}{(1+r)^n}. \quad \text{By (297).}$$

$$301 \quad \text{Discount} = A - P.$$

### ANNUITIES.

$$302 \quad \left. \begin{array}{l} \text{The amount of an Annu-} \\ \text{ity of } \pounds 1 \text{ in } n \text{ years,} \\ \text{at Simple Interest ... } \end{array} \right\} = n + \frac{n(n-1)}{2} r. \quad \text{By (82).}$$

$$303 \quad \text{Present value of same} = \frac{n + \frac{1}{2}n(n-1)r}{1+nr}. \quad \text{By (299).}$$

$$304 \quad \left. \begin{array}{l} \text{Amount at Compound} \\ \text{Interest ... } \end{array} \right\} = \frac{(1+r)^n - 1}{(1+r) - 1}. \quad \text{By (85).}$$

$$\text{Present worth of same} = \frac{1 - (1+r)^{-n}}{(1+r) - 1}. \quad \text{By (300).}$$

$$305 \quad \left. \begin{array}{l} \text{Amount when the pay-} \\ \text{ments of Interest} \\ \text{are made } q \text{ times per} \\ \text{annum ... } \end{array} \right\} = \frac{\left( 1 + \frac{r}{q} \right)^{nq} - 1}{\left( 1 + \frac{r}{q} \right)^q - 1}. \quad \text{By (298).}$$

$$\text{Present value of same} = \frac{1 - \left( 1 + \frac{r}{q} \right)^{-nq}}{\left( 1 + \frac{r}{q} \right)^q - 1}.$$

$$306 \quad \left. \begin{array}{l} \text{Amount when the pay-} \\ \text{ments of the Annuity} \\ \text{are made } m \text{ times per} \\ \text{annum} \quad \dots \quad \dots \quad \dots \end{array} \right\} = \frac{(1+r)^n - 1}{m \left\{ (1+r)^{\frac{1}{m}} - 1 \right\}}.$$

$$\text{Present value of same} = \frac{1 - (1+r)^{-n}}{m \left\{ (1+r)^{\frac{1}{m}} - 1 \right\}}.$$

$$307 \quad \left. \begin{array}{l} \text{Amount when the In-} \\ \text{terest is paid } q \text{ times} \\ \text{and the Annuity } m \\ \text{times per annum} \quad \dots \end{array} \right\} = \frac{\left(1 + \frac{r}{q}\right)^{nq} - 1}{m \left\{ \left(1 + \frac{r}{q}\right)^{\frac{q}{m}} - 1 \right\}}.$$

$$\text{Present value of same} = \frac{1 - \left(1 + \frac{r}{q}\right)^{-nq}}{m \left\{ \left(1 + \frac{r}{q}\right)^{\frac{q}{m}} - 1 \right\}}.$$

## PROBABILITIES.

**309** If all the ways in which an event can happen be  $m$  in number, all being equally likely to occur, and if in  $n$  of these  $m$  ways the event would happen under certain restrictive conditions; then the probability of the restricted event happening is equal to  $n \div m$ .

Thus, if the letters of the alphabet be chosen at random, any letter being equally likely to be taken, the probability of a vowel being selected is equal to  $\frac{5}{26}$ . The number of unrestricted cases here is 26, and the number of restricted ones 5.

**310** If, however, all the  $m$  events are not equally probable, they may be divided into groups of equally probable cases. The probability of the restricted event happening in each group separately must be calculated, and the sum of these probabilities will be the total probability of the restricted event happening at all.

EXAMPLE.—There are three bags *A*, *B*, and *C*.

*A* contains 2 white and 3 black balls.

*B*    "    3       "    4       "

*C*    "    4       "    5       "

A bag is taken at random and a ball drawn from it. Required the probability of the ball being white.

Here the probability of the bag *A* being chosen =  $\frac{1}{3}$ , and the subsequent probability of a white ball being drawn =  $\frac{2}{5}$ .

Therefore the probability of a white ball being drawn from *A*

$$= \frac{1}{3} \times \frac{2}{5} = \frac{2}{15}.$$

Similarly the probability of a white ball being drawn from *B*

$$= \frac{1}{3} \times \frac{3}{7} = \frac{1}{7}.$$

And the probability of a white ball being drawn from *C*

$$= \frac{1}{3} \times \frac{4}{9} = \frac{4}{27}.$$

Therefore the total probability of a white ball being drawn

$$= \frac{2}{15} + \frac{1}{7} + \frac{4}{27} = \frac{401}{945}.$$

If *a* be the number of ways in which an event can happen, and *b* the number of ways in which it can fail, then the

**311**   Probability of the event happening =  $\frac{a}{a+b}.$

**312**   Probability of the event failing       =  $\frac{b}{a+b}.$

Thus                      Certainty = 1.

If *p*, *p'* be the respective probabilities of two *independent* events, then

**313**   Probability of both happening       =  $pp'.$

**314**       "       of not *both* happening =  $1 - pp'.$

**315**       "       of one happening and one failing  
                              =  $p + p' - 2pp'.$

**316**       "       of both failing =  $(1-p)(1-p').$

If the probability of an event happening in one trial be  $p$ , and the probability of its failing  $q$ , then

$$\begin{aligned} \text{317} \quad & \text{Probability of the event happening } r \text{ times in } n \text{ trials} \\ & = C(n, r) p^r q^{n-r}. \end{aligned}$$

$$\begin{aligned} \text{318} \quad & \text{Probability of the event failing } r \text{ times in } n \text{ trials} \\ & = C(n, r) p^{n-r} q^r. \quad [\text{By induction.}] \end{aligned}$$

**319** Probability of the event happening *at least*  $r$  times in  $n$  trials = the sum of the *first*  $n-r+1$  terms in the expansion of  $(p+q)^n$ .

**320** Probability of the event failing *at least*  $r$  times in  $n$  trials = the sum of the *last*  $n-r+1$  terms in the same expansion.

**321** The number of trials in which the probability of the same event happening amounts to  $p'$

$$= \frac{\log(1-p')}{\log(1-p)}.$$

From the equation  $(1-p)^x = 1-p'$ .

**322** DEFINITION.—When a sum of money is to be received if a certain event happens, that sum multiplied into the probability of the event is termed the expectation.

EXAMPLE.—If three coins be taken at random from a bag containing one sovereign, four half-crowns, and five shillings, the expectation will be the sum of the expectations founded upon each way of drawing three coins. But this is also equal to the average value of three coins out of the ten; that is,  $\frac{3}{10}$ ths of 35 shillings, or 10s. 6d.

**323** The probability that, after  $r$  chance selections of the numbers 0, 1, 2, 3 ...  $n$ , the sum of the numbers drawn will be  $s$ , is equal to the coefficient of  $x^s$  in the expansion of

$$(x^0 + x^1 + x^2 + \dots + x^n)^r \div (n+1)^r.$$

**324** The probability of the existence of a certain cause of an observed event out of several known causes, one of which *must* have produced the event, is proportional to the *a priori* probability of the cause existing multiplied by the probability of the event happening from it if it does exist.

Thus, if the *a priori* probabilities of the causes be  $P_1, P_2 \dots$  &c., and the corresponding probabilities of the event happening from those causes  $Q_1, Q_2 \dots$  &c., then the probability of the  $r^{\text{th}}$  cause having produced the event is

$$\frac{P_r Q_r}{\Sigma (PQ)}.$$

**325** If  $P'_1, P'_2 \dots$  &c. be the *a priori* probabilities of a second event happening from the same causes respectively, then, *after* the first event has happened, the probability of the

second happening is  $\frac{\Sigma (PQP')}{\Sigma (PQ)}.$

For this is the sum of such probabilities as  $\frac{P_r Q_r P'_r}{\Sigma (PQ)}$ , which is the probability of the  $r^{\text{th}}$  cause existing multiplied by the probability of the second event happening from it.

Ex. 1.—Suppose there are

4 vases containing each 5 white and 6 black balls,  
2 vases containing each 3 white and 5 black balls,  
and 1 vase containing 2 white and 1 black ball.

A white ball has been drawn, and the probability that it came from the group of 2 vases is required.

$$\begin{array}{lll} \text{Here} & P_1 = \frac{4}{7}, & P_2 = \frac{2}{7}, & P_3 = \frac{1}{7} \\ & Q_1 = \frac{5}{11}, & Q_2 = \frac{3}{8}, & Q_3 = \frac{2}{3}. \end{array}$$

Therefore, by (324), the probability required is

$$\frac{\frac{2 \cdot 3}{7 \cdot 8}}{\frac{4 \cdot 5}{7 \cdot 11} + \frac{2 \cdot 3}{7 \cdot 8} + \frac{1 \cdot 2}{7 \cdot 3}} = \frac{99}{427}.$$

Ex. 2.—After the white ball has been drawn and replaced, a ball is drawn again; required the probability of the ball being black.

Here  $P'_1 = \frac{6}{11}, \quad P'_2 = \frac{5}{8}, \quad P'_3 = \frac{1}{3}.$

The probability, by (325), will be

$$\frac{\frac{4.5.6}{7.11.11} + \frac{2.3.5}{7.8.8} + \frac{1.2.1}{7.3.3}}{\frac{4.5}{7.11} + \frac{2.3}{7.8} + \frac{1.2}{7.3}} = \frac{58639}{112728}.$$

If the probability of the second ball being white is required,  $Q_1 Q_2 Q_3$  must be employed instead of  $P'_1 P'_2 P'_3$ .

**326** The probability of one event *at least* happening out of a number of events whose respective probabilities are  $a, b, c,$

&c., is  $P_1 - P_2 + P_3 - P_4 + \&c.$

where  $P_1$  is the probability of 1 event happening,

$P_2$         „        „        2        „

and so on. For, by (316), the probability is

$$1 - (1-a)(1-b)(1-c) \dots = \Sigma a - \Sigma ab + \Sigma abc - \dots$$

**327** The probability of the occurrence of  $r$  assigned events and no more out of  $n$  events is

$$Q_r - Q_{r+1} + Q_{r+2} - Q_{r+3} + \&c.,$$

where  $Q_r$  is the probability of the  $r$  assigned events;  $Q_{r+1}$  the probability of  $r+1$  events including the  $r$  assigned events.

For if  $a, b, c \dots$  be the probabilities of the  $r$  events, and  $a', b', c' \dots$  the probabilities of the excluded events, the required probability will be

$$\begin{aligned} & abc \dots (1-a')(1-b')(1-c') \dots \\ & = abc \dots (1 - \Sigma a' + \Sigma a'b' - \Sigma a'b'c' + \dots). \end{aligned}$$

**328** The probability of *any*  $r$  events happening and no more is

$$\Sigma Q_r - \Sigma Q_{r+1} + \Sigma Q_{r+2} - \&c.$$

NOTE.—If  $a = b = c = \&c.$ , then  $\Sigma Q_r = C(n, r) Q_r$ , &c.



## INEQUALITIES.

**330**  $\frac{a_1 + a_2 + \dots + a_n}{b_1 + b_2 + \dots + b_n}$  lies between the greatest and least of the fractions  $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_n}{b_n}$ , the denominators being all of the same sign.

PROOF.—Let  $k$  be the greatest of the fractions, and  $\frac{a_r}{b_r}$  any other; then  $a_r < kb_r$ . Substitute in this way for each  $a$ . Similarly if  $k$  be the least fraction.

**331** 
$$\frac{a+b}{2} > \sqrt{ab}.$$

**332** 
$$\frac{a_1 + a_2 + \dots + a_n}{n} > \sqrt[n]{a_1 a_2 \dots a_n};$$

or, Arithmetic mean  $>$  Geometric mean.

PROOF.—Substitute both for the greatest and least factors their Arithmetic mean. The product is thus *increased* in value. Repeat the process indefinitely. The limiting value of the G. M. is the A. M. of the quantities.

**333** 
$$\frac{a^m + b^m}{2} > \left(\frac{a+b}{2}\right)^m,$$

excepting when  $m$  is a positive proper fraction.

PROOF:  $a^m + b^m = \left(\frac{a+b}{2}\right)^m \{(1+x)^m + (1-x)^m\},$

where  $x = \frac{a-b}{a+b}$ . Employ Bin. Th.

**334** 
$$\frac{a_1^m + a_2^m + \dots + a_n^m}{n} > \left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)^m,$$

excepting when  $m$  is a positive proper fraction.

P

Otherwise.—The Arithmetic mean of the  $m^{\text{th}}$  powers is greater than the  $m^{\text{th}}$  power of the Arithmetic mean, excepting when  $m$  is a positive proper fraction.

PROOF.—Similar to (332). Substitute for the greatest and least on the left side, employing (333).

**336** If  $x$  and  $m$  are positive, and  $x$  and  $mx$  less than unity; then

$$(1+x)^{-m} > 1-mx. \quad (125, 240)$$

**337** If  $x$ ,  $m$ , and  $n$  are positive, and  $n$  greater than  $m$ ; then, by taking  $x$  small enough, we can make

$$1+nx > (1+x)^m.$$

For  $x$  may be diminished until  $1+nx$  is  $> (1-mx)^{-1}$ , and this is  $> (1+x)^m$ , by *last*.

**338** If  $x$  be positive,  $\log(1+x) < x.$  (150)

If  $x$  be positive and  $> 1$ ,  $\log(1+x) > x - \frac{x^2}{2}.$  (155, 240)

If  $x$  be positive and  $< 1$ ,  $\log \frac{1}{1-x} > x.$  (156)

**339** When  $n$  becomes infinite in the two expressions

$$\frac{1.3.5 \dots (2n-1)}{2.4.6 \dots 2n} \quad \text{and} \quad \frac{3.5.7 \dots (2n+1)}{2.4.6 \dots 2n},$$

the first vanishes, the second becomes infinite, and their product lies between  $\frac{1}{2}$  and 1.

Shewn by adding 1 to each factor (see 73), and multiplying the result by the original fraction.

**340** If  $m$  be  $> n$ , and  $n > a$ ,

$$\left(\frac{m+a}{m-a}\right)^m \text{ is } < \left(\frac{n+a}{n-a}\right)^n.$$

**341** If  $a, b$  be positive quantities,

$$a^a b^b \text{ is } > \left( \frac{a+b}{2} \right)^{a+b}.$$

Similarly  $a^a b^b c^c > \left( \frac{a+b+c}{3} \right)^{a+b+c}.$

These and similar theorems may be proved by taking logarithms of each side, and employing the Expon. Th. (158), &c.

SCALES OF NOTATION.

**342** If  $N$  be a whole number of  $n+1$  digits, and  $r$  the radix of the scale,  $N = p_n r^n + p_{n-1} r^{n-1} + p_{n-2} r^{n-2} + \dots + p_1 r + p_0$ , where  $p_n, p_{n-1}, \dots p_0$  are the digits.

**343** Similarly a radix-fraction will be expressed by

$$\frac{p_1}{r} + \frac{p_2}{r^2} + \frac{p_3}{r^3} + \&c.,$$

where  $p_1, p_2, \&c.$  are the digits.

EXAMPLES: 3426 in the scale of 7 =  $3 \cdot 7^3 + 4 \cdot 7^2 + 2 \cdot 7 + 6$ ;

1045 in the same scale =  $\frac{1}{7} + \frac{0}{7^2} + \frac{4}{7^3} + \frac{5}{7^4}.$

**344** Ex.—To transform 34268 from the scale of 5 to the scale of 11.

RULE.—*Divide successively by the new radix.*

$$\begin{array}{r} 11 \overline{) 34268} \\ 11 \overline{) 1343} - t \\ 11 \overline{) 40} - 3 \\ \quad 1 - 9 \end{array}$$

Result 193t, in which  $t$  stands for 10.

**345** Ex.—To transform  $\cdot t0e1$  from the scale of 12 to that of 7,  $e$  standing for 11, and  $t$  for 10.

**RULE.**—*Multiply successively by the new radix.*

$$\begin{array}{r}
 \cdot t0e1 \\
 \underline{7} \\
 5\cdot t657 \\
 \underline{7} \\
 6\cdot 1931 \\
 \underline{7} \\
 1\cdot 0497 \\
 \underline{7} \\
 0\cdot 2971
 \end{array}
 \qquad \text{Result } \cdot 5610 \dots$$

**346** Ex.—In what scale does  $2t7$  represent the number 475 in the scale of ten?

$$\begin{array}{ll}
 \text{Solve the equation} & 2r^2 + 10r + 7 = 475. \\
 \text{Result} & r = 13.
 \end{array}
 \qquad [178]$$

**347** The sum of the digits of any number divided by  $r-1$  leaves the same remainder as the number itself divided by  $r-1$ ;  $r$  being the radix of the scale. (401)

**348** The difference between the sums of the digits in the even and odd places divided by  $r+1$  leaves the same remainder as the number itself when divided by  $r+1$ .

## THEORY OF NUMBERS.

**349** If  $a$  is prime to  $b$ ,  $\frac{a}{b}$  is in its lowest terms.

**PROOF.**—Let  $\frac{a}{b} = \frac{a_1}{b_1}$ , a fraction in lower terms.

Divide  $a$  by  $a_1$ , remainder  $a_2$ , quotient  $q_1$ ,

$b$  by  $b_1$ , remainder  $b_2$ , quotient  $q_1$ ;

and so on, as in finding the H. C. F. of  $a$  and  $a_1$ , and of  $b$  and  $b_1$  (see 30).  
Let  $a_n$  and  $b_n$  be the highest common factors thus determined.

---

Then, because  $\frac{a}{b} = \frac{a_1}{b_1}$ ,  $\therefore \frac{a}{b} = \frac{a - q_1 a_1}{b - q_1 b_1} = \frac{a_2}{b_2}$ , (70)

and so on; thus  $\frac{a}{b} = \frac{a_1}{b_1} = \frac{a_2}{b_2} = \&c. \dots\dots = \frac{a_n}{b_n}$ .

Therefore  $a$  and  $b$  are equimultiples of  $a_n$  and  $b_n$ ; that is,  $a$  is not prime to  $b$  if any fraction exists in lower terms.

---

**350** If  $a$  is prime to  $b$ , and  $\frac{a'}{b'} = \frac{a}{b}$ ; then  $a'$  and  $b'$  are equimultiples of  $a$  and  $b$ .

PROOF.—Let  $\frac{a'}{b'}$  reduced to its lowest terms be  $\frac{p}{q}$ . Then  $\frac{p}{q} = \frac{a}{b}$ , and, since  $p$  is now prime to  $q$ , and  $a$  prime to  $b$ , it follows, by 349, that  $\frac{p}{q}$  is neither greater nor less than  $\frac{a}{b}$ ; that is, it is equal to it. Therefore, &c.

---

**351** If  $ab$  is divisible by  $c$ , and  $a$  is not; then  $b$  must be.

PROOF.—Let  $\frac{ab}{c} = q$ ;  $\therefore \frac{a}{c} = \frac{q}{b}$ .

But  $a$  is prime to  $c$ ; therefore, by last,  $b$  is a multiple of  $c$ .

**352** If  $a$  and  $b$  be each of them prime to  $c$ ,  $ab$  is prime to  $c$ . [By (351).]

---

**353** If  $abcd \dots$  is divisible by a prime, one at least of the factors  $a, b, c$ , &c. must also be divisible by it.

Or, if  $p$  be prime to all but one of the factors, that factor is divisible by  $p$ . (351)

**354** Therefore, if  $a^n$  is divisible by  $p$ ,  $p$  cannot be prime to  $a$ ; and if  $p$  be a prime it must divide  $a$ .

---

**355** If  $a$  is prime to  $b$ , any power of  $a$  is prime to any power of  $b$ .

Also, if  $a, b, c$ , &c. are prime to each other, the product of any of their powers is prime to any other product of their powers.

**356** No expression with integral coefficients, such as  $A + Bx + Cx^2 + \dots$ , can represent primes only.

PROOF.—For it is divisible by  $x$  if  $A = 0$ ; and if not, it is divisible by  $A$ , when  $x = A$ .

**357** The number of primes is infinite.

PROOF.—Suppose if possible  $p$  to be the greatest prime. Then the product of all primes up to  $p$ , plus unity, is either a prime, in which case it would be a greater prime than  $p$ , or it must be divisible by a prime; but no prime up to  $p$  divides it, because there is a remainder 1 in each case. Therefore, if divisible at all, it must be by a prime greater than  $p$ . In either case, then, a prime greater than  $p$  exists.

**358** If  $a$  be prime to  $b$ , and the quantities  $a, 2a, 3a, \dots (b-1)a$  be divided by  $b$ , the remainders will be different.

PROOF.—Assume  $ma - nb = m'a - n'b$ ,  $m$  and  $n$  being less than  $b$ ,

$$\therefore \frac{a}{b} = \frac{n - n'}{m - m'}. \quad \text{Then by (350).}$$

**359** A number can be resolved into prime factors in one way only. [By (353).]

**360** To resolve 5040 into its prime factors.

RULE.—Divide by the prime numbers successively.

$$\begin{array}{r}
 2 \times 5 \mid 5040 \\
 \hline
 2 \mid 504 \\
 \hline
 2 \mid 252 \\
 \hline
 2 \mid 126 \\
 \hline
 7 \mid 63 \\
 \hline
 3 \mid 9 \\
 \hline
 3
 \end{array}$$

$$\text{Thus } 5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7.$$

**361** Required the least multiplier of 4704 which will make the product a perfect fourth power.

By (196),

$$4704 = 2^5 \cdot 3 \cdot 7^2.$$

Then

$$2^5 \cdot 3^1 \cdot 7^2 \times 2^3 \cdot 3^3 \cdot 7^1 = 2^8 \cdot 3^4 \cdot 7^3 = 84^4,$$

the indices 8, 4, 4 being the least multiples of 4 which are not less than 5, 1, 2 respectively.

Thus  $2^3 \cdot 3^3 \cdot 7^1 = 3584$  is the multiplier required.



**362** All numbers are of one of the forms  $2n$  or  $2n+1$

„ „  $2n$  or  $2n-1$

„ „  $3n$  or  $3n\pm 1$

„ „  $4n$  or  $4n\pm 1$  or  $4n+2$

„ „  $4n$  or  $4n\pm 1$  or  $4n-2$

„ „  $5n$  or  $5n\pm 1$  or  $5n\pm 2$

and so on.

**363** All square numbers are of the form  $5n$  or  $5n\pm 1$ .

PROOF.—By squaring the forms  $5n$ ,  $5n\pm 1$ ,  $5n\pm 2$ , which comprehend all numbers whatever.

**364** All cube numbers are of the form  $7n$  or  $7n\pm 1$ .

And similarly for other powers.

**365** The highest power of a prime  $p$ , which is contained in the product  $m!$ , is the sum of the integral parts of

$$\frac{m}{p}, \frac{m}{p^2}, \frac{m}{p^3}, \&c.$$

For there are  $\frac{m}{p}$  factors in  $m!$  which  $p$  will divide;  $\frac{m}{p^2}$  which it will divide a second time; and so on. The successive divisions are equivalent to dividing by

$$p^{\frac{m}{p}} \cdot p^{\frac{m}{p^2}} \dots \&c. = p^{\frac{m}{p} + \frac{m}{p^2} + \dots}$$

EXAMPLE.—The highest power of 3 which will divide  $29!$ . Here the factors 3, 6, 9, 12, 15, 18, 21, 24, 27 can be divided by 3. Their number is  $\frac{29}{3} = 9$  (the integral part).

The factors 9, 18, 27 can be divided a second time. Their number is  $\frac{29}{3^2} = 3$  (the integral part).

One factor, 27, is divisible a third time.  $\frac{29}{3^3} = 1$  (integral part).

$9+3+1=13$ ; that is,  $3^{13}$  is the highest power of 3 which will divide  $29!$ .

**366** The product of any  $r$  consecutive integers is divisible by  $r!$ .

PROOF:  $\frac{n(n-1)\dots(n-r+1)}{r!}$  is necessarily an integer, by (96).

**367** If  $n$  be a prime, every coefficient in the expansion of  $(a+b)^n$ , except the first and last, is divisible by  $n$ . By last.

**368** If  $n$  be a prime, the coefficient of every term in the expansion of  $(a+b+c \dots)^n$ , except  $a^n$ ,  $b^n$ , &c., is divisible by  $n$ .

PROOF.—By (367). Put  $\beta$  for  $(b+c+\dots)$ .

**369** *Fermat's Theorem*.—If  $p$  be a prime, and  $N$  prime to  $p$ ; then  $N^{p-1}-1$  is divisible by  $p$ .

PROOF:  $N^p = (1+1+\dots)^p = N + Mp$ . By (368).

**370** If  $p$  be any number, and if  $1, a, b, c, \dots (p-1)$  be all the numbers less than, and prime to  $p$ ; and if  $n$  be their number, and  $x$  any one of them; then  $x^n-1$  is divisible by  $p$ .

PROOF.—If  $x, ax, bx \dots (p-1)x$  be divided by  $p$ , the remainders will be all different and prime to  $p$  [as in (358)]; therefore the remainders will be  $1, a, b, c \dots (p-1)$ ; therefore the product

$$x^n abc \dots (p-1) = abc \dots (p-1) + Mp.$$

**371** *Wilson's Theorem*.—If  $p$  be a prime, and only then,  $1+(p-1)!$  is divisible by  $p$ .

Put  $p-1$  for  $r$  and  $n$  in (285), and apply Fermat's Theorem to each term.

**372** If  $p$  be a prime  $= 2n+1$ , then  $(n!)^2 + (-1)^n$  is divisible by  $p$ .

PROOF.—By multiplying together equi-distant factors of  $(p-1)!$  in Wilson's Theorem, and putting  $2n+1$  for  $p$ .

**373** Let  $N = a^p b^q c^r \dots$  in prime factors; the number of integers, including 1, which are less than  $n$  and prime to it, is

$$N \left(1 - \frac{1}{a}\right) \left(1 - \frac{1}{b}\right) \left(1 - \frac{1}{c}\right) \dots$$

PROOF.—The number of integers prime to  $N$  contained in  $a^p$  is  $a^p - \frac{a^p}{a}$ . Similarly in  $b^q, c^r$ , &c. Take the product of these.

Also the number of integers less than and prime to  $(N \times M \times \&c.)$  is the product of the corresponding numbers for  $N$ ,  $M$ , &c. separately.

**374** The number of divisors of  $N$ , including 1 and  $N$  itself, is  $= (p+1)(q+1)(r+1) \dots$ . For it is equal to the number of terms in the product

$$(1+a+\dots+a^p)(1+b+\dots+b^q)(1+c+\dots+c^r) \dots \&c.$$

**375** The number of ways of resolving  $N$  into two factors is half the number of its divisors (374). If the number be a square the two equal factors must, in this case, be reckoned as two divisors.

**376** If the factors of each pair are to be prime to each other, put  $p$ ,  $q$ ,  $r$ , &c. each equal to one.

**377** The sum of the divisors of  $N$  is

$$\frac{a^{p+1}-1}{a-1} \cdot \frac{b^{q+1}-1}{b-1} \cdot \frac{c^{r+1}-1}{c-1} \dots$$

PROOF.—By the product in (374), and by (85).

**378** If  $p$  be a prime, then the  $p-1^{\text{th}}$  power of any number is of the form  $mp$  or  $mp+1$ . By Fermat's Theorem (369).

Ex.—The  $12^{\text{th}}$  power of any number is of the form  $13m$  or  $13m+1$ .

**379** To find all the divisors of a number; for instance, of 504.

I.	II.						
		1					
504	2	2					
252	2	4					
126	2	8					
63	3	3	6	12	24		
21	3	9	18	36	72		
7	7	7	14	28	56	21	42
		84	168	63	126	252	504

EXPLANATION.—Resolve 504 into its prime factors, placing them in column II.

The divisors of 504 are now formed from the numbers in column II., and placed to the right of that column in the following manner:—

Place the divisor 1 to the right of column II., and follow this rule—*Multiply in order all the divisors which are written down by the next number in column II., which has not already been used as a multiplier: place the first new divisor so obtained and all the following products in order to the right of column II.*

**380**  $S_r$ , the sum of the  $r^{\text{th}}$  powers of the first  $n$  natural numbers is divisible by  $2n+1$ .

PROOF:  $x(x^2-1^2)(x^2-2^2)\dots(x^2-n^2)$

constitutes  $2n+1$  factors divisible by  $2n+1$ , by (366). Multiply out, rejecting  $x$ , which is to be less than  $2n+1$ . Thus, using (372),

$$x^{2n} - S_1 x^{2n-2} + S_2 x^{2n-4} - \dots S_{n-1} x^2 + (-1)^n (\underline{n})^2 = M(2n+1).$$

Put 1, 2, 3 ...  $(n-1)$  in succession for  $x$ , and the solution of the  $(n-1)$  equations is of the form  $S_r = M(2n+1)$ .

# THEORY OF EQUATIONS.

## FACTORS OF AN EQUATION.

*General form of a rational integral equation of the  $n^{\text{th}}$  degree.*

**400**  $p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n = 0.$

The left side will be designated  $f(x)$  in the following summary.

**401** If  $f(x)$  be divided by  $x-a$ , the remainder will be  $f(a)$ .  
By assuming  $f(x) = P(x-a) + R.$

**402** If  $a$  be a root of the equation  $f(x) = 0$ , then  $f(a) = 0.$

**403** To compute  $f(a)$  numerically; divide  $f(x)$  by  $x-a$ ,  
and the remainder will be  $f(a).$  [401]

**404** EXAMPLE.—To find the value of  $4x^5 - 3x^3 + 12x^2 - x^2 + 10$  when  $x = 2.$

$$\begin{array}{r|rrrrrr}
 & 4 & -3 & +12 & +0 & -1 & +0 & +10 \\
 2 & & 8 & +10 & +44 & +88 & +174 & +348 \\
 \hline
 & & & & 4 & +5 & +22 & +44 & +87 & +174 & +358
 \end{array}$$

Thus  $f(2) = 358.$

If  $a, b, c \dots k$  be the roots of the equation  $f(x) = 0$ ;  
then, by (401) and (402),

**405**  $f(x) = p_0(x-a)(x-b)(x-c) \dots (x-k).$

By multiplying out the last equation, and equating coefficients with  
equation (400), considering  $p_0 = 1$ , the following results are obtained:—

---

**406**  $-p_1 =$  the sum of all the roots of  $f(x)$ .

$p_2 = \left\{ \begin{array}{l} \text{the sum of the products of the roots taken} \\ \text{two at a time.} \end{array} \right.$

$-p_3 = \left\{ \begin{array}{l} \text{the sum of the products of the roots taken} \\ \text{three at a time.} \end{array} \right.$

...                  ...                  ...                  ...                  ...  
 ...                  ...                  ...                  ...                  ...

$(-1)^r p_r = \left\{ \begin{array}{l} \text{the sum of the products of the roots taken} \\ \text{r at a time.} \end{array} \right.$

...                  ...                  ...                  ...                  ...

$(-1)^n p_n =$  product of all the roots.

**407** The number of roots of  $f(x)$  is equal to the degree of the equation.

---

**408** Imaginary roots must occur in pairs of the form

$$a + \beta\sqrt{-1}, \quad a - \beta\sqrt{-1}.$$

The quadratic factor corresponding to these roots will then have real coefficients; for it will be

$$x^2 - 2ax + a^2 + \beta^2. \quad [405, 226]$$


---

**409** If  $f(x)$  be of an odd degree, it has at least one real root of the opposite sign to  $p_n$ .

Thus  $x^2 - 1 = 0$  has *at least* one positive root.

**410** If  $f(x)$  be of an even degree, and  $p_n$  negative, there is *at least* one positive and one negative root.

Thus  $x^4 - 1$  has  $+1$  and  $-1$  for roots.

---

**411** If several terms at the beginning of the equation are of one sign, and all the rest of another, there is one, and only one, positive root.

Thus  $x^5 + 2x^4 + 3x^3 + x^2 - 5x - 4 = 0$  has only one positive root.

---

**412** If all the terms are positive there is no positive root.

**413** If all the terms of an even order are of one sign, and all the rest are of another sign, there is no negative root.

**414** Thus  $x^4 - x^3 + x^2 - x + 1 = 0$  has no negative root.

**415** If all the indices are even, and all the terms of the same sign, there is no real root; and if all the indices are odd, and all the terms of the same sign, there is no real root but zero.

Thus  $x^4 + x^2 + 1 = 0$  has no real root, and  $x^5 + x^3 + x = 0$  has no real root but zero. In this last equation there is no absolute term, because such a term would involve the zero power of  $x$ , which is even, and by hypothesis is wanting.

### DESCARTES' RULE OF SIGNS.

**416** In the following theorems every two adjacent terms in  $f(x)$ , which have the same signs, count as one "continuation of sign"; and every two adjacent terms, with different signs, count as one change of sign.

**417**  $f(x)$ , multiplied by  $(x-a)$ , has an *odd* number of changes of sign thereby introduced, and *one at least*.

**418**  $f(x)$  cannot have more positive roots than changes of sign, or more negative roots than continuations of sign.

**419** When all the roots of  $f(x)$  are real, the number of positive roots is equal to the number of changes of sign in  $f(x)$ ; and the number of negative roots is equal to the number of changes of sign in  $f(-x)$ .

**420** Thus, it being known that the roots of the equation

$$x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$$

are all real; the number of positive roots will be equal to the number of changes of sign, which is four. Also  $f(-x) = x^4 + 10x^3 + 35x^2 + 50x + 24 = 0$ , and since there is no change of sign, there is consequently, by the rule, no negative root.

**421** If the degree of  $f(x)$  exceeds the number of changes of sign in  $f(x)$  and  $f(-x)$  together, by  $\mu$ , there are at least  $\mu$  imaginary roots.

**422** If, between two terms in  $f(x)$  of the same sign, there be an odd number of consecutive terms wanting, then there must be at least one more than that number of imaginary roots; and if the missing terms lie between terms of different



sign, there is at least one less than the same number of imaginary roots.

Thus, in the cubic equation  $x^3 + 4x - 7 = 0$ , there must be two imaginary roots.

And in the equation  $x^6 - 1 = 0$  there are, for certain, four imaginary roots.

**423** If an even number of consecutive terms be wanting in  $f(x)$ , there is at least the same number of imaginary roots.

Thus the equation  $x^5 + 1 = 0$  has four terms absent; and therefore four imaginary roots at least.

## THE DERIVED FUNCTIONS OF $f(x)$ .

Rule for forming the derived functions.

**424** Multiply each term by the index of  $x$ , and reduce the index by one; that is, differentiate the function with respect to  $x$ .

EXAMPLE.—Take

$$\begin{aligned} f(x) &= x^5 + x^4 + x^3 - x^2 - x - 1 \\ f'(x) &= 5x^4 + 4x^3 + 3x^2 - 2x - 1 \\ f''(x) &= 20x^3 + 12x^2 + 6x - 2 \\ f'''(x) &= 60x^2 + 24x + 6 \\ f^{(4)}(x) &= 120x + 24 \\ f^{(5)}(x) &= 120 \end{aligned}$$

$f'(x)$ ,  $f''(x)$ , &c. are called the first, second, &c. derived functions of  $f(x)$ .

**425** To form the equation whose roots differ from those of  $f(x)$  by a quantity  $a$ .

Put  $x = y + a$  in  $f(x)$ , and expand each term by the Binomial Theorem, arranging the results in vertical columns in the following manner:—

$$\begin{aligned} f(a+y) &= (a+y)^5 + (a+y)^4 + (a+y)^3 - (a+y)^2 - (a+y) - 1 \\ &= \begin{array}{r} a^5 + a^4 + a^3 - a^2 - a - 1 \\ + (5a^4 + 4a^3 + 3a^2 - 2a - 1)y \\ + (10a^3 + 6a^2 + 3a - 1)y^2 \\ + (10a^2 + 4a + 1)y^3 \\ + (5a + 1)y^4 \\ + y^5 \end{array} \end{aligned}$$

Comparing this result with that seen in (424), it is seen that

$$\begin{aligned} 426 \quad f(a+y) &= f(a) + f'(a)y \\ &+ \frac{f''(a)}{2}y^2 + \frac{f'''(a)}{3}y^3 + \frac{f^{(4)}(a)}{4}y^4 + \frac{f^{(5)}(a)}{5}y^5 \end{aligned}$$

so that the coefficient generally of  $y^r$  in the transformed equation is  $\frac{f^{(r)}(a)}{r!}$ .

**427** To form the equation most expeditiously when  $a$  has a numerical value, divide  $f(x)$  continuously by  $x-a$ , and the successive remainders will furnish the coefficients.

EXAMPLE.—To expand  $f(y+2)$  when, as in (425),

$$f(x) = x^5 + x^4 + x^3 - x^2 - x - 1.$$

Divide repeatedly by  $x-2$ , as follows:—

$$\begin{array}{r|l} 2 & 1 + 1 + 1 - 1 - 1 - 1 \\ & + 2 + 6 + 14 + 26 + 50 \\ \hline 2 & 1 + 3 + 7 + 13 + 25 \quad + 49 = f(2) \\ & + 2 + 10 + 34 + 94 \\ \hline 2 & 1 + 5 + 17 + 47 \quad + 119 = f'(2) \\ & + 2 + 14 + 62 \\ \hline 2 & 1 + 7 + 31 \quad + 109 = \frac{f''(2)}{2} \\ & + 2 + 18 \\ \hline 2 & 1 + 9 \quad + 49 = \frac{f'''(2)}{3} \\ & + 2 \\ \hline 2 & 1 \quad + 11 = \frac{f^{(4)}(2)}{4} \\ & \\ \hline & 1 = \frac{f^{(5)}(2)}{5} \end{array}$$

That these remainders are the required coefficients is seen by inspecting the form of the equation (426); for if that equation be divided by  $x-a=y$  repeatedly, these remainders are obviously produced when  $a=2$ .

Thus the equation, whose roots are each less by 2 than the roots of the proposed equation, is  $y^5 + 11y^4 + 49y^3 + 109y^2 + 119y + 49 = 0$ .

**428** To make any assigned term vanish in the transformed equation,  $a$  must be so determined that the coefficient of that term shall vanish.

EXAMPLE.—In order that there may be no term involving  $y^4$  in equation (426), we must have  $f^{(4)}(a) = 0$ .

Find  $f^{(4)}(a)$  as in (424);

thus  $120a + 24 = 0$ ;  $\therefore a = -\frac{1}{5}$ .

The equation in (424) must now be divided repeatedly by  $x + \frac{1}{5}$  after the manner of (427), and the resulting equation will be minus its second term.

**429** Note, that to remove the second term of the equation  $f(x) = 0$ , the requisite value of  $a$  is  $= -\frac{p_1}{np_0}$ ; that is, the coefficient of the second term, with the sign changed, divided by the coefficient of the first term, and by the number expressing the degree of the equation.

**430** To transform  $f(x)$  into an equation in  $y$  so that  $y = \phi(x)$ , a given function of  $x$ , put  $x = \phi^{-1}(y)$ , the inverse function of  $y$ .

EXAMPLE.—To obtain an equation whose roots are respectively three times the roots of the equation  $x^3 - 6x + 1 = 0$ . Here  $y = 3x$ ; therefore  $x = \frac{y}{3}$ , and the equation becomes  $\frac{y^3}{27} - \frac{6y}{3} + 1 = 0$ , or  $y^3 - 54y + 27 = 0$ .

**431** To transform  $f(x) = 0$  into an equation in which the coefficient of the first term shall be unity, and the other coefficients the least possible integers.

EXAMPLE.—Take the equation

$$288x^3 + 240x^2 - 176x - 21 = 0.$$

Divide by the coefficient of the first term, and reduce the fractions; the equation becomes  $x^3 + \frac{5}{6}x^2 - \frac{11}{18}x - \frac{7}{96} = 0$ .

Substitute  $\frac{y}{k}$  for  $x$ , and multiply by  $k^3$ ; we get

$$y^3 + \frac{5k}{6}y^2 - \frac{11k^2}{18}y - \frac{7k^3}{96} = 0.$$

Next resolve the denominators into their prime factors,

$$y^3 + \frac{5k}{2 \cdot 3}y^2 - \frac{11k^2}{2 \cdot 3^2}y - \frac{7k^3}{2^5 \cdot 3} = 0.$$

The smallest value must now be assigned to  $k$ , which will suffice to make each coefficient an integer. This is easily seen by inspection to be  $2^5 \cdot 3 = 12$ , and the resulting equation is  $y^3 + 10y^2 - 88y - 126 = 0$ , the roots of which are connected with the roots of the original equation by the relation  $y = 12x$ .

## EQUAL ROOTS OF AN EQUATION.

By expanding  $f(x+z)$  in powers of  $z$  by (405), and also by (426), and equating the coefficients of  $z$  in the two ex-



The solution of the equations  $X_1 = 0$ ,  $X_2 = 0$ , &c. will furnish all the roots of  $f(x)$ ; those which occur twice being found from  $X_2$ ; those which occur three times each, from  $X_3$ ; and so on.

**446** If  $f(x)$  has all its coefficients commensurable,  $X_1, X_2, X_3$ , &c. have likewise their coefficients commensurable.

Hence, if only one root be repeated  $r$  times, that root must be commensurable.

**447** In all the following theorems, unless otherwise stated,  $f(x)$  is understood to have unity for the coefficient of its first term.

### LIMITS OF THE ROOTS.

**448** If the greatest negative coefficients in  $f(x)$  and  $f(-x)$  be  $p$  and  $q$  respectively, then  $p+1$  and  $-(q+1)$  are limits of the roots.

**449** If  $x^{n-r}$  and  $x^{n-s}$  are the highest negative terms in  $f(x)$  and  $f(-x)$  respectively,  $(1 + \sqrt[r]{p})$  and  $-(1 + \sqrt[s]{q})$  are limits of the roots.

**450** If  $k$  be a superior limit to the positive roots of  $f\left(\frac{1}{x}\right)$ , then  $\frac{1}{k}$  will be an inferior limit to the positive roots of  $f(x)$ .

**451** If each negative coefficient be divided by the sum of all the preceding positive coefficients, the greatest of the fractions so formed + unity will be a superior limit to the positive roots.

**452** *Newton's method.*—Put  $x = h + y$  in  $f(x)$ ; then, by (426),

$$f(h+y) = f(h) + yf'(h) + \frac{y^2}{2}f''(h) + \dots + \frac{y^n}{n}f^{(n)}(h) = 0.$$

Take  $h$  so that  $f(h), f'(h), f''(h) \dots f^{(n)}(h)$  are all positive; then  $h$  is a superior limit to the positive roots.

**453** According as  $f(a)$  and  $f(b)$  have the same or different signs, the number of roots intermediate between  $a$  and  $b$  is even or odd.

**454 Rolle's Theorem.**—One real root of the equation  $f'(x)$  lies between every two adjacent real roots of  $f(x)$ .

**455** COR. 1.— $f(x)$  cannot have more than one root greater than the greatest root in  $f'(x)$ ; or more than one less than the least root in  $f'(x)$ .

**456** COR. 2.—If  $f(x)$  has  $m$  real roots,  $f'(x)$  has at least  $m - r$  real roots.

**457** COR. 3.—If  $f'(x)$  has  $\mu$  imaginary roots,  $f(x)$  has also  $\mu$  at least.

**458** COR. 4.—If  $\alpha, \beta, \gamma \dots \kappa$  be the roots of  $f'(x)$ ; then the number of changes of sign in the series of terms

$$f(\infty), f(\alpha), f(\beta), f(\gamma) \dots f(-\infty)$$

is equal to the number of roots of  $f(x)$ .

## NEWTON'S METHOD OF DIVISORS.

**459** To discover the integral roots of an equation.

EXAMPLE.—To ascertain if 5 be a root of

$$x^4 - 6x^3 + 86x^2 - 176x + 105 = 0.$$

If 5 be a root it will divide 105. Add the quotient to the next coefficient. Result,  $-155$ .

If 5 be a root it will divide  $-155$ . Add the quotient to the next coefficient; and so on.

If the number tried be a root, the divisions will be effectible to the end, and the last quotient will be  $-1$ , or  $-p_0$ , if  $p_0$  be not unity.

$$\begin{array}{r} 5 \overline{) 105} \\ \underline{21} \\ -176 \\ 5 \overline{) -155} \\ \underline{-31} \\ 86 \\ 5 \overline{) 55} \\ \underline{11} \\ -6 \\ 5 \overline{) -5} \\ \underline{-1} \end{array}$$

**460** In employing this method, limits of the roots may first be found, and divisors chosen between those limits.

**461** Also, to lessen the number of trial divisors, take any integer  $m$ ; then any divisor  $a$  of the last term can be rejected if  $a - m$  does not divide  $f(m)$ .

In practice take  $m = +1$  and  $-1$ .

To find whether any of the roots determined as above are repeated, divide  $f(x)$  by the factors corresponding to them, and then apply the method of divisors to the resulting equation.



EXAMPLE.—Take the equation

$$x^6 + 2x^5 - 17x^4 - 26x^3 + 88x^2 + 72x - 144 = 0.$$

Putting  $x = 1$ , we find  $f(1) = -24$ . The divisors of 144 are

1, 2, 3, 4, 6, 8, 9, 12, 16, 24, &c.

The values of  $a - m$  (since  $m = 1$ ) are therefore

0, 1, 2, 3, 5, 7, 8, 11, 15, 23, &c.

Of these last numbers only 1, 2, 3, and 8 will divide 24. Hence 2, 3, 4, and 9 are the only divisors of 144 which it is of use to try. The only integral roots of the equation will be found to be  $\pm 2$  and  $\pm 3$ .

**462** If  $f(x)$  and  $F(X)$  have common roots, they are contained in the greatest common measure of  $f(x)$  and  $F(X)$ .

**463** If  $f(x)$  has for its roots  $a, \phi(a), b, \phi(b)$  amongst others; then the equations  $f(x) = 0$  and  $f\{\phi(x)\} = 0$  have the common roots  $a$  and  $b$ .

**464** But, if all the roots occur in pairs in this way, these equations coincide.

For example, suppose that each pair of roots,  $a$  and  $b$ , satisfies the equation  $a + b = 2r$ . We may then assume  $a - b = 2z$ . Therefore  $f(z + r) = 0$ . This equation involves only even powers of  $z$ , and may be solved for  $z^2$ .

**465** Otherwise: Let  $ab = z$ ; then  $f(x)$  is divisible by  $(x - a)(x - b) = x^2 - 2rx + z$ . Perform the division until a remainder is obtained of the form  $Px + Q$ , where  $P$  and  $Q$  only involve  $z$ .

The equations  $P = 0, Q = 0$  determine  $z$ , by (462); and  $a$  and  $b$  are found from  $a + b = 2r, ab = z$ .

## RECIPROCAL EQUATIONS.

**466** A reciprocal equation has its roots in pairs of the form  $a, \frac{1}{a}$ ; also the relation between the coefficients is

$$p_r = p_{n-r}, \text{ or else } p_r = -p_{n-r}.$$

**467** A reciprocal equation of an even degree, with its last term positive, may be made to depend upon the solution of an equation of half the same degree.



**468** EXAMPLE:  $4x^6 - 24x^5 + 57x^4 - 73x^3 + 57x^2 - 24x + 4 = 0$  is a reciprocal equation of an even degree, with its last term positive.

Any reciprocal equation which is not of this form may be reduced to it by dividing by  $x+1$  if the last term be positive; and, if the last term be negative, by dividing by  $x-1$  or  $x^2-1$ , so as to bring the equation to an even degree. Then proceed in the following manner:—

**469** First bring together equidistant terms, and divide the equation by  $x^3$ ; thus

$$4\left(x^3 + \frac{1}{x^3}\right) - 24\left(x^2 + \frac{1}{x^2}\right) + 57\left(x + \frac{1}{x}\right) - 73 = 0.$$

By putting  $x + \frac{1}{x} = y$ , and by making repeated use of the relation  $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$ , the equation is reduced to a cubic in  $y$ , the degree being one-half that of the original equation.

Put  $p$  for  $x + \frac{1}{x}$ , and  $p_m$  for  $x_m + \frac{1}{x_m}$ .

**470** The relation between the successive factors of the form  $p_m$  may be expressed by the equation

$$p_m = pp_{m-1} - p_{m-2}.$$

**471** The equation for  $p_m$ , in terms of  $p$ , is

$$p_m = p^m - mp^{m-2} + \frac{m(m-3)}{1 \cdot 2} p^{m-4} - \dots$$

$$+ (-1)^r \frac{m(m-r-1) \dots (m-2r+1)}{r} p^{m-2r} + \dots$$

By (545), putting  $q = 1$ .

## BINOMIAL EQUATIONS.

**472** If  $a$  be a root of  $x^n - 1 = 0$ , then  $a^m$  is likewise a root where  $m$  is any positive or negative integer.

**473** If  $a$  be a root of  $x^n + 1 = 0$ , then  $a^{2m+1}$  is likewise a root.

**474** If  $m$  and  $n$  be prime to each other,  $x^m - 1$  and  $x^n - 1$  have no common root but unity.

Take  $pm - qn = 1$  for an indirect proof.

**475** If  $n$  be a prime number, and if  $a$  be a root of  $x^n - 1 = 0$ , the other roots are  $a, a^2, a^3 \dots a^n$ .

These are all roots, by (472). Prove, by (474), that no two can be equal.

**476** If  $n$  be not a prime number, other roots besides these may exist. The successive powers, however, of some root will furnish all the rest.

**477** If  $x^n - 1 = 0$  has the index  $n = mpq$ ;  $m, p, q$  being prime factors; then the roots are the terms of the product

$$(1 + a + a^2 + \dots + a^{m-1}) (1 + \beta + \beta^2 + \dots + \beta^{p-1}) \\ \times (1 + \gamma + \gamma^2 + \dots + \gamma^{q-1}),$$

where  $a$  is a root of  $x^m - 1$ ,

$\beta$  „  $x^p - 1$ ,

$\gamma$  „  $x^q - 1$ ,

but neither  $a, \beta$ , nor  $\gamma = 1$ .

Proof as in (475).

**478** If  $n = m^3$ , and

$a$  be a root of  $x^m - 1 = 0$ ,

$\beta$  „  $x^m - a = 0$ ,

$\gamma$  „  $x^m - \beta = 0$ ;

then the roots of  $x^n - 1 = 0$  will be the terms of the product

$$(1 + a + a^2 + \dots + a^{m-1}) (1 + \beta + \beta^2 + \dots + \beta^{m-1}) \\ \times (1 + \gamma + \gamma^2 + \dots + \gamma^{m-1}).$$

**479**  $x^n \pm 1 = 0$  may be treated as a reciprocal equation, and depressed in degree after the manner of (468).

**480** The complete solution of the equation

$$x^n - 1 = 0$$

is obtained by De Moivre's Theorem.

(757)

The  $n$  different roots are given by the formula

$$x = \cos \frac{2r\pi}{n} \pm \sqrt{-1} \sin \frac{2r\pi}{n}$$

in which  $r$  must have the successive values 0, 1, 2, 3, &c., concluding with  $\frac{n}{2}$ , if  $n$  be even; and with  $\frac{n-1}{2}$ , if  $n$  be odd.

**481** Similarly the  $n$  roots of the equation

$$x^n + 1 = 0$$

are given by the formula

$$x = \cos \frac{(2r+1)\pi}{n} \pm \sqrt{-1} \sin \frac{(2r+1)\pi}{n}$$

$r$  taking the successive values 0, 1, 2, 3, &c., up to  $\frac{n-2}{2}$ , if  $n$  be even; and up to  $\frac{n-3}{2}$ , if  $n$  be odd.

**482** The number of different values of the product

$$A^{\frac{1}{m}} B^{\frac{1}{n}}$$

is equal to the least common multiple of  $m$  and  $n$ , when  $m$  and  $n$  are integers.

## CUBIC EQUATIONS.

**483** To solve the general cubic equation

$$x^3 + px^2 + qx + r = 0.$$

Remove the term  $px^2$  by the method of (429). Let the transformed equation be  $x^3 + qx + r = 0$ .

**484** *Cardan's method.*—The complete theoretical solution of this equation by Cardan's method is as follows:—

$$\text{Put} \quad x = y + z \quad (\text{i.})$$

$$y^3 + z^3 + (3yz + q)(y + z) + r = 0.$$

$$\text{Put} \quad 3yz + q = 0; \quad \therefore y = -\frac{q}{3z}.$$

Substitute this value of  $y$ , and solve the resulting quadratic in  $y^3$ . The roots are equal to  $y^3$  and  $z^3$  respectively; and we have, by (i.),

$$\text{485} \quad x = \left\{ -\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}} \right\}^{\frac{1}{3}} + \left\{ -\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}} \right\}^{\frac{1}{3}}.$$

The cubic must have one real root at least, by (409).

Let  $m$  be one of the three values of  $\left\{-\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right\}^{\frac{1}{3}}$ , and  $n$  one of the three values of  $\left\{-\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right\}^{\frac{1}{3}}$ .

**486** Let  $1, \alpha, \alpha^2$  be the three cube roots of unity, so that

$$\alpha = -\frac{1}{2} + \frac{1}{2}\sqrt{-3}, \text{ and } \alpha^2 = -\frac{1}{2} - \frac{1}{2}\sqrt{-3}. \quad [472]$$

**487** Then, since  $\sqrt[3]{m^3} = m\sqrt[3]{1}$ , the roots of the cubic will be

$$m+n, \quad \alpha m + \alpha^2 n, \quad \alpha^2 m + \alpha n.$$

Now, if in the expansion of

$$\left\{-\frac{r}{2} \pm \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right\}^{\frac{1}{3}}$$

by the Binomial Theorem, we put

$\mu$  = the sum of the odd terms, and

$\nu$  = the sum of the even terms;

then we shall have  $m = \mu + \nu$ , and  $n = \mu - \nu$ ;

or else  $m = \mu + \nu\sqrt{-1}$ , and  $n = \mu - \nu\sqrt{-1}$ ;

according as  $\sqrt{\frac{r^2}{4} + \frac{q^3}{27}}$  is real or imaginary.

By substituting these expressions for  $m$  and  $n$  in (487), it appears that—

**488** (i.) If  $\frac{r^2}{4} + \frac{q^3}{27}$  be positive, the roots of the cubic will be

$$2\mu, \quad -\mu + \nu\sqrt{-3}, \quad -\mu - \nu\sqrt{-3}.$$

(ii.) If  $\frac{r^2}{4} + \frac{q^3}{27}$  be negative, the roots will be

$$2\mu, \quad -\mu + \nu\sqrt{3}, \quad -\mu - \nu\sqrt{3}.$$

(iii.) If  $\frac{r^2}{4} + \frac{q^3}{27} = 0$ , the roots are

$$2m, \quad -m, \quad -m;$$

since  $m$  is now equal to  $\mu$ .

**489** *The Trigonometrical method.*—The equation

$$x^3 + qx + r = 0$$

may be solved in the following manner, by Trigonometry, when  $\frac{r^2}{4} + \frac{q^3}{27}$  is negative.

Assume  $x = n \cos a$ . Divide the equation by  $n^3$ ; thus

$$\cos^3 a + \frac{q}{n^2} \cos a + \frac{r}{n^3} = 0.$$

But

$$\cos^3 a - \frac{3}{4} \cos a - \frac{\cos 3a}{4} = 0. \quad \text{By (657)}$$

Equate coefficients in the two equations; the result is

$$n = \left(\frac{4q}{3}\right)^{\frac{1}{3}}, \quad \cos 3a = -4r\left(-\frac{3}{4q}\right)^{\frac{1}{3}},$$

$a$  must now be found with the aid of the Trigonometrical tables.

**490** The roots of the cubic will be

$$n \cos a, \quad n \cos \left(\frac{2}{3}\pi + a\right), \quad n \cos \left(\frac{2}{3}\pi - a\right).$$

**491** Observe that, according as  $\frac{r^2}{4} + \frac{q^3}{27}$  is positive or negative, Cardan's method or the Trigonometrical will be practicable. In the former case, there will be *one real and two imaginary roots*; in the latter case, *three real roots*.

## BIQUADRATIC EQUATIONS.

**492** *Descartes' Solution.*—To solve the equation

$$x^4 + qx^2 + rx + s = 0 \quad \dots\dots\dots (i.)$$

the term in  $x^3$  having been removed by the method of (429).

$$\text{Assume} \quad (x^2 + ex + f)(x^2 - ex + g) = 0 \quad \dots\dots\dots (ii.)$$

Multiply out, and equate coefficients with (i.); and the following equations for determining  $f$ ,  $g$ , and  $e$  are obtained

$$g + f = q + e^2, \quad g - f = \frac{r}{e}, \quad gf = s \quad \dots\dots\dots (iii.)$$

$$\mathbf{493} \quad e^6 + 2qe^4 + (q^2 - 4s)e^2 - r^2 = 0 \quad \dots\dots\dots (iv.)$$

**494** The cubic in  $e^2$  is reducible by Cardan's method, when the biquadratic has two real and two imaginary roots. For proof, take  $a \pm i\beta$  and  $-a \pm \gamma$  as the roots of (i.), since their sum must be zero. Form the sum of each pair for the values of  $e$  [see (ii.)], and apply the rules in (488) to the cubic in  $e^2$ .

If the biquadratic has all its roots real, or all imaginary, the cubic will have all its roots real. Take  $a \pm i\beta$  and  $-a \pm i\gamma$  for four imaginary roots of (i.), and form the values of  $e$  as before.

**495** If  $\alpha^2$ ,  $\beta^2$ ,  $\gamma^2$  be the roots of the cubic in  $e^2$ , the roots of the biquadratic will be  $-\frac{1}{2}(\alpha + \beta + \gamma)$ ,  $\frac{1}{2}(\alpha + \beta - \gamma)$ ,  $\frac{1}{2}(\beta + \gamma - \alpha)$ ,  $\frac{1}{2}(\gamma + \alpha - \beta)$ .

For proof, take  $w, x, y, z$  for the roots of the biquadratic; then, by (ii.), the sum of each pair must give a value of  $e$ . Hence, we have only to solve the symmetrical equations

$$\begin{array}{ll} y+z = \alpha, & w+x = -\alpha, \\ z+x = \beta, & w+y = -\beta, \\ x+y = \gamma, & w+z = -\gamma. \end{array}$$

**496 Ferrari's solution.**—To the left member of the equation

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

add the quantity  $ax^3 + bx + \frac{b^3}{4a}$ , and assume the result

$$= \left(x^2 + \frac{p}{2}x + m\right)^2.$$

**497** Expanding and equating coefficients, the following cubic equation for determining  $m$  is obtained

$$8m^3 - 4qm^2 + (2pr - 8s)m + 4qs - p^2s - r = 0.$$

Then  $x$  is given by the two quadratics

$$x^2 + \frac{p}{2}x + m = \pm \frac{2ax+b}{2\sqrt{a}}$$

**498** The cubic in  $m$  is reducible by Cardan's method when the biquadratic has two real and two imaginary roots. Assume  $\alpha, \beta, \gamma, \delta$  for the roots of the biquadratic; then  $\alpha\beta$  and  $\gamma\delta$  are the respective products of roots of the two quadratics above. From this find  $m$  in terms of  $\alpha\beta\gamma\delta$ .

**499 Euler's solution.**—Remove the term in  $x^3$ ; then we have

$$x^4 + qx^2 + rx + s = 0.$$

**500** Assume  $x = y + z + u$ , and it may be shewn that  $y^3, z^3$ , and  $u^3$  are the roots of the equation

$$t^3 + \frac{q}{2}t^2 + \frac{q^2 - 4s}{16}t - \frac{r^2}{64} = 0.$$

**501** The six values of  $y, z$ , and  $u$ , thence obtained, are restricted by the relation  $yzu = -\frac{r}{8}$ .

Thus  $x = y + z + u$  will take four different values.

## COMMENSURABLE ROOTS.

**502** To find the commensurable roots of an equation.

First transform it by putting  $x = \frac{y}{k}$  into an equation of the form  $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n = 0$ , having  $p_0 = 1$ , and the remaining coefficients integers. (431)

**503** This equation cannot have a rational fractional root, and the integral roots may be found by Newton's method of Divisors (459).

These roots, divided each by  $k$ , will furnish the commensurable roots of the original equation.

**504** EXAMPLE.—To find the commensurable roots of the equation

$$81x^5 - 207x^4 - 9x^3 + 89x^2 + 2x - 8 = 0.$$

Dividing by 81, and proceeding as in (431), we find the requisite substitution to be  $x = \frac{y}{9}$ .

The transformed equation is

$$y^5 - 23y^4 - 9y^3 + 801y^2 + 162y - 5832 = 0.$$

The roots all lie between 24 and  $-34$ , by (451).

The method of divisors gives the integral roots

$$6, -4, \text{ and } 3.$$

Therefore, dividing each by 9, we find the commensurable roots of the original equation to be  $\frac{2}{3}, -\frac{4}{9}, \text{ and } \frac{1}{3}$ .

**505** To obtain the remaining roots; diminish the transformed equation by the roots 6,  $-4$ , and 3, in the following manner (see 427):—

	1-23- 9+801+162-5832
6	6-102-666+810+5832
	1-17-111+135+972
-4	- 4+ 84+108-972
	1-21- 27+243
3	3- 54-243
	1-18- 81

The depressed equation is therefore

$$y^2 - 18y - 81 = 0.$$

The roots of which are  $9(1 + \sqrt{2})$  and  $9(1 - \sqrt{2})$ ; and, consequently, the incommensurable roots of the proposed equation are  $1 + \sqrt{2}$  and  $1 - \sqrt{2}$ .



## INCOMMENSURABLE ROOTS.

**506 Sturm's Theorem.**—If  $f(x)$ , freed from equal roots, be divided by  $f'(x)$ , and the last divisor by the last remainder, *changing the sign* of each remainder before dividing by it, until a remainder independent of  $x$  is obtained, or else a remainder which cannot change its sign; then  $f(x)$ ,  $f'(x)$ , and the successive remainders constitute Sturm's functions, and are denoted by  $f(x)$ ,  $f_1(x)$ ,  $f_2(x)$ , &c. ....  $f_m(x)$ .

The operation may be exhibited as follows:—

$$\begin{aligned} f(x) &= q_1 f_1(x) - f_2(x), \\ f_1(x) &= q_2 f_2(x) - f_3(x), \\ f_2(x) &= q_3 f_3(x) - f_4(x), \\ &\dots \dots \dots \dots \dots \\ f_{m-2}(x) &= q_{m-1} f_{m-1}(x) - f_m(x). \end{aligned}$$

**507 NOTE.**—Any constant factor of a remainder may be rejected, and the quotient may be set down for the corresponding function.

**508** An inspection of the foregoing equations shews—

(1) That  $f_m(x)$  cannot be zero; for, if it were,  $f(x)$  and  $f_1(x)$  would have a common factor, and therefore  $f(x)$  would have equal roots, by (432).

(2) Two consecutive functions, after the first, cannot vanish together; for this would make  $f_m(x)$  zero.

(3) When any function, after the first, vanishes, the two adjacent ones have contrary signs.

**509** If, as  $x$  increases,  $f(x)$  passes through the value zero, Sturm's functions lose one change of sign.

For, before  $f(x)$  takes the value zero,  $f(x)$  and  $f_1(x)$  have contrary signs, and afterwards they have the same sign; as may be shown by making  $h$  small, and changing its sign in the expansion of  $f(x+h)$ , by (426).

**510** If any other of Sturm's functions vanishes, there is neither loss nor gain in the number of changes of sign.

This will appear on inspecting the equations.

**511 RESULT.**—The number of roots of  $f(x)$  between  $a$  and  $b$  is equal to the difference in the number of changes of sign in Sturm's functions, when  $x=a$  and when  $x=b$ .

**512** COR.—The total number of roots of  $f(x)$  will be found by taking  $a = +\infty$  and  $b = -\infty$ ; the sign of each function will then be the same as that of its first term.

When the number of functions exceeds the degree of  $f(x)$  by unity, the two following theorems hold:—

**513** *If the first terms in all the functions, after the first, are positive; all the roots of  $f(x)$  are real.*

**514** *If the first terms are not all positive; then, for every change of sign, there will be a pair of imaginary roots.*

For the proof put  $x = +\infty$  and  $-\infty$ , and examine the number of changes of sign in each case, applying Descartes' rule. (416).

**515** If  $\phi(x)$  has no factor in common with  $f(x)$ , and if  $\phi(x)$  and  $f'(x)$  take the same sign when  $f(x) = 0$ ; then the rest of Sturm's functions may be found from  $f(x)$  and  $\phi(x)$ , instead of  $f'(x)$ . For the reasoning in (509) and (510) will apply to the new functions.

**516** If Sturm's functions be formed without first removing equal roots from  $f(x)$ , the theorem will still give the number of distinct roots, without repetitions, between assigned limits.

For if  $f(x)$  and  $f_1(x)$  be divided by their highest common factor (see 444), and if the quotients be used instead of  $f(x)$  and  $f_1(x)$  to form Sturm's functions; then, by (515), the theorem will apply to the new set of functions, which will differ only from those formed from  $f(x)$  and  $f_1(x)$  by the absence of the same factor in every term of the series.

**517** EXAMPLE.—To find the position of the roots of the equation

$$x^4 - 4x^3 + x^2 + 6x + 2 = 0.$$

Sturm's functions, formed according to the rule given above, are here calculated.

The first terms of the functions are all positive; therefore there is no imaginary root.

$$\begin{aligned} f(x) &= x^4 - 4x^3 + x^2 + 6x + 2 \\ f_1(x) &= 2x^3 - 6x^2 + x + 3 \\ f_2(x) &= 5x^2 - 10x - 7 \\ f_3(x) &= x - 1 \\ f_4(x) &= 12 \end{aligned}$$

The changes of sign in the functions, as  $x$  passes through integral values, are exhibited in the adjoining table. There are two changes of sign lost while  $x$  passes from  $-1$  to  $0$ , and two more lost while  $x$  passes from  $2$  to  $3$ . There

$x =$	-2	-1	0	1	2	3	4
$f(x) =$	+	+	+	+	+	+	+
$f_1(x) =$	-	-	+	+	-	+	+
$f_2(x) =$	+	+	-	-	-	+	+
$f_3(x) =$	-	-	-	+	+	+	+
$f_4(x) =$	+	+	+	+	+	+	+
No. of changes of sign .....	4	4	2	2	2	0	0

are therefore two roots lying between 0 and  $-1$ ; and two roots also between 2 and 3.

These roots are all incommensurable, by (503).

**518** *Fourier's Theorem.*—Fourier's functions are the following quantities  $f(x)$ ,  $f'(x)$ ,  $f''(x)$  .....  $f^n(x)$ .

**519** Properties of Fourier's functions.—As  $x$  increases, Fourier's functions lose *one* change of sign for each root of the equation  $f(x) = 0$ , through which  $x$  passes, and  $r$  changes of sign for  $r$  repeated roots.

**520** If any of the other functions vanish, an *even* number of changes of sign is lost.

**521** RESULTS.—*The number of real roots of  $f(x)$  between  $\alpha$  and  $\beta$  cannot be more than the difference between the number of changes of sign in Fourier's functions when  $x = \alpha$ , and the number of changes when  $x = \beta$ .*

**522** When that difference is *odd*, the number of intermediate roots is *odd*, and therefore *one at least*.

**523** When the same difference is *even*, the number of intermediate roots is either *even* or *zero*.

**524** Descartes' rule of signs follows from the above for the signs of Fourier's functions, when  $x = 0$  are the signs of the terms in  $f(x)$ ; and when  $x = \infty$ , Fourier's functions are all positive.

**525** *Lagrange's method of approximating to the incommensurable roots of an equation.*

Let  $a$  be the greatest integer less than an incommensurable root of  $f(x)$ . Diminish the roots of  $f(x)$  by  $a$ . Take the reciprocal of the resulting equation. Let  $b$  be the greatest integer less than a positive root of this equation. Diminish the roots of this equation by  $b$ , and proceed as before.

**526** Let  $a, b, c$ , &c. be the quantities thus determined; then, an approximation to the incommensurable root of  $f(x)$  will be

the continued fraction 
$$x = a + \frac{1}{b + \frac{1}{c + \dots}}$$

**527 Newton's method of approximation.**—If  $c_1$  be a quantity a little less than one of the roots of the equation  $f(x) = 0$ , so that  $f(c_1 + h) = 0$ ; then  $c_1$  is a first approximation to the value of the root. Also because

$$f(c_1 + h) = f(c_1) + hf'(c_1) + \frac{h^2}{2} f''(c_1) + \&c. \dots \dots (426),$$

and  $h$  is but small, a second approximation to the root will be

$$c_1 - \frac{f(c_1)}{f'(c_1)} = c_2.$$

In the same way a third approximation may be obtained from  $c_2$ , and so on.

**528 Fourier's limitation of Newton's method.**—To ensure that  $c_1, c_2, c_3$ , &c. shall successively increase up to the value  $c_1 + h$  without passing beyond it, it is necessary for all values of  $x$  between  $c_1$  and  $c_1 + h$ .

- (i.) That  $f(x)$  and  $f'(x)$  should have contrary signs.
- (ii.) That  $f(x)$  and  $f''(x)$  should have the same sign.

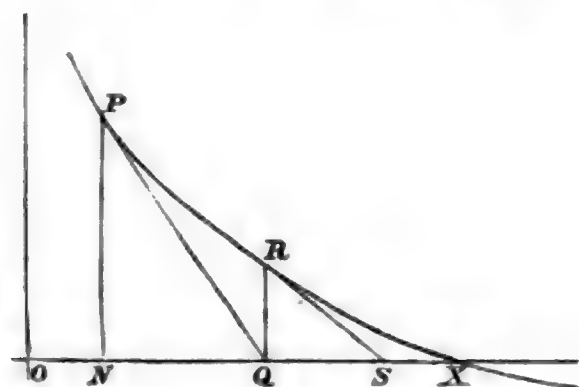


FIG. 1.

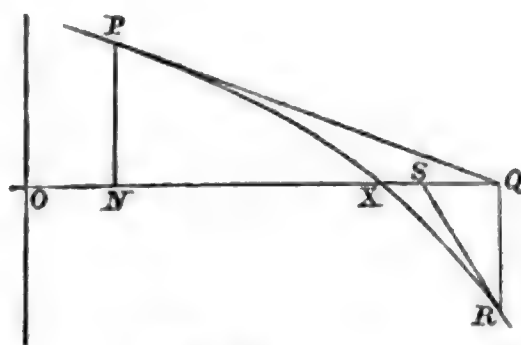


FIG. 2.

A proof may be obtained from the figure. Draw the curve  $y = f(x)$ . Let  $OX$  be a root of the equation, and  $ON = c_1$ ; draw the successive ordinates and tangents  $NP, PQ, QR$ , &c. Then  $OQ = c_2$ ,  $OS = c_3$ , and so on.

Fig. (2) represents  $c_2 > OX$ , and the subsequent approximations decreasing towards the root.

**530 Newton's Rule for Limits of the Roots.**—Let the coefficients of  $f(x)$  be respectively divided by the Binomial coefficients, and let  $a_0, a_1, a_2 \dots a_n$  be the quotients, so that

$$f(x) = a_0 x^n + na_1 x^{n-1} + \frac{n(n-1)}{1 \cdot 2} a_2 x^{n-2} + \dots + na_{n-1} x + a_n.$$

Let  $A_1, A_2, A_3 \dots A_n$  be formed by the law  $A_r = a_r^2 - a_{r-1}a_{r+1}$ . Write the first series of quantities over the second, in the following manner:—

$$\begin{array}{ccccccccc} a_0, & a_1, & a_2, & a_3 & \dots & a_{n-1}, & a_n, \\ A_0, & A_1, & A_2, & A_3 & \dots & A_{n-1}, & A_n. \end{array}$$

Whenever two adjacent terms in the first series have the same sign, and the two corresponding terms below them in the second series also the same sign; let this be called a *double permanence*. When two adjacent terms above have different signs, and the two below the same sign, let this be known as a *variation-permanence*.

**531** RULE.—*The number of double permanences in the associated series is a superior limit to the number of negative roots of  $f(x)$ .*

*The number of variation-permanences is a superior limit to the number of positive roots.*

*The number of imaginary roots cannot be less than the number of variations of sign in the second series.*

**532** *Sylvester's Theorem.*—Let  $f(x+\lambda)$  be expanded by (426) in powers of  $x$ , and let the two series be formed as in Newton's Rule (530).

Let  $P(\lambda)$  denote the number of double permanences.

Then  $P(\lambda) \sim P(\mu)$  is either equal to the number of roots of  $f(x)$ , or surpasses that number by an even integer.

NOTE.—The first series may be multiplied by  $\lfloor n$ , and will then stand thus,

$$f^n(\lambda), \quad f^{n-1}(\lambda), \quad \lfloor 2 f^{n-2}(\lambda), \quad \lfloor 3 f^{n-3}(\lambda) \dots \lfloor n f(\lambda).$$

The second series may be reduced to

$$G_n(\lambda), \quad G_{n-1}(\lambda), \quad G_{n-2}(\lambda) \dots G(\lambda),$$

where 
$$G_r(\lambda) \equiv \{f^r(\lambda)\}^2 - \frac{n-r+1}{n-r} f^{r-1}(\lambda) f^{r+1}(\lambda).$$

**533** *Horner's Method.*—To find the numerical values of the roots of an equation. Take, for example, the equation

$$x^4 - 4x^3 + x^2 + 6x + 2 = 0,$$

and find limits of the roots by Sturm's Method or otherwise.



It has been shewn in (517) that this equation has two incommensurable roots between 2 and 3. The process of calculating the least of these roots is here exhibited.

$\begin{array}{r} -4 \\ 2 \\ \hline -2 \\ 2 \\ \hline 0 \\ 2 \\ \hline 2 \\ 2 \\ \hline 40 \\ 4 \\ \hline 44 \\ 4 \\ \hline 48 \\ 4 \\ \hline 52 \\ 4 \\ \hline 560 \\ 1 \\ \hline 561 \\ 1 \\ \hline 562 \\ 1 \\ \hline 563 \\ 1 \\ \hline 5640 \\ 4 \\ \hline 5644 \\ 4 \\ \hline 5648 \\ 4 \\ \hline 5652 \\ 4 \\ \hline 5,656 \end{array}$	$\begin{array}{r} +1 \\ -4 \\ \hline -3 \\ 0 \\ \hline -3 \\ 4 \\ \hline C_1 \quad 100 \\ 176 \\ \hline 276 \\ 192 \\ \hline 468 \\ 208 \\ \hline C_2 \quad 67600 \\ 561 \\ \hline 68161 \\ 562 \\ \hline 68723 \\ 563 \\ \hline C_3 \quad 6928600 \\ 22576 \\ \hline 6951176 \\ 22592 \\ \hline 6973768 \\ 22608 \\ \hline C_4 \quad 6996376 \\ 11 \\ \hline 69974 \\ 11 \\ \hline 69985 \\ 11 \\ \hline C_5 \quad 69996 \\ C_6 \quad 7 \end{array}$	$\begin{array}{r} +6 \\ -6 \\ \hline 0 \\ -6 \\ \hline B_1 \quad -6000 \\ 1104 \\ \hline -4896 \\ 1872 \\ \hline -3024000 \\ 68161 \\ \hline -2955839 \\ 68723 \\ \hline B_2 \quad -2887116000 \\ 27804704 \\ \hline -2859311296 \\ 27895072 \\ \hline B_3 \quad -2831416224 \\ 139948 \\ \hline -283001674 \\ 139970 \\ \hline B_4 \quad -282861704 \\ 700 \\ \hline -28285470 \\ 700 \\ \hline B_5 \quad -28284770 \\ 21 \\ \hline -2828456 \\ 21 \\ \hline B_6 \quad -2828435 \end{array}$	$\begin{array}{r} +2 (2.414213 \\ 0 \\ \hline A_1 \quad 20000 \\ -19584 \\ \hline A_2 \quad 4160000 \\ -2955839 \\ \hline A_3 \quad 12041610000 \\ -11437245184 \\ \hline A_4 \quad 604364816 \\ -566003348 \\ \hline A_5 \quad 38361468 \\ -28285470 \\ \hline A_6 \quad 10075998 \\ -8485368 \\ \hline A_7 \quad 1590630 \end{array}$
			$\begin{array}{r} 282843 \overline{) 1590630} \quad (562372 \\ \underline{1414215} \\ 176415 \\ 28284 \overline{) 176415} \\ \underline{169706} \\ 6709 \\ 2828 \overline{) 6709} \\ \underline{5657} \\ 1052 \\ 282 \overline{) 1052} \\ \underline{848} \\ 204 \\ 28 \overline{) 204} \\ \underline{197} \\ 7 \\ 2 \overline{) 7} \\ \underline{5} \\ 2 \end{array}$

Root = 2.414213562372.

METHOD.—Diminish the roots by 2 in the manner of (427).

The resulting coefficients are indicated by  $A_1, B_1, C_1, D_1$ .

By Newton's rule (527),  $-\frac{f(c)}{f'(c)}$ ; that is,  $-\frac{A_1}{B_1}$  is an approximation to the remaining part of the root. This gives .3 for the next figure; .4 will be found to be the correct one. The highest figure must be taken which will not change the sign of  $A$ .

Diminish the roots by .4. This is accomplished most easily by affixing ciphers to  $A_1, B_1, C_1, D_1$ , in the manner shewn, and then employing 4 instead of .4.

Having obtained  $A_7$ , and observing that its sign is +, retrace the steps,

T

trying 5 instead of 4. This gives  $A_2$  with a minus sign, thereby proving the existence of a root between 2.4 and 2.5. The new coefficients are  $A_2, B_2, C_2, D_2$

$-\frac{A_2}{B_2}$  gives 1 for the next figure of the root.

Affix ciphers as before, and diminish the roots by 1, distinguishing the new coefficients as  $A_3, B_3, C_3, D_3$ .

Note that at every stage of the work  $A$  and  $B$  must preserve their signs unchanged. If a change of sign takes place it shews that too large a figure has been tried.

To abridge the calculation proceed thus:—After a certain number of figures of the root have been obtained (in this example four), instead of adding ciphers cut off one digit from  $B_4$ , two from  $C_4$ , and three from  $D_4$ . This amounts to the same thing as adding the ciphers, and then dividing each number by 10000.

Continue the work with the numbers so reduced, and cut off digits in like manner at each stage until the  $D$  and  $C$  columns have disappeared.

$A_7$  and  $B_7$  now alone remain, and six additional figures of the root are determined correctly by the division of  $A_7$  by  $B_7$ .

To find the other root which lies between 2 and 3, we proceed as follows:—After diminishing the roots by 2, try 6 for the next figure. This gives  $A_2$  negative; 7 does the same, but 8 makes  $A_2$  positive. That is to say,  $f(2.7)$  is negative, and  $f(2.8)$  positive. Therefore a root exists between 2.7 and 2.8, and its value may be approximated to, in the manner shewn.

Throughout this last calculation  $A$  will preserve the negative sign. Observe also that the trial number for the next figure of the root given at each stage of the process by the formula  $-\frac{f(c)}{f'(c)}$ , will in this case be always too great, as in the former case it was always too small.

## SYMMETRICAL FUNCTIONS OF THE ROOTS OF AN EQUATION.

NOTATION.—Let  $a, b, c \dots$  be the roots of the equation  $f(x) = 0$ .

Let  $s_m$  denote  $a^m + b^m + \dots$ , the sum of the  $m^{\text{th}}$  powers of the roots.

Let  $s_{m,p}$  denote  $a^m b^p + b^m a^p + a^m c^p + \dots$  through all the permutations of the roots, two at a time.

Similarly let  $s_{m,p,q}$  denote  $a^m b^p c^q + a^m b^q c^p + \dots$ , taking all the permutations of the roots three at a time; and so on.



### 534 SUMS OF THE POWERS OF THE ROOTS.

$$s_m + p_1 s_{m-1} + p_2 s_{m-2} + \dots + p_{m-1} s_1 + m p_m = 0,$$

where  $m$  is less than  $n$ , the degree of  $f(x)$ .

Obtained by expanding by division each term in the value of  $f'(x)$  given at (432), arranging the whole in powers of  $x$ , and equating coefficients in the result and in the value of  $f'(x)$ , found by differentiation as in (424).

535 If  $m$  be greater than  $n$ , the formula will be

$$s_m + p_1 s_{m-1} + p_2 s_{m-2} + \dots + p_n s_{m-n} = 0.$$

Obtained by multiplying  $f(x) = 0$  by  $x^{m-n}$ , substituting for  $x$  the roots  $a, b, c$ , &c. in succession, and adding the results.

By these formulæ  $s_1, s_2, s_3$ , &c. may be calculated successively.

536 To find the sum of the negative powers of the roots, put  $m$  equal to  $n-1, n-2, n-3$ , &c. successively in (535), in order to obtain  $s_{-1}, s_{-2}, s_{-3}$ , &c.

537 To calculate  $s_r$  independently.

RULE:  $s_r = -r \times \text{coefficient of } x^{-r} \text{ in the expansion of } \log \frac{f(x)}{x^n} \text{ in descending powers of } x.$

Proved by taking  $f(x) = (x-a)(x-b)(x-c) \dots$ , dividing by  $x^n$ , and expanding the logarithm of the right side of the equation by (156).

### 538 SYMMETRICAL FUNCTIONS WHICH ARE NOT POWERS OF THE ROOTS.

These are expressed in terms of the sums of powers of the roots as under, and thence, by (534), in terms of the roots explicitly,

$$s_{m,p} = s_m s_p - s_{m+p},$$

$$539 \quad s_{m,p,q} = s_m s_p s_q - s_{m+p} s_q - s_{m+q} s_p - s_{p+q} s_m + 2s_{m+p+q}.$$

The last equation may be proved by multiplying  $s_{m,p}$  by  $s_q$ ; and expansions of other symmetrical functions may be obtained in a similar way.

540 If  $\phi(x)$  be a rational integral function of  $x$ , then the symmetrical function of the roots of  $f(x)$ , denoted by

$\phi(a) + \phi(b) + \phi(c) + \&c.$ , is equal to the coefficient of  $x^{n-1}$  in the remainder obtained by dividing  $\phi(x) f''(x)$  by  $f(x)$ .

Proved by multiplying the equation (432) by  $\frac{\phi(x)}{f(x)}$ , and by theorem (401).

**541** To find the equation whose roots are the squares of the differences of the roots of a given equation.

Let  $F(x)$  be the given equation, and  $S_r$  the sum of the  $r^{\text{th}}$  powers of its roots. Let  $f(x)$  and  $s_r$  have the same meaning with regard to the required equation.

The coefficients of the required equation can be calculated from those of the given one as follows:—

*The coefficients of each equation may be connected with the sums of the powers of its roots by (534); and the sums of the powers of the roots of the two equations are connected by the formula*

$$\mathbf{542} \quad 2s_r = nS_{2r} - 2rS_1S_{2r-1} + \frac{2r(2r-1)}{1 \cdot 2} S_2S_{2r-2} - \dots + nS_{2r}.$$

**RULE.**— $2s_r$  is equal to the formal expansion of  $(S-S)^{2r}$  by the Binomial Theorem, with the first and last terms each multiplied by  $n$ , and the indices all changed to suffices. As the equi-distant terms are equal we can divide by 2, and take half the series.

**DEMONSTRATION.**—Let  $a, b, c \dots$  be the roots of  $F(x)$ .

Let  $\phi(x) = (x-a)^{2r} + (x-b)^{2r} + \dots \dots \dots$  (i.)

Expand each term on the right by the Bin. Theor., and add, substituting  $S_1, S_2, \&c.$  In the result change  $x$  into  $a, b, c \dots$  successively, and add the  $n$  equations to obtain the formula, observing that, by (i.),

$$\phi(a) + \phi(b) + \dots = 2s_r.$$

If  $n$  be the degree of  $F(x)$ , then  $\frac{1}{2}n(n-1)$  is the degree of  $f(x)$ . By (96).

**543** The last term of the equation  $f(x) = 0$  is equal to

$$n^n F(a) F(\beta) F(\gamma) \dots$$

where  $a, \beta, \gamma, \dots$  are the roots of  $F'(x)$ . Proved by shewing that

$$F'(a) F'(b) \dots = n^n F(a) F(\beta) \dots$$

**544** If  $F(x)$  has negative or imaginary roots,  $f(x)$  must have imaginary roots.

**545** The sum of the  $m^{\text{th}}$  powers of the roots of the quadratic equation  $x^2 - px + q = 0$ .

$$s_m = p^m - mp^{m-2}q + \frac{m(m-3)}{2} p^{m-4}q^2 - \dots$$

$$\dots + (-1)^r \frac{m(m-r-1) \dots (m-2r+1)}{r} p^{m-2r}q^r + \&c.$$

By (537) expanding the logarithm by (156).

**546** The sum of the  $m^{\text{th}}$  powers of the roots of  $x^n - 1 = 0$  is  $n$  if  $m$  be a multiple of  $n$ , and zero if it be not.

By (537); expanding the logarithm by (156).

**547** If  $\phi(x) = a_0 + a_1x + a_2x^2 + \&c. \dots\dots\dots (i.)$ , then the sum of the selected terms

$$a_mx^m + a_{m+n}x^{m+n} + a_{m+2n}x^{m+2n} + \&c.$$

will be  $s = \frac{1}{n} \{ \alpha^{n-m} \phi(\alpha x) + \beta^{n-m} \phi(\beta x) + \gamma^{n-m} \phi(\gamma x) + \&c. \}$

where  $\alpha, \beta, \gamma, \&c.$  are the  $n^{\text{th}}$  roots of unity.

For proof, multiply (i.) by  $\alpha^{n-m}$ , and change  $x$  into  $\alpha x$ ; so with  $\beta, \gamma, \&c.$ , and add the resulting equations.

**548** To approximate to the root of an equation by means of the sums of the powers of the roots.

By taking  $m$  large enough, the fraction  $\frac{s_{m+1}}{s_m}$  will approximate to the value of the numerically greatest root, unless there be a modulus of imaginary roots greater than any real root, in which case the fraction has no limiting value.

**549** Similarly the fraction  $\frac{s_ms_{m+2} - s_{m+1}^2}{s_{m-1}s_{m+1} - s_m^2}$  approximates, as  $m$  increases, to the *greatest product* of any pair of roots, real or imaginary; excepting in the case in which the product of the pair of imaginary roots, though less than the product of the two real roots, is greater than the square of the least of them, for then the fraction has no limiting value.

**550** Similarly the fraction  $\frac{s_m s_{m+3} - s_{m+1} s_{m+2}}{s_m s_{m+2} - s_{m+1}^2}$  approximates, as  $m$  increases, to the sum of the two numerically greatest roots, or to the sum of the two imaginary roots with the greatest modulus.

### EXPANSION OF AN IMPLICIT FUNCTION OF $x$ .

Let  $y^a(Ax^a+) + y^b(B_1x^b+) + \dots + y^s(Sx^s+) = 0 \dots (1)$

be an equation arranged in descending powers of  $y$ , the coefficients being functions of  $x$ , the highest powers only of  $x$  in each coefficient being written.

It is required to obtain  $y$  in a series of descending powers of  $x$ .

First form the fractions

$$-\frac{a-b}{a-\beta}, \quad -\frac{a-c}{a-\gamma}, \quad -\frac{a-d}{a-\delta} \dots \dots -\frac{a-s}{a-\sigma} \dots \dots (2).$$

Let  $-\frac{a-k}{a-n} = t$  be the greatest of these algebraically, or if several are equal and greater than the rest, let it be the last of such. Then, with the letters corresponding to these equal and greatest fractions, form the equation

$$Au^a + \dots + Ku^k = 0 \dots \dots (3).$$

Each value of  $u$  in this equation corresponds to a value of  $y$ , commencing with  $ux'$ .

Next select the greatest of the fractions

$$-\frac{k-l}{\kappa-\lambda}, \quad -\frac{k-m}{\kappa-\mu} \dots \dots -\frac{k-s}{\kappa-\sigma} \dots \dots (4).$$

Let  $-\frac{k-n}{\kappa-\nu} = t'$  be the last of the greatest ones. Form the corresponding equation  $Ku^k + \dots + Nu^v = 0 \dots \dots (5)$ . Then each value of  $u$  in this equation gives a corresponding value of  $y$ , commencing with  $ux''$ .

Proceed in this way until the last fraction of the series (2) is reached.

To obtain the second term in the expansion of  $y$ , put

$$y = x^t(u + y_1) \text{ in (1) ..... (6),}$$

employing the different values of  $u$ , and again of  $t'$  and  $u$ ,  $t''$  and  $u$ , &c. in succession; and in each case this substitution will produce an equation in  $y$  and  $x$  similar to the original equation in  $y$ .

Repeat the foregoing process with the new equation in  $y$ , observing the following additional rule:—

*When all the values of  $t$ ,  $t'$ ,  $t''$ , &c. have been obtained, the negative ones only must be employed in forming the equations in  $u$ .* (7).

**552** To obtain  $y$  in a series of *ascending* powers of  $x$ .

Arrange equation (1) so that  $\alpha$ ,  $\beta$ ,  $\gamma$ , &c. may be in *ascending* order of magnitude, and  $a$ ,  $b$ ,  $c$ , &c. the *lowest* powers of  $x$  in the respective coefficients.

Select  $t$ , the greatest of the fractions in (2), and proceed exactly as before, with the one exception of substituting the word *positive* for *negative* in (7).

**553** EXAMPLE.—Take the equation

$$(x^3 + x^4) + (3x^3 - 5x^3)y + (-4x + 7x^2 + x^3)y^2 - y^3 = 0.$$

It is required to expand  $y$  in ascending powers of  $x$ .

The fractions (2) are  $-\frac{3-2}{0-1}$ ,  $-\frac{3-1}{0-2}$ ,  $-\frac{3-0}{0-5}$ ; or 1, 1, and  $\frac{3}{5}$ .

The first two being equal and greatest, we have  $t = 1$ .

The fractions (4) reduce to  $-\frac{1-0}{2-5} = \frac{1}{3} = t'$ .

Equation (3) is  $1 + 3u - 4u^2 = 0$ ,

which gives  $u = 1$  and  $-\frac{1}{4}$ , with  $t = 1$ .

Equation (5) is  $-4u^3 - u^5 = 0$ ,

and from this  $u = 0$  and  $-4^{\frac{1}{4}}$ , with  $t' = \frac{1}{4}$ .

We have now to substitute for  $y$ , according to (6), either

$$x(1 + y_1), \quad x(-\frac{1}{4} + y_1), \quad x^{\frac{1}{4}}y, \quad \text{or} \quad x^{\frac{1}{4}}(-4^{\frac{1}{4}} + y_1).$$

Put  $y = x(1 + y_1)$ , the first of these values, in the original equation, and arrange in ascending powers of  $y$ , thus

$$-4x^4 + (-5x^3 + )y_1 + (-4x^3 + )y_1^2 - 10x^2y_1^3 - 5x^3y_1^4 - x^5y_1^5 = 0.$$

The lowest power only of  $x$  in each coefficient is here written.

The fractions (2) now become

$$-\frac{4-3}{0-1}, -\frac{4-3}{0-2}, -\frac{4-5}{0-3}, -\frac{4-5}{0-4}, -\frac{4-5}{0-5};$$

or

$$1, \frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{5}.$$

From these  $t=1$ , and equation (3) becomes

$$-4-5u=0; \therefore u=-\frac{4}{5}.$$

Hence one of the values of  $y_1$  is, as in (6),  $y_1 = x(-\frac{4}{5} + y_2)$ .

Therefore  $y = x\{1 + x(-\frac{4}{5} + y_2)\} = x - \frac{4}{5}x^2 + \dots$

Thus the first two terms of one of the expansions have been obtained.

## DETERMINANTS.

**554 Definitions.**—The determinant  $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$  is equivalent to  $a_1b_2 - a_2b_1$ , and is called a determinant of the second order. A determinant of the third order is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \equiv a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1).$$

Another notation is  $\Sigma \pm a_1b_2c_3$ , or simply  $(a_1b_2c_3)$ .

The letters are named *constituents*, and the terms are called *elements*. The determinant is composed of all the elements obtained by permutations of the suffixes 1, 2, 3.

The coefficients of the constituents are determinants of the next lower order, and are termed *minors* of the original determinant. Thus, the first determinant above is the minor of  $c_3$  in the second determinant. It is denoted by  $C_3$ . So the minor of  $a_1$  is denoted by  $A_1$ , and so on.

**555** A determinant of the  $n^{\text{th}}$  order may be written in either of the forms below

$$\begin{vmatrix} a_1 & a_2 & \dots & a_r & \dots & a_n \\ b_1 & b_2 & \dots & b_r & \dots & b_n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ l_1 & l_2 & \dots & l_r & \dots & l_n \end{vmatrix} \quad \text{OR} \quad \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1r} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2r} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nr} & \dots & a_{nn} \end{vmatrix}$$

In the latter, or double suffix notation, the first suffix indicates the row, and the second the column. The former notation will be adopted in these pages.



A *Composite determinant* is one in which the number of columns exceeds the number of rows,  $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$  and it is written as in the annexed example. Its value is the sum of all the determinants obtained by taking a number of rows in every possible way.

A *Simple determinant* has single terms for its constituents.

A *Compound determinant* has more than one term in some or all of its constituents. See (570) for an example.

For the definitions of *Symmetrical*, *Reciprocal*, *Partial*, and *Complementary* determinants; see (574), (575), and (576).

### General Theory.

**556** The number of constituents is  $n^2$ .

The number of elements in the complete determinant is  $\lfloor n$ .

**557** The first or leading element is  $a_1 b_2 c_3 \dots l_n$ . Any element may be derived from the first by permutation of the suffixes.

The sign of an element is + or - according as it has been obtained from the diagonal element by an even or odd number of permutations of the suffixes.

Hence the following rule for determining the sign of an element.

**RULE.**—Take the suffixes in order, and put them back to their places in the first element. Let  $m$  be the whole number of places passed over; then  $(-1)^m$  will give the sign required.

**Ex.**—To find the sign of the element  $a_4 b_3 c_5 d_1 e_2$  of the determinant  $(a_1 b_2 c_3 d_4 e_5)$ .

			$a_1$	$b_2$	$c_3$	$d_4$	$e_5$
Move the suffix 1, three places...	...		1	4	3	5	2
" " 2, three places...	...		1	2	4	3	5
" " 3, one place	...		1	2	3	4	5

In all, seven places; therefore  $(-1)^7 = -1$  gives the sign required.

**558** If two suffixes in any element be transposed, the sign of the element is changed.

Half of the elements are plus, and half are minus.

**559** The elements are not altered by changing the rows into columns.

If two rows or columns are transposed, the sign of the



determinant is changed. Because each element changes its sign.

If two rows or columns are identical, the determinant vanishes.

**560** If all the constituents but one in a row or column vanish, the determinant becomes the product of that constituent and a determinant of the next lower order.

**561** A cyclical interchange is effected by  $n-1$  successive transpositions of adjacent rows or columns, until the top row has been brought to the bottom, or the left column to the right side. Hence

A cyclical interchange changes the sign of a determinant of an even order only.

The  $r^{\text{th}}$  row may be brought to the top by  $r-1$  cyclical interchanges.

**562** If each constituent in a row or column be multiplied by the same factor, the determinant becomes multiplied by it.

If each constituent of a row or column is the sum of  $m$  terms, the compound determinant becomes the sum of  $m$  simple determinants of the same order.

Also, if every constituent of the determinant consists of  $m$  terms, the compound determinant is resolvable into the sum of  $m^2$  simple determinants.

**563** To express the minor of the  $r^{\text{th}}$  row and  $k^{\text{th}}$  column as a determinant of the  $n-1^{\text{th}}$  order.

Put all the constituents in the  $r^{\text{th}}$  row and  $k^{\text{th}}$  column equal to 0, and then make  $r-1$  cyclical interchanges in the rows and  $k-1$  in the columns, and multiply by  $(-1)^{(r+k)(n-1)}$ .

$$[\therefore = (-1)^{(r-1+k-1)(n-1)}.$$

**564** To express a determinant as a determinant of a higher order.

Continue the diagonal with constituents of "ones," and fill up with zeros on one side, and with any quantities whatever ( $a, \beta, \gamma$ , &c.) on the other.

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 \\ \beta & \epsilon & a & h & g \\ \gamma & \zeta & h & b & f \\ \delta & \eta & g & f & c \end{vmatrix}$$

**565** The sum of the products of each constituent of a column by the corresponding minor in another given column is zero. And the same is true if we read 'row' instead of 'column.' Thus, referring to the determinant in (555),

$$\text{Taking the } p^{\text{th}} \text{ and } q^{\text{th}} \text{ columns,} \quad \text{Taking the } a \text{ and } c \text{ rows,}$$

$$a_p A_q + b_p B_q + \dots + l_p L_q = 0. \quad a_1 C_1 + a_2 C_2 + \dots + a_n C_n = 0.$$

For in each case we have a determinant with two columns identical.

**566** In any row or column the sum of the products of each constituent by its minor is the determinant itself. That is,

$$\text{Taking the } p^{\text{th}} \text{ column,} \quad \text{Or taking the } c \text{ row,}$$

$$a_p A_p + b_p B_p + \dots + l_p L_p = \Delta. \quad c_1 C_1 + c_2 C_2 + \dots + c_n C_n = \Delta.$$

**567** The last equation may be expressed by  $\Sigma c_p C_p = \Delta$ .

Also, if  $(a_p c_q)$  express the determinant  $\begin{vmatrix} a_p & a_q \\ c_p & c_q \end{vmatrix}$ ; then

$\Sigma(a_p c_q)$  will represent the sum of all the determinants of the second order which can be formed by taking any two columns out of the  $a$  and  $c$  rows. The minor of  $(a_p, c_q)$  may be written  $(A_p, C_q)$ , and signifies the determinant obtained by suppressing the two rows and two columns of  $a_p$  and  $c_q$ . Thus  $\Delta = \Sigma(a_p, c_q)(A_p, C_q)$ . And a similar notation when three or more rows and columns are selected.

## 568 Analysis of a determinant.

**RULE.**—To resolve into its elements a determinant of the  $n^{\text{th}}$  order. Express it as the sum of  $n$  determinants of the  $(n-1)^{\text{th}}$  order by (560), and repeat the process with each of the new determinants.

**EXAMPLE:**

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

$$\text{Again,} \quad \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = b_1 \begin{vmatrix} c_2 & c_3 \\ d_2 & d_3 \end{vmatrix} + b_2 \begin{vmatrix} c_3 & c_1 \\ d_3 & d_1 \end{vmatrix} + b_3 \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix}$$

and so on. In the first series the determinants have alternately plus and minus signs, by the rule for cyclical interchanges (561), the order being even.

## 569

*Synthesis of a determinant.*

The process is facilitated by making use of two evident rules. Those constituents which belong to the row and column of a given constituent  $a$ , will be designated " $a$ 's constituents." Also, two pairs of constituents such as  $a_p, c_q$  and  $a_q, c_p$ , forming the corners of a rectangle, will be said to be "conjugate" to each other.

**RULE I.**—No constituent will be found in the same term with one of its own constituents.

**RULE II.**—The conjugates of any two constituents  $a$  and  $b$  will be common to  $a$ 's and  $b$ 's constituents.

**Ex.**—To write the following terms in the form of a determinant :

$$abcd + bfgl + f^2h^2 + ledf + cghp + lahr + elpr \\ - fhpr - ablr - ach^2 - lfhg - bdf^2 - efhl - cedp.$$

The determinant will be of the fourth order; and since every term must contain four constituents, the constituent 1 is supplied to make up the number in some of the terms. Select any term, as  $abcd$ , for the leading diagonal.

Now apply Rule I.,

$a$  is not found with  $e, f, f, g, p, 0 \dots (1)$ .  $c$  is not found with  $f, f, l, r, 1, 0 \dots (3)$ .  
 $b$  is not found with  $e, h, h, p, 1, 0 \dots (2)$ .  $d$  is not found with  $g, h, h, l, r, 0 \dots (4)$ .

Each constituent has  $2(n-1)$ , that is, 6 constituents belonging to it, since  $n=4$ . Assuming, therefore, that the above letters are the constituents of  $a, b, c$ , and  $d$ , and that there are no more, we supply a sixth zero constituent in each case.

Now apply Rule II.—The constituents common

to  $a$  and  $b$  are  $e, p$ ; to  $a$  and  $c$ — $f, f$ ; to  $b$  and  $c$ — $1, 0$ ;  
to  $a$  and  $d$ — $g, 0$ ; to  $b$  and  $d$ — $h, h, 0$ ; to  $c$  and  $d$ — $l, r, 0$ .

The determinant may now be formed. The diagonal being  $abcd$ ; place  $e, p$ , the conjugates of  $a$  and  $b$ , either as in the diagram or transposed.

$$\begin{vmatrix} a & e & f & g \\ p & b & 1 & h \\ f & 0 & c & r \\ 0 & h & l & d \end{vmatrix}$$

Then  $f$  and  $f$ , the conjugates of  $a$  and  $c$ , may be written.

$1$  and  $0$ , the conjugates of  $b$  and  $c$ , must be placed as indicated, because  $1$  is one of  $p$ 's constituents, since it is not found in any term with  $p$ , and must therefore be in the second row.

Similarly the places of  $g$  and  $0$ , and of  $l$  and  $r$ , are assigned.

In the case of  $b$  and  $d$  we have  $h, h, 0$  from which to choose the two conjugates, but we see that  $0$  is not one of them because that would assign two zero constituents to  $b$ , whereas  $b$  has but one, which is already placed.

By similar reasoning the ambiguity in selecting the conjugates  $l, r$  is removed.

The foregoing method is rigid in the case of a complete determinant

having different constituents. It becomes uncertain when the zero constituents increase in number, and when several constituents are identical. But even then, in the majority of cases, it will soon afford a clue to the required arrangement.

**570**    *PRODUCT OF TWO DETERMINANTS OF  
THE  $n^{\text{th}}$  ORDER.*

$$\begin{array}{c} (P) \\ \left| \begin{array}{cccc} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \\ \dots & \dots & \dots & \dots \\ l_1 & l_2 & \dots & l_n \end{array} \right| \end{array} \quad \begin{array}{c} (Q) \\ \left| \begin{array}{cccc} \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \beta_1 & \beta_2 & \dots & \beta_n \\ \dots & \dots & \dots & \dots \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \end{array} \right| \end{array} = \begin{array}{c} (S) \\ \left| \begin{array}{cccc} A_1 & A_2 & \dots & A_n \\ B_1 & B_2 & \dots & B_n \\ \dots & \dots & \dots & \dots \\ L_1 & L_2 & \dots & L_n \end{array} \right| \end{array}$$

The values of  $A_1, B_1 \dots L_1$  in the first column of  $S$  are annexed. For the second column write  $b$ 's in the place of  $a$ 's. For the third column write  $c$ 's, and so on.

For proof substitute the values of  $A_1, B_1, \&c.$  in the determinant  $S$ , and then resolve  $S$  into the sum of a number of determinants by (562), and note the determinants which vanish through having identical columns.

**RULE.**—To form the determinant  $S$ , which is the product of two determinants  $P$  and  $Q$ . First connect by plus signs the constituents in the rows of both the determinants  $P$  and  $Q$ .

Now place the first row of  $P$  upon each row of  $Q$  in turn, and let each two constituents as they touch become products. This is the first column of  $S$ .

Perform the same operation upon  $Q$  with the second row of  $P$  to obtain the second column of  $S$ ; and again with the third row of  $P$  to obtain the third column of  $S$ , and so on.

**571** If the number of columns, both in  $P$  and  $Q$ , be  $n$ , and the number of rows  $r$ , and if  $n$  be  $> r$ , then the determinant  $S$ , found in the same way from  $P$  and  $Q$ , is equal to the sum of the  $C(n, r)$  products of pairs of determinants obtained by taking any  $r$  columns out of  $P$ , and the corresponding  $r$  columns out of  $Q$ .

But if  $n$  be  $< r$  the determinant  $S$  vanishes.

For in that case, in every one of the component determinants, there will be two columns identical.

**572** The product of the determinants  $P$  and  $Q$  may be formed in four ways by changing the rows into columns in either or both  $P$  and  $Q$ .

**573** Let the following system of  $n$  equations in  $x_1 x_2 \dots x_n$  be transformed by substituting the accompanying values of the variables,

$$\begin{array}{ll} a_1 x_1 + a_2 x_2 + \dots + a_n x_n = 0, & x_1 = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \dots + \alpha_n \xi_n, \\ b_1 x_1 + b_2 x_2 + \dots + b_n x_n = 0, & x_2 = \beta_1 \xi_1 + \beta_2 \xi_2 + \dots + \beta_n \xi_n, \\ \dots\dots\dots & \dots\dots\dots \\ l_1 x_1 + l_2 x_2 + \dots + l_n x_n = 0, & x_n = \lambda_1 \xi_1 + \lambda_2 \xi_2 + \dots + \lambda_n \xi_n. \end{array}$$

The eliminant of the resulting equations in  $\xi_1 \xi_2 \dots \xi_n$  is the determinant  $S$  in (570), and is therefore equal to the product of the determinants  $P$  and  $Q$ . The determinant  $Q$  is then termed the modulus of transformation.

**574** A *Symmetrical determinant* is symmetrical about the leading diagonal. If the  $R$ 's form the  $r^{\text{th}}$  row, and the  $K$ 's the  $k^{\text{th}}$  row; then  $R_k = K_r$  throughout a symmetrical determinant.

The square of a determinant is a symmetrical determinant.

**575** A *Reciprocal determinant* has for its constituents the first minors of the original determinant, and is equal to its  $n-1^{\text{th}}$  power; that is,

$$\begin{vmatrix} A_1 & \dots & A_n \\ \dots\dots\dots & & \\ \dots\dots\dots & & \\ L_1 & \dots & L_n \end{vmatrix} = \begin{vmatrix} a_1 & \dots & a_n \\ \dots\dots\dots & & \\ \dots\dots\dots & & \\ l_1 & \dots & l_n \end{vmatrix}^{n-1}$$

PROOF.—Multiply both sides of the equation by the original determinant (555). The constituents on the left side all vanish excepting the diagonal of  $\Delta$ 's.

**576** *Partial and Complementary determinants.*

If  $r$  rows and the same number of columns be selected from a determinant, and if the rows be brought to the top, and the columns to the left side, without changing their order, then the elements common to the selected rows and columns form a Partial determinant of the order  $r$ , and the elements *not* found in any of those rows and columns form the Complementary determinant, its order being  $n-r$ .





equal. Therefore it is divisible by each of the factors on the left; therefore by their product. And the quotient is seen to be unity, for both sides of the equation are of the same degree; viz.,  $\frac{1}{2}n(n-1)$ .

**579** The product of the squares of the differences of the same  $n$  quantities  $\left. \vphantom{\begin{matrix} s_0 & s_1 & \dots & s_{n-1} \\ s_1 & s_2 & \dots & s_n \\ \dots & \dots & \dots & \dots \\ s_{n-1} & s_n & \dots & s_{2n-2} \end{matrix}} \right\} = \begin{vmatrix} s_0 & s_1 & \dots & s_{n-1} \\ s_1 & s_2 & \dots & s_n \\ \dots & \dots & \dots & \dots \\ s_{n-1} & s_n & \dots & s_{2n-2} \end{vmatrix}$

PROOF.—Square the determinant in (578), and write  $s_r$  for the sum of the  $r^{\text{th}}$  powers of the roots.

**580** With the same meaning for  $s_1, s_2, \dots$ , the same determinant taken of an order  $r$ , less than  $n$ , is equal to the sum of the products of the squares of the differences of  $r$  of the  $n$  quantities taken in every possible way; that is, in  $C(n, r)$  ways.

$$\begin{aligned} \text{Ex.:} \quad \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix} &= (a_1 - a_2)^2 + (a_1 - a_3)^2 + \&c. \equiv \Sigma (a_1 - a_2)^2, \\ \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} &= \Sigma (a_1 - a_2)^2 (a_1 - a_3)^2 (a_1 - a_4)^2. \end{aligned}$$

The next determinant in order

$$= \Sigma (a_1 - a_2)^2 (a_1 - a_3)^2 (a_1 - a_4)^2 (a_2 - a_3)^2 (a_2 - a_4)^2 (a_3 - a_4)^2.$$

And so on until the equation (579) is reached.

Proved by substituting the values of  $s_1, s_2, \dots$  &c., and resolving the determinant into its partial determinants by (571).

**581** The quotient of

$$\frac{a_0 x^m + a_1 x^{m-1} + \dots + a_r x^{m-r} + \dots}{b_0 x^n + b_1 x^{n-1} + \dots + b_r x^{n-r} + \dots}$$

is given by the formula

$$q_0 x^{m-n} + q_1 x^{m-n-1} + \dots + q_r x^{m-n-r} + \dots,$$

where

$$q_r = \frac{1}{b_0^{r+1}} \begin{vmatrix} b_0 & 0 & 0 & \dots & a_0 \\ b_1 & b_0 & 0 & \dots & a_1 \\ b_2 & b_1 & b_0 & \dots & a_2 \\ \dots & \dots & \dots & \dots & \dots \\ b_r & b_{r-1} & b_{r-2} & \dots & b_1 a_r \end{vmatrix}$$

Proved by Induction.



ELIMINATION.

**582**     *Solution of  $n$  linear equations in  $n$  variables.*

The equations and the values of the variables are arranged below :

$$\begin{array}{ll} a_1 x_1 + a_2 x_2 + \dots + a_n x_n = \xi_1, & x_1 \Delta = A_1 \xi_1 + B_1 \xi_2 + \dots + L_1 \xi_n \\ b_1 x_1 + b_2 x_2 + \dots + b_n x_n = \xi_2, & x_2 \Delta = A_2 \xi_1 + B_2 \xi_2 + \dots + L_2 \xi_n \\ \dots\dots\dots & \dots\dots\dots \\ l_1 x_1 + l_2 x_2 + \dots + l_n x_n = \xi_n, & x_n \Delta = A_n \xi_1 + B_n \xi_2 + \dots + L_n \xi_n \end{array}$$

where  $\Delta$  is the determinant annexed, and  $A_1, B_1, \dots, L_1$  &c. are its first minors.

$$\left| \begin{array}{cccc} a_1 & \dots & a_n \\ \dots & \dots & \dots \\ l_1 & \dots & l_n \end{array} \right|$$

To find the value of one of the unknowns  $x_r$ .

**RULE.**—Multiply the equations respectively by the minors of the  $r^{\text{th}}$  column, and add the results.  $x_r$  will be equal to the fraction whose numerator is the determinant  $\Delta$ , with its  $r^{\text{th}}$  column replaced by  $\xi_1, \xi_2 \dots \xi_n$ , and whose denominator is  $\Delta$  itself.

**583** If  $\xi_1, \xi_2 \dots \xi_n$  and  $\Delta$  all vanish, then  $x_1, x_2 \dots x_n$  are in the ratios of the minors of any row of the determinant  $\Delta$ . For example, in the ratios  $C_1 : C_2 : C_3 : \dots : C_n$ .

The eliminant of the given equations is now  $\Delta = 0$ .

**584**     *Orthogonal Transformation.*

If the two sets of variables in the  $n$  equations (582) be connected by the relation

$$x_1 + x_2^2 + \dots + x_n^2 = \xi_1^2 + \xi_2^2 + \dots + \xi_n^2 \dots\dots (1),$$

then the changing from one set of variables to the other, by substituting the values of the  $\xi$ 's in terms of the  $x$ 's in any function of the former, or *vice versâ*, is called orthogonal transformation.

When equation (1) is satisfied, two results follow.

I. The determinant  $\Delta = \pm 1$ .

X

II. Each of the constituents of  $\Delta$  is equal to the corresponding minor, or else to minus that minor according as  $\Delta$  is positive or negative.

PROOF.—Substitute the values of  $\xi_1, \xi_2 \dots \xi_n$  in terms of  $x_1, x_2 \dots x_n$  in equation (1), and equate coefficients of the squares and products of the new variables. We get the  $n^2$  equations

$$\left. \begin{array}{l} a_1^2 + b_1^2 + = 1 \\ a_1 a_2 + b_1 b_2 + = 0 \\ a_1 a_3 + b_1 b_3 + = 0 \\ \dots\dots\dots \\ a_1 a_n + b_1 b_n + = 0 \end{array} \right\} \left. \begin{array}{l} a_2 a_1 + b_2 b_1 + = 0 \\ a_2^2 + b_2^2 + = 1 \\ a_2 a_3 + b_2 b_3 + = 0 \\ \dots\dots\dots \\ a_2 a_n + b_2 b_n + = 0 \end{array} \right\} \left. \begin{array}{l} a_3 a_1 + b_3 b_1 + = 0 \\ a_3 a_2 + b_3 b_2 + = 0 \\ a_3^2 + b_3^2 + = 1 \\ \dots\dots\dots \\ a_3 a_n + b_3 b_n + = 0 \end{array} \right\}$$

Also  $\Delta = \begin{vmatrix} a_1 b_1 \dots l_1 \\ a_2 b_2 \dots l_2 \\ a_3 b_3 \dots l_3 \\ \dots\dots\dots \\ a_n b_n \dots l_n \end{vmatrix}$

Form the square of the determinant  $\Delta$  by the rule (570), and these equations show that the product is a determinant in which the only constituents that do not vanish constitute a diagonal of 'ones.' Therefore

$$\Delta^2 = 1 \text{ and } \Delta = \pm 1.$$

Again, solving the first set of equations for  $a_1$  (writing  $a_1^2$  as  $a_1 a_1$ , &c.), the second set for  $a_2$ , the third for  $a_3$ , and so on, we have, by (582), the results annexed; which proves the second proposition.

$$\left\{ \begin{array}{l} a_1 \Delta = A_1 + A_1 0 + A_1 0 + = A_1 \\ a_2 \Delta = A_1 0 + A_2 + A_2 0 + = A_2 \\ a_3 \Delta = A_1 0 + A_2 0 + A_3 + = A_3 \\ \quad \quad \quad \text{\&c.} \quad \quad \quad \text{\&c.} \end{array} \right.$$

**585 Theorem.**—The  $n-2^{\text{th}}$  power of a determinant of the  $n^{\text{th}}$  order multiplied by any constituent is equal to the corresponding minor of the reciprocal determinant.

PROOF.—Let  $\rho$  be the reciprocal determinant of  $\Delta$ , and  $\beta_r$  the minor of  $B_r$  in  $\rho$ . Write the transformed equations (582) for the  $x$ 's in terms of the  $\xi$ 's, and solve them for  $\xi_r$ . Then equate the coefficient of  $x_r$  in the result with its coefficient in the original value of  $\xi_r$ .

$$\text{Thus } \rho \xi_r = \Delta (\beta_1 x_1 + \dots + \beta_r x_r + \dots), \text{ and } \xi_r = b_1 x_1 + \dots + b_r x_r + \dots;$$

$$\therefore \Delta \beta_r = \rho b_r = \Delta^{n-1} b_r \text{ by (575); } \therefore \beta_r = \Delta^{n-1} b_r.$$

**586** To eliminate  $x$  from the two equations

$$ax^m + bx^{m-1} + cx^{m-2} + \dots = 0 \dots\dots\dots (1),$$

$$a'x^n + b'x^{n-1} + c'x^{n-2} + \dots = 0 \dots\dots\dots (2).$$

If it is desired that the equation should be homogeneous in  $x$  and  $y$ ; put  $\frac{x}{y}$  instead of  $x$ , and clear of fractions. The following methods will still be applicable.

**I. Bezout's Method.**—Suppose  $m > n$ .

**RULE.**—Bring the equations to the same degree by multiplying (2) by  $x^{m-n}$ . Then multiply (1) by  $a'$ , and (2) by  $a$ , and subtract.

Again, multiply (1) by  $a'x + b'$ , and (2) by  $(ax + b)$ , and subtract.

Again, multiply (1) by  $a'x^2 + b'x + c'$ , and (2) by  $(ax^2 + bx + c)$ , and subtract, and so on until  $n$  equations have been obtained. Each will be of the degree  $m - 1$ .

Write under these the  $m - n$  equations obtained by multiplying (2) successively by  $x$ . The eliminant of the  $m$  equations is the result required.

Ex.—Let the equations be 
$$\begin{cases} ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0, \\ a'x^5 + b'x^4 + c'x^3 + d'x^2 + e'x + f' = 0. \end{cases}$$

The five equations obtained by the method, and their eliminant, by (583), are, writing capital letters for the functions of  $a, b, c, d, e, f$ ,

$$\left. \begin{aligned} A_1x^4 + B_1x^3 + C_1x^2 + D_1x + E_1 &= 0 \\ A_2x^4 + B_2x^3 + C_2x^2 + D_2x + E_2 &= 0 \\ A_3x^4 + B_3x^3 + C_3x^2 + D_3x + E_3 &= 0 \\ a'x^4 + b'x^3 + c'x^2 + d'x &= 0 \\ a'x^3 + b'x^2 + c'x + d' &= 0 \end{aligned} \right\} \text{ and } \begin{vmatrix} A_1 & B_1 & C_1 & D_1 & E_1 \\ A_2 & B_2 & C_2 & D_2 & E_2 \\ A_3 & B_3 & C_3 & D_3 & E_3 \\ a' & b' & c' & d' & 0 \\ 0 & a' & b' & c' & d' \end{vmatrix} = 0.$$

Should the equations be of the same degree, the eliminant will be a symmetrical determinant.

587

**II. Sylvester's Dialytic Method.**

**RULE.**—Multiply equation (1) successively by  $x, n - 1$  times; and equation (2)  $m - 1$  times; and eliminate  $x$  from the  $m + n$  resulting equations.

Ex.—To eliminate  $x$  from 
$$\begin{vmatrix} ax^3 + bx^2 + cx + d = 0 \\ px + qx + r = 0 \end{vmatrix}.$$

The  $m + n$  equations and their eliminant are

$$\left. \begin{aligned} px^3 + qx^2 + rx &= 0 \\ px^2 + qx + r &= 0 \\ px + q &= 0 \\ ax^3 + bx^2 + cx + d &= 0 \\ ax^2 + bx + c &= 0 \\ ax + b &= 0 \end{aligned} \right\} \text{ and } \begin{vmatrix} 0 & 0 & p & q & r \\ 0 & p & q & r & 0 \\ p & q & r & 0 & 0 \\ 0 & a & b & c & d \\ a & b & c & d & 0 \end{vmatrix} = 0.$$

**588** III. *Method of elimination by Symmetrical Functions.*

Divide the two equations in (586) respectively by the coefficients of their first terms, thus reducing them to the forms

$$f(x) \equiv x^m + p_1 x^{m-1} + \dots + p_m = 0,$$

$$\phi(x) \equiv x^n + q_1 x^{n-1} + \dots + q_n = 0.$$

**RULE.**—Let  $a, b, c \dots$  represent the roots of  $f(x)$ . Form the equation  $\phi(a) \phi(b) \phi(c) \dots = 0$ . This will contain symmetrical functions only of the roots  $a, b, c \dots$ .

Express these functions in terms of  $p_1, p_2 \dots$  by (538), &c., and the equation becomes the eliminant.

**Reason of the rule.**—The eliminant is the condition for a common root of the two equations. That root must make one of the factors  $\phi(a), \phi(b) \dots$  vanish, and therefore it makes their product vanish.

**589** The eliminant expressed in terms of the roots  $a, b, c \dots$  of  $f(x)$ , and the roots  $\alpha, \beta, \gamma \dots$  of  $\phi(x)$ , will be

$$(a - \alpha)(a - \beta)(a - \gamma) \dots (b - \alpha)(b - \beta)(b - \gamma) \dots \&c.,$$

being the product of all possible differences between a root of one equation and a root of another.

**590** The eliminant is a homogeneous function of the coefficients of either equation, being of the  $n^{\text{th}}$  degree in the coefficients of  $f(x)$ , and of the  $m^{\text{th}}$  degree in the coefficients of  $\phi(x)$ .

**591** The sum of the suffixes of  $p$  and  $q$  in each term of the eliminant  $= mn$ . Also, if  $p, q$  contain  $z$ ; if  $p_2, q_2$  contain  $z^2$ ; if  $p_3, q_3$  contain  $z^3$ , and so on, the eliminant will contain  $z^{mn}$ .

Proved by the fact that  $p_r$  is a homogeneous function of  $r$  dimensions of the roots  $a, b, c \dots$ , by (406).

**592** If the two equations involve  $x$  and  $y$ , the elimination may be conducted with respect to  $x$ ; and  $y$  will be contained in the coefficients  $p_1, p_2 \dots, q_1, q_2 \dots$ .

**593** *Elimination by the Method of Highest Common Factor.*

Let two algebraical equations in  $x$  and  $y$  be represented by  $A = 0$  and  $B = 0$ .

It is required to eliminate  $x$ .

Arrange  $A$  and  $B$  according to descending powers of  $x$ , and, having rejected any factor which is a function of  $y$  only, proceed to find the Highest Common Factor of  $A$  and  $B$ .

The process may be exhibited as follows :

$$\left. \begin{aligned} c_1 A &= q_1 B + r_1 R_1 \\ c_2 B &= q_2 R_1 + r_2 R_2 \\ c_3 R_1 &= q_3 R_2 + r_3 R_3 \\ c_4 R_2 &= q_4 R_3 + r_4 \end{aligned} \right\} \begin{aligned} &c_1, c_2, c_3, c_4 \text{ are the multipliers re-} \\ &\text{quired at each stage in order to avoid} \\ &\text{fractional quotients; and these must} \\ &\text{be constants or functions of } y \text{ only.} \\ &q_1, q_2, q_3, q_4 \text{ are the successive quo-} \\ &\text{tients.} \end{aligned}$$

$r_1 R_1, r_2 R_2, r_3 R_3, r_4$  are the successive remainders;  $r_1, r_2, r_3, r_4$  being functions of  $y$  only.

The process terminates as soon as a remainder is obtained which is a function of  $y$  only;  $r_4$  is here supposed to be such a remainder.

Now, the simplest factors having been taken for  $c_1, c_2, c_3, c_4$ , we see that

$$\left. \begin{aligned} 1 \text{ is the H. C. F. of } &c_1 \text{ and } r_1 \\ d_2 \text{ ,, ,, } &c_1 \text{ and } r_2 \\ d_3 \text{ ,, ,, } &\frac{c_1 c_2}{d_2} \text{ and } r_3 \\ d_4 \text{ ,, ,, } &\frac{c_1 c_2 c_3}{d_2 d_3} \text{ and } r_4 \end{aligned} \right\} \begin{aligned} &\text{The values of } x \text{ and } y, \\ &\text{which satisfy simulta-} \\ &\text{neously the equations} \\ &A=0 \text{ and } B=0, \text{ are those} \\ &\text{obtained by the four pairs} \\ &\text{of simultaneous equations} \\ &\text{following:} \end{aligned}$$

$$\left. \begin{aligned} r_1 &= 0 \text{ and } B = 0 \dots\dots\dots (1) \\ \frac{r_2}{d_2} &= 0 \text{ and } R_1 = 0 \dots\dots\dots (2) \\ \frac{r_3}{d_3} &= 0 \text{ and } R_2 = 0 \dots\dots\dots (3) \\ \frac{r_4}{d_4} &= 0 \text{ and } R_3 = 0 \dots\dots\dots (4) \end{aligned} \right\} \begin{aligned} &\text{The final equation in } y, \\ &\text{which gives all admissible} \\ &\text{values, is} \\ &\frac{r_1 r_2 r_3 r_4}{d_2 d_3 d_4} = 0, \end{aligned}$$

If it should happen that the remainder  $r_4$  is zero, the simultaneous equations (1), (2), (3), and (4) reduce to

$$r_1 = 0 \text{ and } \frac{B}{R_3} = 0; \quad \frac{r_2}{d_2} = 0 \text{ and } \frac{R_1}{R_3} = 0; \quad \frac{r_3}{d_3} = 0 \text{ and } \frac{R_2}{R_3} = 0.$$

**594** To find infinite values of  $x$  or  $y$  which satisfy the given equations.

Put  $x = \frac{1}{z}$ . Clear of fractions, and make  $z = 0$ .

If the two resulting equations in  $y$  have any common roots, such roots, together with  $x = \infty$ , satisfy simultaneously the equations proposed.

Similarly we may put  $y = \frac{1}{z}$ .

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# PLANE TRIGONOMETRY.

## ANGULAR MEASUREMENT.

**600** The unit of Circular measure is a Radian, and is the angle at the centre of a circle which subtends an arc equal to the radius. Hence

**601** Circular measure of an angle  $= \frac{\text{arc}}{\text{radius}}$ .

**602** Circular measure of two right angles  $= 3.14159 \dots \equiv \pi$ .

**603** The unit of Centesimal measure is a Grade, and is the one-hundredth part of a right angle.

**604** The unit of Sexagesimal measure is a Degree, and is the one-ninetieth part of a right angle.

To change degrees into grades, or circular measure, or *vice versâ*, employ one of the three equations included in

**605** 
$$\frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi},$$

where  $D$ ,  $G$ , and  $C$  are respectively the numbers of degrees, grades, and radians in the angle considered.

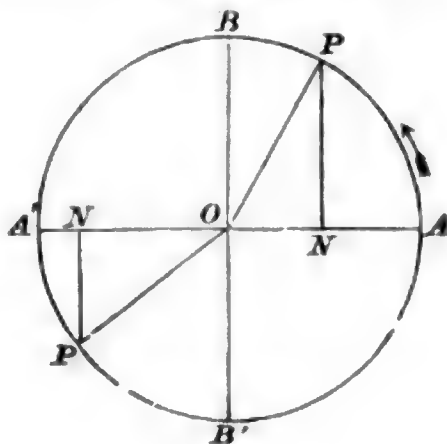
## TRIGONOMETRICAL RATIOS.

**606** Let  $OA$  be fixed, and let the revolving line  $OP$  describe a circle round  $O$ . Draw  $PN$  always perpendicular to  $AA'$ . Then, in all positions of  $OP$ ,

$\frac{PN}{OP}$  = the sine of the angle  $AOP$ ,

$\frac{ON}{OP}$  = the cosine of the angle  $AOP$ ,

$\frac{PN}{ON}$  = the tangent of the angle  $AOP$ .



- 607** If  $P$  be *above* the line  $AA'$ ,  $\sin AOP$  is positive.  
 If  $P$  be *below* the line  $AA'$ ,  $\sin AOP$  is negative.
- 608** If  $P$  lies to the *right* of  $BB'$ ,  $\cos AOP$  is positive.  
 If  $P$  lies to the *left* of  $BB'$ ,  $\cos AOP$  is negative.

**609** Note, that by the angle  $AOP$  is meant the angle through which  $OP$  has revolved from  $OA$ , its initial position; and this angle of revolution may have any magnitude. If the revolution takes place in the opposite direction, the angle described is reckoned negative.

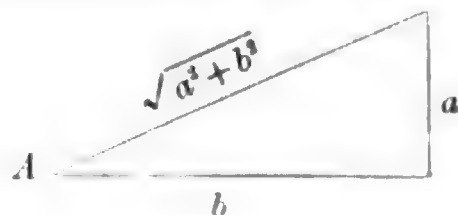
- 610** The secant of an angle is the reciprocal of its cosine,  
 or  $\cos A \sec A = 1$ .
- 611** The cosecant of an angle is the reciprocal of its sine,  
 or  $\sin A \operatorname{cosec} A = 1$ .
- 612** The cotangent of an angle is the reciprocal of its tangent,  
 or  $\tan A \cot A = 1$ .

*Relations between the trigonometrical functions of the same angle.*

- 613**  $\sin^2 A + \cos^2 A = 1$ . [I. 47]
- 614**  $\sec^2 A = 1 + \tan^2 A$ .
- 615**  $\operatorname{cosec}^2 A = 1 + \cot^2 A$ .
- 616**  $\tan A = \frac{\sin A}{\cos A}$ . [606]

If  $\tan A = \frac{a}{b}$ .

- 617**  $\sin A = \frac{a}{\sqrt{a^2 + b^2}}$ ,  
 $\cos A = \frac{b}{\sqrt{a^2 + b^2}}$ .



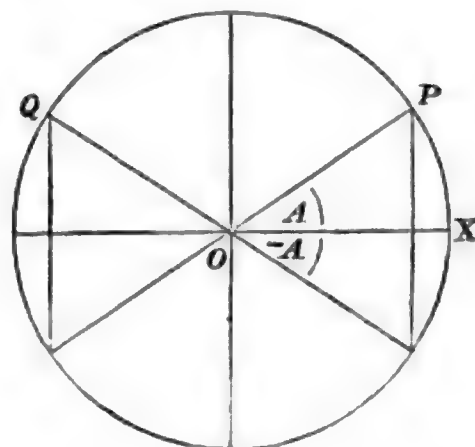
- 618**  $\sin A = \frac{\tan A}{\sqrt{1 + \tan^2 A}}$ ,  $\cos A = \frac{1}{\sqrt{1 + \tan^2 A}}$ . [617]

619 The Complement of  $A$  is  $= 90^\circ - A$ .

620 The Supplement of  $A$  is  $= 180^\circ - A$ .

621  $\sin(90^\circ - A) = \cos A$ ,  
 $\tan(90^\circ - A) = \cot A$ ,  
 $\sec(90^\circ - A) = \operatorname{cosec} A$ .

622  $\sin(180^\circ - A) = \sin A$ ,  
 $\cos(180^\circ - A) = -\cos A$ ,  
 $\tan(180^\circ - A) = -\tan A$ .



In the figure

$$\angle QOX = 180^\circ - A. \quad [607, 608]$$

623  $\sin(-A) = -\sin A$ .

624  $\cos(-A) = \cos A$ .

By Fig., and (607), (608).

The secant, cosecant, and cotangent of  $180^\circ - A$ , and of  $-A$ , will follow the same rule as their reciprocals, the cosine, sine, and tangent. [610–612]

625 To reduce any ratio of an angle greater than  $90^\circ$  to the ratio of an angle less than  $90^\circ$ .

RULE.—Determine the sign of the ratio by the rules (607), and then substitute for the given angle the *acute* angle formed by its two bounding lines, produced if necessary.

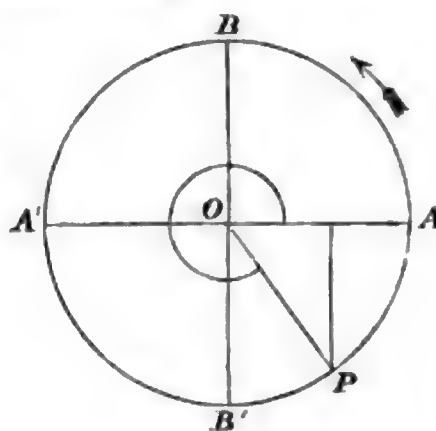
Ex.—To find all the ratios of  $660^\circ$ .  
 Measuring  $300^\circ (= 660^\circ - 360^\circ)$  round the circle from  $A$  to  $P$ , we find the acute angle  $AOP$  to be  $60^\circ$ , and  $P$  lies *below*  $AA'$ , and to the *right* of  $BB'$ .

Therefore

$$\sin 660^\circ = -\sin 60^\circ = -\frac{\sqrt{3}}{2},$$

$$\cos 660^\circ = \cos 60^\circ = \frac{1}{2},$$

and from the sine and cosine all the remaining ratios may be found by (610–616).



INVERSE NOTATION.—The angle whose sine is  $x$  is denoted by  $\sin^{-1} x$ .

**626** All the angles which have a given sine, cosine, or tangent, are given by the formulæ

$$\sin^{-1} x = n\pi + (-1)^n \theta \dots\dots\dots (1),$$

$$\cos^{-1} x = 2n\pi \pm \theta \dots\dots\dots (2),$$

$$\tan^{-1} x = n\pi + \theta \dots\dots\dots (3).$$

In these formulæ  $\theta$  is any angle which has  $x$  for its sine, cosine, or tangent respectively, and  $n$  is any integer.

$\operatorname{Cosec}^{-1} x$ ,  $\sec^{-1} x$ ,  $\cot^{-1} x$  have similar general values, by (610–612).

These formulæ are verified by taking  $A$ , in Fig. 622, for  $\theta$ , and making  $n$  an odd or even integer successively.

## FORMULÆ INVOLVING TWO ANGLES, AND MULTIPLE ANGLES.

**627**  $\sin (A+B) = \sin A \cos B + \cos A \sin B,$

**628**  $\sin (A-B) = \sin A \cos B - \cos A \sin B,$

**629**  $\cos (A+B) = \cos A \cos B - \sin A \sin B,$

**630**  $\cos (A-B) = \cos A \cos B + \sin A \sin B.$

PROOF.—By (700) and (701), we have

$$\sin C = \sin A \cos B + \cos A \sin B,$$

and

$$\sin C = \sin (A+B), \text{ by (622).}$$

To obtain  $\sin (A-B)$  change the sign of  $B$  in (627), and employ (623), (624),  $\cos (A+B) = \sin \{ (90^\circ - A) - B \}$ , by (621).

Expand by (628), and use (621), (623), (624). For  $\cos (A-B)$  change the sign of  $B$  in (629).

**631**  $\tan (A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$

**632**  $\tan (A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$

**633**  $\cot (A+B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}.$

**634**  $\cot (A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}.$

Obtained from (627–630).

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$$635 \quad \sin 2A = 2 \sin A \cos A. \quad [627. \text{ Put } B=A$$

$$636 \quad \cos 2A = \cos^2 A - \sin^2 A,$$

$$637 \quad = 2 \cos^2 A - 1,$$

$$638 \quad = 1 - 2 \sin^2 A. \quad [629, 613$$

$$639 \quad 2 \cos^2 A = 1 + \cos 2A. \quad [637$$

$$640 \quad 2 \sin^2 A = 1 - \cos 2A. \quad [638$$

$$641 \quad \sin \frac{A}{2} = \sqrt{\frac{1 - \cos A}{2}}. \quad [640$$

$$642 \quad \cos \frac{A}{2} = \sqrt{\frac{1 + \cos A}{2}}. \quad [639$$

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$$643 \quad \tan \frac{A}{2} = \sqrt{\frac{1 - \cos A}{1 + \cos A}} = \frac{1 - \cos A}{\sin A} = \frac{\sin A}{1 + \cos A}. \quad [641, 642, 613$$

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$$646 \quad \cos A = \frac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}, \quad \sin A = \frac{2 \tan \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}. \quad [643, 613$$

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$$648 \quad \cos A = \frac{1}{1 + \tan A \tan \frac{A}{2}}.$$

$$649 \quad \sin \left(45^\circ + \frac{A}{2}\right) = \cos \left(45^\circ - \frac{A}{2}\right) = \sqrt{\frac{1 + \sin A}{2}}. \quad [641$$

$$650 \quad \cos \left(45^\circ + \frac{A}{2}\right) = \sin \left(45^\circ - \frac{A}{2}\right) = \sqrt{\frac{1 - \sin A}{2}}. \quad [642$$

$$651 \quad \tan \left(45^\circ + \frac{A}{2}\right) = \sqrt{\frac{1 + \sin A}{1 - \sin A}} = \frac{1 + \sin A}{\cos A} = \frac{\cos A}{1 - \sin A}.$$

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$$652 \quad \tan 2A = \frac{2 \tan A}{1 - \tan^2 A}. \quad [631. \text{ Put } B=A$$

$$653 \quad \cot 2A = \frac{\cot^2 A - 1}{2 \cot A}.$$

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$$654 \quad \tan(45^\circ + A) = \frac{1 + \tan A}{1 - \tan A}.$$

$$655 \quad \tan(45^\circ - A) = \frac{1 - \tan A}{1 + \tan A}. \quad [631, 632]$$


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$$656 \quad \sin 3A = 3 \sin A - 4 \sin^3 A.$$

$$657 \quad \cos 3A = 4 \cos^3 A - 3 \cos A.$$

$$658 \quad \tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.$$

By putting  $B = 2A$  in (627), (629), and (631).

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$$659 \quad \begin{aligned} \sin(A+B) \sin(A-B) &= \sin^2 A - \sin^2 B \\ &= \cos^2 B - \cos^2 A. \end{aligned}$$

$$660 \quad \begin{aligned} \cos(A+B) \cos(A-B) &= \cos^2 A - \sin^2 B \\ &= \cos^2 B - \sin^2 A. \end{aligned}$$

From (627), &c.

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$$661 \quad \sin \frac{A}{2} + \cos \frac{A}{2} = \pm \sqrt{1 + \sin A}.$$

[Proved by squaring.]

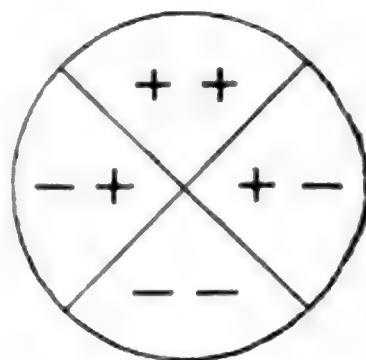
$$662 \quad \sin \frac{A}{2} - \cos \frac{A}{2} = \pm \sqrt{1 - \sin A}.$$

$$663 \quad \sin \frac{A}{2} = \frac{1}{2} \{ \sqrt{1 + \sin A} - \sqrt{1 - \sin A} \}.$$

$$664 \quad \cos \frac{A}{2} = \frac{1}{2} \{ \sqrt{1 + \sin A} + \sqrt{1 - \sin A} \},$$

when  $\frac{A}{2}$  lies between  $-45^\circ$  and  $+45^\circ$ .

**665** In the accompanying diagram the signs exhibited in each quadrant are the signs to be prefixed to the two surds in the value of  $\sin \frac{A}{2}$  according to the quadrant in which  $\frac{A}{2}$  lies.



For  $\cos \frac{A}{2}$  change the second sign.

PROOF.—By examining the changes of sign in (661) and (662) by (607).



$$666 \quad 2 \sin A \cos B = \sin (A + B) + \sin (A - B).$$

$$667 \quad 2 \cos A \sin B = \sin (A + B) - \sin (A - B).$$

$$668 \quad 2 \cos A \cos B = \cos (A + B) + \cos (A - B).$$

$$669 \quad 2 \sin A \sin B = \cos (A - B) - \cos (A + B).$$

[627-630]

$$670 \quad \sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}.$$

$$671 \quad \sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}.$$

$$672 \quad \cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}.$$

$$673 \quad \cos B - \cos A = 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}.$$

Obtained by changing  $A$  into  $\frac{A+B}{2}$ , and  $B$  into  $\frac{A-B}{2}$ , in (666-669).

It is advantageous to commit the foregoing formulæ to memory, in words, thus—

2 sin cos = sin sum + sin difference,  
2 cos sin = sin sum - sin difference,  
2 cos cos = cos sum + cos difference,  
2 sin sin = cos difference - cos sum.

sin first + sin second = 2 sin half sum cos half difference,  
sin first - sin second = 2 cos half sum sin half difference,  
cos first + cos second = 2 cos half sum cos half difference,  
cos second - cos first = 2 sin half sum sin half difference.

$$674 \quad \sin (A + B + C)$$

$$= \sin A \cos B \cos C + \sin B \cos C \cos A \\ + \sin C \cos A \cos B - \sin A \sin B \sin C.$$

$$675 \quad \cos (A + B + C)$$

$$= \cos A \cos B \cos C - \cos A \sin B \sin C \\ - \cos B \sin C \sin A - \cos C \sin A \sin B.$$

$$676 \quad \tan (A + B + C)$$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}.$$

PROOF.—Put  $B + C$  for  $B$  in (627), (629), and (631).

If  $A + B + C = 180^\circ$ ,

$$677 \quad \sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

$$\sin A + \sin B - \sin C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}.$$

$$678 \quad \cos A + \cos B + \cos C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + 1.$$

$$\cos A + \cos B - \cos C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2} - 1.$$

$$679 \quad \tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

$$680 \quad \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}.$$

$$681 \quad \sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

$$682 \quad \cos 2A + \cos 2B + \cos 2C = -4 \cos A \cos B \cos C - 1.$$

General formulæ, including the foregoing, obtained by applying (666-673).

If  $A + B + C = \pi$ , and  $n$  be any integer,

$$683 \quad 4 \sin \frac{nA}{2} \sin \frac{nB}{2} \sin \frac{nC}{2}$$

$$= \sin \left( \frac{n\pi}{2} - nA \right) + \sin \left( \frac{n\pi}{2} - nB \right) + \sin \left( \frac{n\pi}{2} - nC \right) - \sin \frac{n\pi}{2}.$$

$$684 \quad 4 \cos \frac{nA}{2} \cos \frac{nB}{2} \cos \frac{nC}{2}$$

$$= \cos \left( \frac{n\pi}{2} - nA \right) + \cos \left( \frac{n\pi}{2} - nB \right) + \cos \left( \frac{n\pi}{2} - nC \right) + \cos \frac{n\pi}{2}.$$

If  $A + B + C = 0$ ,

$$685 \quad 4 \sin \frac{nA}{2} \sin \frac{nB}{2} \sin \frac{nC}{2} = -\sin nA - \sin nB - \sin nC.$$

$$686 \quad 4 \cos \frac{nA}{2} \cos \frac{nB}{2} \cos \frac{nC}{2} = \cos nA + \cos nB + \cos nC + 1.$$

*RULE.*—If, in formulæ (683) to (686), two factors on the left be changed by writing *sin* for *cos*, or *cos* for *sin*, then, on the right side, change the signs of those terms which do not contain the angles of the altered factors.

Thus, from (683), we obtain

$$\begin{aligned} 687 \quad & 4 \sin \frac{nA}{2} \cos \frac{nB}{2} \cos \frac{nC}{2} \\ &= -\sin \left( \frac{n\pi}{2} - nA \right) + \sin \left( \frac{n\pi}{2} - nB \right) + \sin \left( \frac{n\pi}{2} - nC \right) + \sin \frac{n\pi}{2}. \end{aligned}$$

A Formula for the construction of Tables of sines, co-sines, &c.—

$$688 \quad \sin(n+1)\alpha - \sin n\alpha = \sin n\alpha - \sin(n-1)\alpha - k \sin n\alpha,$$

where  $\alpha = 10''$ , and  $k = 2(1 - \cos \alpha) = .0000000023504$ .

689 Formulae for verifying the tables—

$$\begin{aligned} \sin A + \sin(72^\circ + A) - \sin(72^\circ - A) &= \sin(36^\circ + A) - \sin(36^\circ - A), \\ \cos A + \cos(72^\circ + A) + \cos(72^\circ - A) &= \cos(36^\circ + A) + \cos(36^\circ - A), \\ \sin(60^\circ + A) - \sin(60^\circ - A) &= \sin A. \end{aligned}$$

## RATIOS OF CERTAIN ANGLES.

$$690 \quad \sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}}, \quad \tan 45^\circ = 1.$$

$$691 \quad \sin 60^\circ = \frac{\sqrt{3}}{2}, \quad \cos 60^\circ = \frac{1}{2}, \quad \tan 60^\circ = \sqrt{3}.$$

$$692 \quad \sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}, \quad \cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}},$$

$$\left. \begin{aligned} \tan 15^\circ &= 2 - \sqrt{3} \\ \cot 15^\circ &= 2 + \sqrt{3} \end{aligned} \right\}.$$

$$693 \quad \sin 18^\circ = \frac{\sqrt{5}-1}{4}, \quad \cos 18^\circ = \frac{\sqrt{5}+\sqrt{5}}{2\sqrt{2}},$$

$$\tan 18^\circ = \sqrt{\frac{5-2\sqrt{5}}{5}}.$$

$$694 \quad \sin 54^\circ = \frac{\sqrt{5}+1}{4}, \quad \cos 54^\circ = \frac{\sqrt{5}-\sqrt{5}}{2\sqrt{2}},$$

$$\tan 54^\circ = \sqrt{\frac{5+2\sqrt{5}}{5}}.$$

695 By taking the complements of these angles, the same table gives the ratios of  $30^\circ$ ,  $75^\circ$ ,  $72^\circ$ , and  $36^\circ$ .

**696** PROOFS.— $\sin 15^\circ$  is obtained from  $\sin (45^\circ - 30^\circ)$ , expanded by (628).

**697**  $\sin 18^\circ$  from the equation  $\sin 2x = \cos 3x$ , where  $x = 18^\circ$ .

**698**  $\sin 54^\circ$  from  $\sin 3x = 3 \sin x - 4 \sin^3 x$ , where  $x = 18^\circ$ .

**699** And the ratios of various angles may be obtained by taking the sum, difference, or some multiple of the angles in the table, and making use of known formulæ. Thus

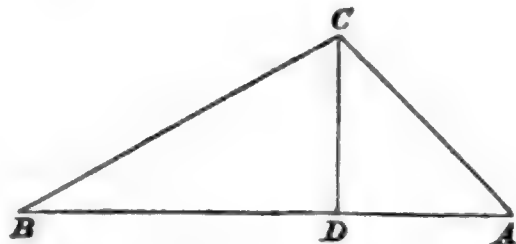
$$12^\circ = 30^\circ - 18^\circ, \quad 7\frac{1}{2}^\circ = \frac{15^\circ}{2}, \quad \&c., \quad \&c.$$

### PROPERTIES OF THE TRIANGLE.

**700**  $c = a \cos B + b \cos A.$

**701**  $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$

**702**  $a^2 = b^2 + c^2 - 2bc \cos A.$



PROOF.—By Euc. II. 12 and 13,  $a^2 = b^2 + c^2 - 2c \cdot AD.$

**703**  $\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$

If  $s \equiv \frac{a+b+c}{2}$ , and  $\Delta$  denote the area  $ABC$ ,

**704**  $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}.$

[641, 642, 703, 9, 10, 1.

**705**  $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$

**706**  $\sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}.$  [635, 704

**707**  $\Delta = \frac{bc}{2} \sin A = \sqrt{s(s-a)(s-b)(s-c)},$  [707, 706

**708**  $= \frac{1}{4} \sqrt{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}.$

*The Triangle and Circle.*

Let

$r$  = radius of inscribed circle,

$r_a$  = radius of escribed circle touching the side  $a$ ,

$R$  = radius of circumscribing circle.

$$709 \quad r = \frac{\Delta}{s}.$$

[From Fig.,  $\Delta = \frac{ra}{2} + \frac{rb}{2} + \frac{rc}{2}$ .

$$710 \quad r = \frac{a \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}}.$$

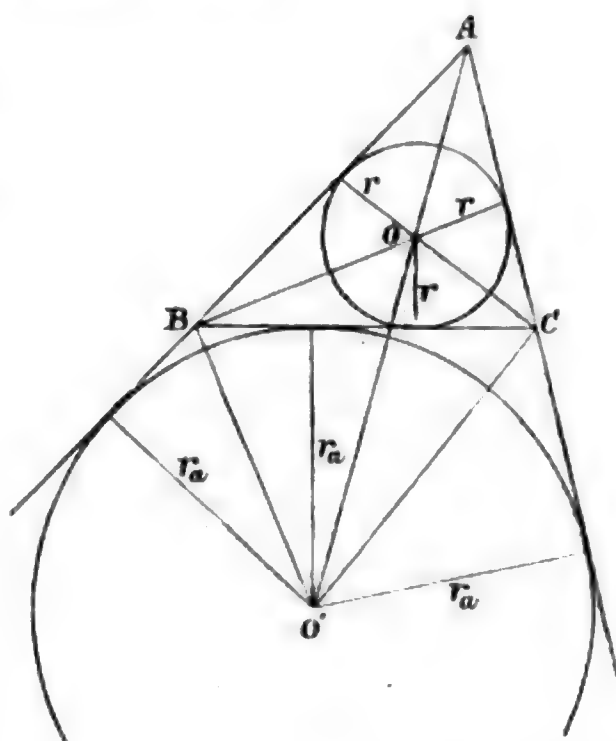
[By  $a = r \cot \frac{B}{2} + r \cot \frac{C}{2}$ .

$$711 \quad r_a = \frac{\Delta}{s-a}.$$

[By  $\Delta = \frac{r_a b}{2} + \frac{r_a c}{2} - \frac{r_a a}{2}$ .

$$712 \quad r_a = \frac{a \cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}}.$$

[From  $a = r_a \tan \frac{B}{2} + r_a \tan \frac{C}{2}$ .

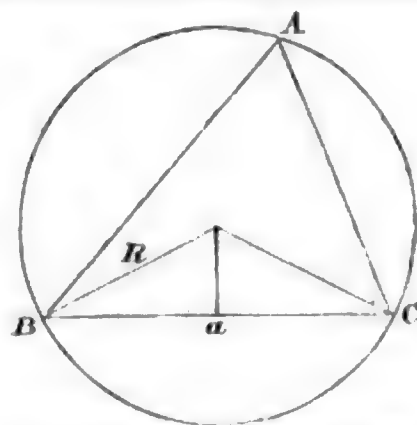


$$713 \quad R = \frac{a}{2 \sin A} = \frac{abc}{4\Delta}.$$

[By (III. 20) and (706)

$$715 \quad = \frac{1}{4} \sqrt{\left\{ (b+c)^2 \sec^2 \frac{A}{2} + (b-c)^2 \operatorname{cosec}^2 \frac{A}{2} \right\}}$$

[702



Distance between the centres of inscribed and circumscribed circles

$$716 \quad = \sqrt{R^2 - 2Rr}. \quad [936$$

Radius of circle touching  $b, c$  and the inscribed circle

$$717 \quad r' = r \tan^2 \frac{1}{4} (B+C). \quad [\text{By } \sin \frac{A}{2} = \frac{r-r'}{r+r'}.$$

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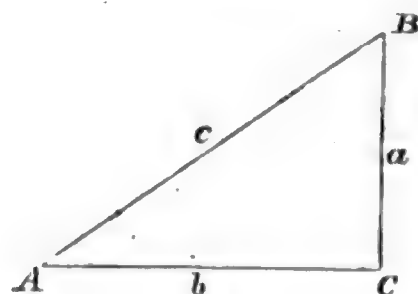
SOLUTION OF TRIANGLES.

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Right-angled triangles are solved by the formulæ

**718**  $c^2 = a^2 + b^2;$

**719** 
$$\begin{cases} a = c \sin A, \\ b = c \cos A, \\ a = b \tan A, \\ \text{\&c.} \end{cases}$$




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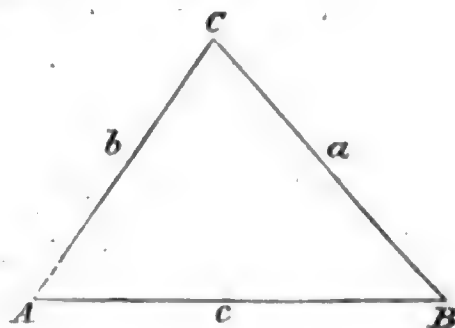
*Scalene Triangles.*


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**720** CASE I.—The equation

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad [701]$$

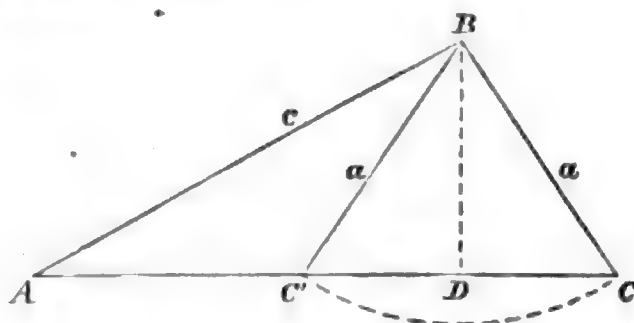
will determine any one of the four quantities  $A, B, a, b$  when the remaining three are known.



**721** *The Ambiguous Case.*

When, in Case I., two sides and an acute angle opposite to one of them are given, we have, from the figure,

$$\sin C = \frac{c \sin A}{a}.$$



Then  $C$  and  $180^\circ - C$  are the values of  $C$  and  $C'$ , by (622).

Also 
$$b = c \cos A \pm \sqrt{a^2 - c^2 \sin^2 A},$$

because 
$$= AD \pm DC.$$

**722** When an angle  $B$  is to be determined from the equation

$$\sin B = \frac{b}{a} \sin A,$$

and  $\frac{b}{a}$  is a small fraction; the circular measure of  $B$  may be approximated to by putting  $\sin(B+C)$  for  $\sin A$ , and using theorem (796).



**723** CASE II.—When two sides  $b, c$  and the included angle  $A$  are known, the third side  $a$  is given by the formula

$$a^2 = b^2 + c^2 - 2bc \cos A, \quad [702]$$

when logarithms are not used.

Otherwise, employ the following formula with logarithms,

$$\mathbf{724} \quad \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}.$$

Obtained from  $\frac{b-c}{b+c} = \frac{\sin B - \sin C}{\sin B + \sin C}$  (701), and then applying (670) and (671).

$\frac{B-C}{2}$  having been found from the above equation, and  $\frac{B+C}{2}$  being equal to  $90^\circ - \frac{A}{2}$ , we have

$$\mathbf{725} \quad B = \frac{B+C}{2} + \frac{B-C}{2}, \quad C = \frac{B+C}{2} - \frac{B-C}{2}.$$

$B$  and  $C$  having been determined,  $a$  can be found by Case I.

**726** If the logarithms of  $b$  and  $c$  are known, the trouble of taking out  $\log(b-c)$  and  $\log(b+c)$  may be avoided by employing the subsidiary angle  $\theta = \tan^{-1} \frac{b}{c}$ , and the formula

$$\mathbf{727} \quad \tan \frac{1}{2}(B-C) = \tan \left( \theta - \frac{\pi}{4} \right) \cot \frac{A}{2}, \quad [655]$$

Or else the subsidiary angle  $\theta = \cos^{-1} \frac{c}{b}$ , and the formula

$$\mathbf{728} \quad \tan \frac{1}{2}(B-C) = \tan^2 \frac{\theta}{2} \cot \frac{A}{2}. \quad [643]$$

If  $a$  be required without calculating the angles  $B$  and  $C$ , we may use the formula

$$\mathbf{729} \quad a = \frac{(b+c) \sin \frac{A}{2}}{\cos \frac{1}{2}(B-C)}. \quad \left[ \text{From the figure in 960, by drawing a perpendicular from } B \text{ to } EC \text{ produced.} \right]$$

**730** If  $a$  be required in terms of  $b, c$ , and  $A$  alone, and in a form adapted to logarithmic computation, employ the subsidiary angle

$$\theta = \sin^{-1} \left( \frac{4bc}{(b+c)^2} \cos^2 \frac{A}{2} \right),$$

and the formula

$$a = (b+c) \cos \theta. \quad [702, 637]$$

CASE III.—When the three sides are known, the angles may be found without employing logarithms, from the formula

$$731 \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}. \quad [703]$$

732 If logarithms are to be used, take the formulæ for  $\sin \frac{A}{2}$ ,  $\cos \frac{A}{2}$ , or  $\tan \frac{A}{2}$ ; (704) and (705).

QUADRILATERAL INSCRIBED IN A CIRCLE.

$$733 \quad \cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}.$$

From  $AC^2 = a^2 + b^2 - 2ab \cos B = c^2 + d^2 + 2cd \cos B$ , by (702), and  $B + D = 180^\circ$ .

$$734 \quad \sin B = \frac{2Q}{ab + cd}. \quad [613, 733]$$

$$735 \quad Q = \sqrt{(s-a)(s-b)(s-c)(s-d)} \\ = \text{area of } ABCD,$$

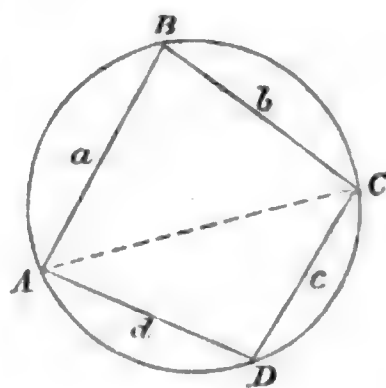
and  $s = \frac{1}{2}(a + b + c + d).$

Area =  $\frac{1}{2}ab \sin B + \frac{1}{2}cd \sin B$ ; substitute  $\sin B$  from last.

$$736 \quad AC^2 = \frac{(ac + bd)(ad + bc)}{(ab + cd)}. \quad [702, 733]$$

Radius of circumscribed circle

$$737 \quad = \frac{1}{4Q} \sqrt{(ab + cd)(ac + bd)(ad + bc)}. \quad [713, 734, 736]$$



If  $AD$  bisect the side of the triangle  $ABC$  in  $D$ ,

$$738 \quad \tan BDA = \frac{4\Delta}{b^2 - c^2}.$$

$$739 \quad \cot BAD = 2 \cot A + \cot B.$$

$$740 \quad AD^2 = \frac{1}{4}(b^2 + c^2 + 2bc \cos A) = \frac{1}{2}(b^2 + c^2 - \frac{1}{2}a^2).$$

If  $AD$  bisect the angle  $A$  of a triangle  $ABC$ ,

$$742 \quad \tan BDA = \cot \frac{B-C}{2} = \frac{b+c}{b-c} \tan \frac{A}{2}.$$

$$743 \quad AD = \frac{2bc}{b+c} \cos \frac{A}{2}.$$

If  $AD$  be perpendicular to  $BC$ ,

$$744 \quad AD = \frac{bc \sin A}{a} = \frac{b^2 \sin C + c^2 \sin B}{b + c}.$$

$$745 \quad BD \sim CD = \frac{b^2 - c^2}{a} = a \frac{\tan B - \tan C}{\tan B + \tan C}.$$

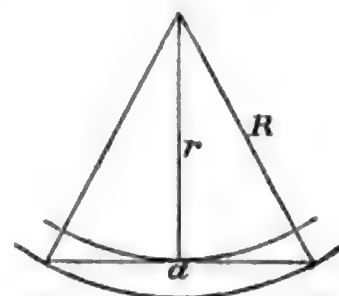
### REGULAR POLYGON AND CIRCLE.

Radius of circumscribing circle  $= R$ .

Radius of inscribed circle  $= r$ .

Side of polygon  $= a$ .

Number of sides  $= n$ .



$$746 \quad R = \frac{a}{2} \operatorname{cosec} \frac{\pi}{n}, \quad r = \frac{a}{2} \cot \frac{\pi}{n}.$$

Area of Polygon

$$748 \quad = \frac{1}{2} na^2 \cot \frac{\pi}{n} = \frac{1}{2} nR^2 \sin \frac{2\pi}{n} = nr^2 \tan \frac{\pi}{n}.$$

### USE OF SUBSIDIARY ANGLES.

749 To adapt  $a \pm b$  to logarithmic computation.

Take  $\theta = \tan^{-1} \sqrt{\frac{b}{a}}$ ; then  $a + b = a \sec^2 \theta$ .

750 For  $a - b$  take  $\theta = \tan^{-1} \left( \frac{b}{a} \right)$ ; thus

$$a - b = \frac{a \sqrt{2} \cos (\theta + 45^\circ)}{\cos \theta}.$$

751 To adapt  $a \cos C \pm b \sin C$  to logarithmic computation.

Take  $\theta = \tan^{-1} \frac{a}{b}$ ; then

$$a \cos C \pm b \sin C = \sqrt{(a^2 + b^2)} \sin (\theta \pm C). \quad [\text{By 617}]$$

For similar instances of the use of a subsidiary angle, see (726) to (730).

752 To solve a quadratic equation by employing a subsidiary angle.

If  $x^2 - 2px + q = 0$  be the equation,

$$x = p \left( 1 \pm \sqrt{1 - \frac{q}{p^2}} \right). \quad [\text{By 45}]$$

CASE I.—If  $q$  be  $< p^2$ , put  $\frac{q}{p^2} = \sin^2 \theta$ ; then

$$x = 2p \cos^2 \frac{\theta}{2}, \text{ and } 2p \sin^2 \frac{\theta}{2}. \quad [639, 640]$$

CASE II.—If  $q$  be  $> p^2$ , put  $\frac{q}{p^2} = \sec^2 \theta$ ; then

$$x = p (1 \pm i \tan \theta), \text{ imaginary roots.} \quad [614]$$

CASE III.—If  $q$  be negative, put  $\frac{q}{p^2} = \tan^2 \theta$ ; then

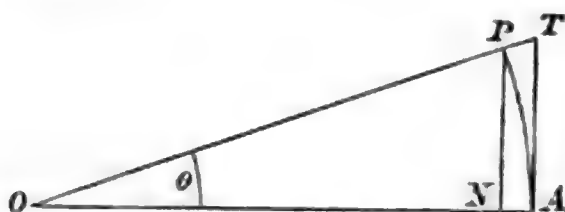
$$x = \sqrt{q} \cot \frac{\theta}{2} \text{ and } -\sqrt{q} \tan \frac{\theta}{2}. \quad [644, 645]$$

### LIMITS OF RATIOS.

**753**  $\frac{\sin \theta}{\theta} = \frac{\tan \theta}{\theta} = 1,$

when  $\theta$  vanishes.

For ultimately  $\frac{PN}{AP} = \frac{AT}{AP} = 1$ . [601, 606]



**754**  $n \sin \frac{\theta}{n} = \theta$  when  $n$  is infinite. By putting  $\frac{\theta}{n}$  for  $\theta$  in last.

**755**  $\left(\cos \frac{\theta}{n}\right)^n = 1$  when  $n$  is infinite.

PROOF.—Put  $\left(1 - \sin^2 \frac{\theta}{n}\right)^{\frac{n}{2}}$ , and expand the logarithm by (156).

### DE MOIVRE'S THEOREM.

**756**  $(\cos \alpha + i \sin \alpha) \cos \beta + i \sin \beta) \dots \&c.$   
 $= \cos (\alpha + \beta + \gamma + \dots) + i \sin (\alpha + \beta + \gamma + \dots),$

where  $i = \sqrt{-1}$ .

[Proved by Induction.]

**757**  $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$

PROOF.—By Induction, or by putting  $\alpha, \beta, \&c.$  each  $= \theta$  in (756).

*Expansion of  $\cos n\theta$ , &c., in powers  $\sin \theta$  and  $\cos \theta$ .*

**758**  $\cos n\theta = \cos^n \theta - C(n, 2) \cos^{n-2} \theta \sin^2 \theta$   
 $+ C(n, 4) \cos^{n-4} \theta \sin^4 \theta - \&c.$

**759**  $\sin n\theta = n \cos^{n-1} \theta \sin \theta - C(n, 3) \cos^{n-3} \theta \sin^3 \theta + \&c.$

PROOF.—Expand (757) by Bin. Th., and equate real and imaginary parts.

$$760 \quad \tan n\theta = \frac{n \tan \theta - C(n, 3) \tan^3 \theta + \&c.}{1 - C(n, 2) \tan^2 \theta + C(n, 4) \tan^4 \theta - \&c.}$$

In series (758, 759), stop at, and exclude, all terms with indices greater than  $n$ . Note,  $n$  is here an integer.

Let  $s_r$  = sum of the  $C(n, r)$  products of  $\tan \alpha, \tan \beta, \tan \gamma$ , &c. to  $n$  terms.

$$761 \quad \sin(\alpha + \beta + \gamma + \&c.) = \cos \alpha \cos \beta \dots (s_1 - s_3 + s_5 - \&c.).$$

$$762 \quad \cos(\alpha + \beta + \gamma + \&c.) = \cos \alpha \cos \beta \dots (1 - s_2 + s_4 - \&c.).$$

PROOF.—By equating real and imaginary parts in (756).

$$763 \quad \tan(\alpha + \beta + \gamma + \&c.) = \frac{s_1 - s_3 + s_5 - s_7 + \&c.}{1 - s_2 + s_4 - s_6 + \&c.}$$

*Expansions of the sine and cosine in powers of the angle.*

$$764 \quad \sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \&c. \quad \cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \&c.$$

PROOF.—Put  $\frac{\theta}{n}$  for  $\theta$  in (757) and  $n = \infty$ , employing (754) and (755).

$$766 \quad e^{i\theta} = \cos \theta + i \sin \theta. \quad e^{-i\theta} = \cos \theta - i \sin \theta. \quad \text{By (150)}$$

$$768 \quad e^{i\theta} + e^{-i\theta} = 2 \cos \theta. \quad e^{i\theta} - e^{-i\theta} = 2i \sin \theta.$$

$$770 \quad i \tan \theta = \frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}. \quad \frac{1 + i \tan \theta}{1 - i \tan \theta} = e^{2i\theta}.$$

*Expansion of  $\cos^n \theta$  and  $\sin^n \theta$  in cosines or sines of multiples of  $\theta$ .*

$$772 \quad 2^{n-1} \cos^n \theta = \cos n\theta + n \cos(n-2)\theta + C(n, 2) \cos(n-4)\theta + C(n, 3) \cos(n-6)\theta + \&c.$$

773 When  $n$  is even,

$$2^{n-1} (-1)^{\frac{n}{2}} \sin^n \theta = \cos n\theta - n \cos(n-2)\theta + C(n, 2) \cos(n-4)\theta - C(n, 3) \cos(n-6)\theta + \&c.,$$

774 And when  $n$  is odd,

$$2^{n-1} (-1)^{\frac{n-1}{2}} \sin^n \theta = \sin n\theta - n \sin(n-2)\theta + C(n, 2) \sin(n-4)\theta - C(n, 3) \sin(n-6)\theta + \&c.$$

Observe that in these series the coefficients are those of the Binomial Theorem, with this exception: *If  $n$  be even, the last term must be divided by 2.*

The series are obtained by expanding  $(e^{i\theta} \pm e^{-i\theta})^n$  by the Binomial Theorem, collecting the equidistant terms in pairs, and employing (768) and (769).

*Expansion of  $\cos n\theta$  and  $\sin n\theta$  in powers of  $\sin \theta$ .*

**775** When  $n$  is even,

$$\begin{aligned}\cos n\theta = 1 - \frac{n^2}{2!} \sin^2 \theta + \frac{n^2(n^2-2^2)}{4!} \sin^4 \theta \\ - \frac{n^2(n^2-2^2)(n^2-4^2)}{6!} \sin^6 \theta + \&c.\end{aligned}$$

**776** When  $n$  is odd,

$$\begin{aligned}\cos n\theta = \cos \theta \left\{ 1 - \frac{n^2-1}{2!} \sin^2 \theta + \frac{(n^2-1)(n^2-3^2)}{4!} \sin^4 \theta \right. \\ \left. - \frac{(n^2-1)(n^2-3^2)(n^2-5^2)}{6!} \sin^6 \theta + \&c. \right\}.\end{aligned}$$

**777** When  $n$  is even,

$$\begin{aligned}\sin n\theta = n \cos \theta \left\{ \sin \theta - \frac{n^2-2^2}{3!} \sin^3 \theta + \right. \\ \left. \frac{(n^2-2^2)(n^2-4^2)}{5!} \sin^5 \theta - \frac{(n^2-2^2)(n^2-4^2)(n^2-6^2)}{7!} \sin^7 \theta + \&c. \right\}.\end{aligned}$$

**778** When  $n$  is odd,

$$\begin{aligned}\sin n\theta = n \sin \theta - \frac{n(n^2-1)}{3!} \sin^3 \theta + \frac{n(n^2-1)(n^2-3^2)}{5!} \sin^5 \theta \\ - \frac{n(n^2-1)(n^2-3^2)(n^2-5^2)}{7!} \sin^7 \theta + \&c.\end{aligned}$$

**PROOF.**—By (758), we may assume, when  $n$  is an even integer,

$$\cos n\theta = 1 + A_2 \sin^2 \theta + A_4 \sin^4 \theta + \dots + A_{2r} \sin^{2r} \theta + \dots$$

Put  $\theta + x$  for  $\theta$ , and in  $\cos n\theta \cos nx - \sin n\theta \sin nx$  substitute for  $\cos nx$  and  $\sin nx$  their values in powers of  $nx$  from (764). Each term on the right is of the type  $A_{2r} (\sin \theta \cos x + \cos \theta \sin x)^{2r}$ . Make similar substitutions for  $\cos x$  and  $\sin x$  in powers of  $x$ . Collect the two coefficients of  $x^2$  in each term by the multinomial theorem (137) and equate them all to the coefficient of  $x^2$  on the left. In this equation write  $\cos^2 \theta$  for  $1 - \sin^2 \theta$  everywhere, and then equate the coefficients of  $\sin^{2r} \theta$  to obtain the relation between the successive quantities  $A_{2r}$  and  $A_{2r+2}$  for the series (775).

To obtain the series (777) equate the coefficients of  $x$  instead of those of  $x^2$ .

When  $n$  is an odd integer, begin by assuming, by (759),

$$\sin n\theta = A_1 \sin \theta + A_3 \sin^3 \theta + \&c.$$

**779** The expansions of  $\cos n\theta$  and  $\sin n\theta$  in powers of  $\cos \theta$  are obtained by changing  $\theta$  into  $\frac{1}{2}\pi - \theta$  in (775) to (778).

**780** *Expansion of  $\cos n\theta$  in descending powers of  $\cos \theta$ .*

$$2 \cos n\theta = (2 \cos \theta)^n - n (2 \cos \theta)^{n-2} + \frac{n(n-3)}{2!} (2 \cos \theta)^{n-4} - \dots + (-1)^r \frac{n(n-r-1)(n-r-2) \dots (n-2r+1)}{r!} (2 \cos \theta)^{n-2r} +$$

up to the last positive power of  $2 \cos \theta$ .

PROOF.—By expanding each term of the identity

$$\log(1-zx) + \log\left(1 - \frac{z}{x}\right) = \log\left\{1 - z\left(x + \frac{1}{x} - z\right)\right\}$$

by (156), equating coefficients of  $z^n$ , and substituting from (768).

**783**  $\sin \alpha + c \sin(\alpha + \beta) + c^2 \sin(\alpha + 2\beta) + \&c.$  to  $n$  terms  
 $= \frac{\sin \alpha - c \sin(\alpha - \beta) - c^n \sin(\alpha + n\beta) + c^{n+1} \sin\{\alpha + \overline{n-1}\beta\}}{1 - 2c \cos \beta + c^2}.$

If  $c$  be  $< 1$  and  $n$  infinite, this becomes

**784**  $= \frac{\sin \alpha - c \sin(\alpha - \beta)}{1 - 2c \cos \beta + c^2}.$

**785**  $\cos \alpha + c \cos(\alpha + \beta) + c^2 \cos(\alpha + 2\beta) + \&c.$  to  $n$  terms  
 $=$  a similar result, changing  $\sin$  into  $\cos$  in the numerator.

**786** Similarly when  $c$  is  $< 1$  and  $n$  infinite.

**787** *Method of summation.*—Substitute for the sines or cosines their exponential values (768). Sum the two resulting geometrical series, and substitute the sines or cosines again for the exponential values by (766).

**788**  $c \sin(\alpha + \beta) + \frac{c^2}{2!} \sin(\alpha + 2\beta) + \frac{c^3}{3!} \sin(\alpha + 3\beta) + \&c.$  to infinity  
 $= e^{c \cos \beta} \sin(\alpha + c \sin \beta) - \sin \alpha.$

**789**  $c \cos(\alpha + \beta) + \frac{c^2}{2!} \cos(\alpha + 2\beta) + \frac{c^3}{3!} \cos(\alpha + 3\beta) + \&c.$  to infinity  
 $= e^{c \cos \beta} \cos(\alpha + c \sin \beta) - \cos \alpha.$

Obtained by the rule in (787).



**790** If, in the series (783) to (789),  $\beta$  be changed into  $\beta + \pi$ , the signs of the alternate terms will thereby be changed.

*Expansion of  $\theta$  in powers of  $\tan \theta$  (Gregory's series).*

$$\mathbf{791} \quad \theta = \tan \theta - \frac{\tan^3 \theta}{3} + \frac{\tan^5 \theta}{5} - \&c.$$

The series converges if  $\tan \theta$  be not  $> 1$ .

PROOF.—By expanding the logarithm of the value of  $e^{2\theta}$  in (771) by (158).

Formulae for the calculation of the value of  $\pi$  by Gregory's series.

$$\mathbf{792} \quad \frac{\pi}{4} = \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239} \quad [791]$$

$$\mathbf{794} \quad = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{70} + \tan^{-1} \frac{1}{99}.$$

PROOF.—By employing the formula for  $\tan (A \pm B)$ , (631).

*To prove that  $\pi$  is incommensurable.*

**795** Convert the value of  $\tan \theta$  in terms of  $\theta$  from (764) and (765) into a continued fraction, thus  $\tan \theta = \frac{\theta}{1 - \frac{\theta^2}{3 - \frac{\theta^2}{5 - \frac{\theta^2}{7 - \&c.}}}$ ; or this result may be obtained by putting  $i\theta$  for  $y$  in (294), and by (770). Hence

$$1 - \frac{\theta}{\tan \theta} = \frac{\theta^2}{3 - \frac{\theta^2}{5 - \frac{\theta^2}{7 - \&c.}}}$$

Put  $\frac{\pi}{2}$  for  $\theta$ , and assume that  $\pi$ , and therefore  $\frac{\pi^2}{4}$ , is commensurable. Let

$$\frac{\pi^2}{4} = \frac{m}{n}, \quad m \text{ and } n \text{ being integers. Then we shall have } 1 = \frac{m}{3n - \frac{mn}{5n - \frac{mn}{7n - \&c.}}}$$

The continued fraction is incommensurable, by (177). But unity cannot be equal to an incommensurable quantity. Therefore  $\pi$  is not commensurable.

$$\mathbf{796} \quad \text{If } \sin x = n \sin (x + a), \quad x = n \sin a + \frac{n^2}{2} \sin 2a + \frac{n^3}{3} \sin 3a + \&c.$$

$$\mathbf{797} \quad \text{If } \tan x = n \tan y, \quad x = y - m \sin 2y + \frac{m^2}{2} \sin 4y - \frac{m^3}{3} \sin 6y + \&c.,$$

where  $m = \frac{1-n}{1+n}$ .

PROOF.—By substituting the exponential values of the sine or tangent (769) and (770), and then eliminating  $x$ .

$$\mathbf{798} \quad \text{Coefficient of } x^n \text{ in the expansion of } e^{ax} \cos bx = \frac{(a^2 + b^2)^{\frac{n}{2}}}{n!} \cos n\theta,$$

where  $a = r \cos \theta$  and  $b = r \sin \theta$ .

For proof, substitute for  $\cos bx$  from (768); expand by (150); put  $a = r \cos \theta$ ,  $b = r \sin \theta$  in the coefficient of  $x^n$ , and employ (757).

**799** When  $e$  is  $< 1$ ,  $\frac{\sqrt{1-e^2}}{1-e \cos \theta} = 1 + 2b \cos \theta + 2b^2 \cos 2\theta + 2b^3 \cos 3\theta + \dots$ ,

where  $b = \frac{e}{1 + \sqrt{1-e^2}}$ .

For proof, put  $e = \frac{2b}{1+b^2}$  and  $2 \cos \theta = x + \frac{1}{x}$ , expand the fraction in two series of powers of  $x$  by the method of (257), and substitute from (768).

$$\begin{aligned} \mathbf{800} \quad & \sin \alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots + \sin\{\alpha + (n-1)\beta\} \\ &= \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right) \sin \frac{n}{2}\beta}{\sin \frac{\beta}{2}}. \end{aligned}$$

$$\begin{aligned} \mathbf{801} \quad & \cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots + \cos\{\alpha + (n-1)\beta\} \\ &= \frac{\cos\left(\alpha + \frac{n-1}{2}\beta\right) \sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}}. \end{aligned}$$

**802** If the terms in these series have the signs  $+$  and  $-$  alternately, change  $\beta$  into  $\beta + \pi$  in the results.

PROOF.—Multiply the series by  $2 \sin \frac{\beta}{2}$ , and apply (669) and (666).

**803** If  $\beta = \frac{2\pi}{n}$  in (800) and (801), each series vanishes.

**804** Generally, If  $\beta = \frac{2\pi}{n}$ , and if  $r$  be an integer not a multiple of  $n$ , the sum of the  $r^{\text{th}}$  powers of the sines or cosines in (800) or (801) is zero if  $r$  be odd; and if  $r$  be even it is  $= \frac{n}{2^r} C\left(r, \frac{r}{2}\right)$ ; by (772) to (774).

*General Theorem.*—Denoting the sum of the series

**805**  $c + c_1 x + c_2 x^2 + \dots + c_n x^n$  by  $F(x)$ ;

then  $c \cos \alpha + c_1 \cos(\alpha + \beta) + \dots + c_n \cos(\alpha + n\beta) = \frac{1}{2} \{e^{i\alpha} F(e^{i\beta}) + e^{-i\alpha} F(e^{-i\beta})\}$ ,  
and

**806**  $c \sin \alpha + c_1 \sin(\alpha + \beta) + \dots + c_n \sin(\alpha + n\beta) = \frac{1}{2i} \{e^{i\alpha} F(e^{i\beta}) - e^{-i\alpha} F(e^{-i\beta})\}$ .

Proved by substituting for the sines and cosines their exponential values (766), &c.

*Expansion of the sine and cosine in factors.*

$$807 \quad x^{2n} - 2x^n y^n \cos n\theta + y^{2n} \\ = \left\{ x^2 - 2xy \cos \theta + y^2 \right\} \left\{ x^2 - 2xy \cos \left( \theta + \frac{2\pi}{n} \right) + y^2 \right\} \dots$$

to  $n$  factors, adding  $\frac{2\pi}{n}$  to the angle successively.

PROOF.—By solving the quadratic on the left, we get  $x = y(\cos n\theta + i \sin n\theta)^{\frac{1}{n}}$ . The  $n$  values of  $x$  are found by (757) and (626), and thence the factors. For the factors of  $x^n \pm y^n$  see (480).

$$808 \quad \sin n\phi = 2^{n-1} \sin \phi \sin \left( \phi + \frac{\pi}{n} \right) \sin \left( \phi + \frac{2\pi}{n} \right) \dots$$

as far as  $n$  factors of sines.

PROOF.—By putting  $x = y = 1$  and  $\theta = 2\phi$  in the last.

809 If  $n$  be even,

$$\sin n\phi = 2^{n-1} \sin \phi \cos \phi \left( \sin^2 \frac{\pi}{n} - \sin^2 \phi \right) \left( \sin^2 \frac{2\pi}{n} - \sin^2 \phi \right) \&c.$$

810 If  $n$  be odd, omit  $\cos \phi$  and make up  $n$  factors, reckoning two factors for each pair of terms in brackets.

PROOF.—From (808), by collecting equidistant factors in pairs, and applying (659).

$$811 \quad \cos n\phi = 2^{n-1} \sin \left( \phi + \frac{\pi}{2n} \right) \sin \left( \phi + \frac{3\pi}{2n} \right) \dots \text{ to } n \text{ factors.}$$

PROOF.—Put  $\phi + \frac{\pi}{2n}$  for  $\phi$  in (808).

812 Also, if  $n$  be odd,

$$\cos n\phi = 2^{n-1} \cos \phi \left( \sin^2 \frac{\pi}{2n} - \sin^2 \phi \right) \left( \sin^2 \frac{3\pi}{2n} - \sin^2 \phi \right) \dots$$

813 If  $n$  be even, omit  $\cos \phi$ .

PROOF.—As in (809).

$$814 \quad n = 2^{n-1} \sin \frac{\pi}{n} \sin \frac{2\pi}{n} \sin \frac{3\pi}{n} \dots \sin \frac{(n-1)\pi}{n}.$$

PROOF.—Divide (809) by  $\sin \phi$ , and make  $\phi$  vanish; then apply (754).

$$815 \quad \sin \theta = \theta \left\{ 1 - \left( \frac{\theta}{\pi} \right)^2 \right\} \left\{ 1 - \left( \frac{\theta}{2\pi} \right)^2 \right\} \left\{ 1 - \left( \frac{\theta}{3\pi} \right)^2 \right\} \dots$$

$$816 \quad \cos \theta = \left\{ 1 - \left( \frac{2\theta}{\pi} \right)^2 \right\} \left\{ 1 - \left( \frac{2\theta}{3\pi} \right)^2 \right\} \left\{ 1 - \left( \frac{2\theta}{5\pi} \right)^2 \right\} \dots$$

PROOF.—Put  $\phi = \frac{\theta}{n}$  in (809) and (812); divide by (814) and make  $n$  infinite.

$$817 \quad e^z - 2 \cos \theta + e^{-z} \\ = 4 \sin^2 \frac{\theta}{2} \left\{ 1 + \frac{z^2}{\theta^2} \right\} \left\{ 1 + \frac{z^2}{(2\pi \pm \theta)^2} \right\} \left\{ 1 + \frac{z^2}{(4\pi \pm \theta)^2} \right\} \dots$$

Proved by substituting  $x = 1 + \frac{z}{2n}$ ,  $y = 1 - \frac{z}{2n}$ , and  $\frac{\theta}{n}$  for  $\theta$  in (807), making  $n$  infinite, and reducing one series of factors to  $4 \sin^2 \frac{\theta}{2}$  by putting  $z = 0$ .

*De Moivre's Property of the Circle.*—Take  $P$  any point, and  $POB = \theta$  any angle,

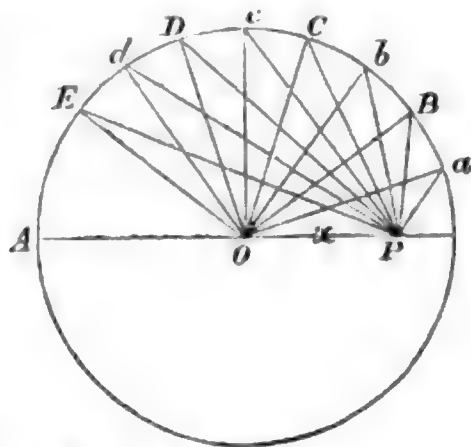
$$BOC = COD = \&c. = \frac{2\pi}{n};$$

$$OP = x; \quad OB = r.$$

$$819 \quad x^{2n} - 2x^n r^n \cos n\theta + r^{2n} \\ = PB^2 PC^2 PD^2 \dots \text{to } n \text{ factors.}$$

By (807) and (702), since  $PB^2 = x^2 - 2xr \cos \theta + r^2$ , &c.

$$820 \quad \text{If } x = r, \quad 2r^n \sin \frac{n\theta}{2} = PB \cdot PC \cdot PD \dots \&c.$$



$$821 \quad \text{Cotes's properties.}—\text{If } \theta = \frac{2\pi}{n},$$

$$x^n - r^n = PB \cdot PC \cdot PD \dots \&c.$$

$$822 \quad x^n + r^n = Pa \cdot Pb \cdot Pc \dots \&c.$$

ADDITIONAL FORMULÆ.

$$823 \quad \cot A + \tan A = 2 \operatorname{cosec} 2A = \sec A \operatorname{cosec} A.$$

$$824 \quad \operatorname{cosec} 2A + \cot 2A = \cot A. \quad \sec A = 1 + \tan A \tan \frac{A}{2}.$$

$$826 \quad \cos A = \cos^4 \frac{A}{2} - \sin^4 \frac{A}{2}.$$

$$827 \quad \tan A + \sec A = \tan \left( 45^\circ + \frac{A}{2} \right).$$

$$828 \quad \frac{\tan A + \tan B}{\cot A + \cot B} = \tan A \tan B.$$

$$829 \quad \sec^2 A \operatorname{cosec}^2 A = \sec^2 A + \operatorname{cosec}^2 A.$$

$$830 \quad \text{If } A+B+C = \frac{\pi}{2},$$

$$\tan B \tan C + \tan C \tan A + \tan A \tan B = 1.$$

$$831 \quad \text{If } A+B+C = \pi,$$

$$\cot B \cot C + \cot C \cot A + \cot A \cot B = 1.$$

$$832 \quad \sin^{-1} \frac{3}{5} + \sin^{-1} \frac{4}{5} = \frac{\pi}{2}. \quad \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{\pi}{4}.$$

In a right-angled triangle  $ABC$ ,  $C$  being the right angle,

$$833 \quad \cos 2B = \frac{a^2 - b^2}{a^2 + b^2}. \quad \tan 2B = \frac{2ab}{a^2 - b^2}.$$

$$834 \quad \tan \frac{1}{2}A = \sqrt{\left(\frac{c-b}{c+b}\right)}. \quad R+r = \frac{1}{2}(a+b).$$

In any triangle,

$$835 \quad \sin \frac{1}{2}(A-B) = \frac{a-b}{c} \cos \frac{1}{2}C.$$

$$\cos \frac{1}{2}(A-B) = \frac{a+b}{c} \sin \frac{1}{2}C.$$

$$836 \quad \frac{\sin A - B}{\sin A + B} = \frac{a^2 - b^2}{c^2}. \quad \frac{\tan \frac{1}{2}A + \tan \frac{1}{2}B}{\tan \frac{1}{2}A - \tan \frac{1}{2}B} = \frac{c}{a-b}.$$

$$837 \quad \frac{1}{2}(a^2 + b^2 + c^2) = bc \cos A + ca \cos B + ab \cos C.$$

$$838 \quad \text{Area of triangle } ABC = \frac{1}{2}bc \sin A \\ = \frac{1}{2}a^2 \frac{\sin B \sin C}{\sin A} = \frac{1}{2}(a^2 - b^2) \frac{\sin A \sin B}{\sin(A-B)}.$$

$$839 \quad = \frac{2abc}{a+b+c} \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C.$$

$$840 \quad = \frac{1}{4}(a+b+c)^2 \tan \frac{1}{2}A \tan \frac{1}{2}B \tan \frac{1}{2}C.$$

With the notation of (709),

$$841 \quad r = \frac{1}{2}(a+b+c) \tan \frac{1}{2}A \tan \frac{1}{2}B \tan \frac{1}{2}C.$$

$$842 \quad 2Rr = \frac{abc}{a+b+c}. \quad \Delta = \sqrt{rr_a r_b r_c}.$$

$$843 \quad a \cos A + b \cos B + c \cos C = 4R \sin A \sin B \sin C.$$

844  $R+r = \frac{1}{2}(a \cot A + b \cot B + c \cot C) =$  sum of perpendiculars on the sides from centre of circumscribing circle.

This may also be shown by applying Enc. VI. D. to the circle described on  $R$  as diameter and the quadrilateral so formed.

845  $r_a r_b r_c = abc \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C.$

846  $r = \sqrt{(r_b r_c)} + \sqrt{(r_c r_a)} + \sqrt{(r_a r_b)}.$

847  $\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}. \quad \tan \frac{1}{2}A = \sqrt{\frac{r r_a}{r_b r_c}}.$

849 If  $O$  be the centre of inscribed circle,

$$OA = \frac{2bc}{a+b+c} \cos \frac{1}{2}A.$$

850  $a(b \cos C - c \cos B) = b^2 - c^2.$

851  $b \cos B + c \cos C = c \cos (B - C).$

852  $a \cos A + b \cos B + c \cos C = 2a \sin B \sin C.$

853  $\cos A + \cos B + \cos C = 1 + \frac{2a \sin B \sin C}{a+b+c}.$

854 If  $s = \frac{1}{2}(a+b+c),$

$$1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c \\ = 4 \sin s \sin (s-a) \sin (s-b) \sin (s-c).$$

855  $-1 + \cos^2 a + \cos^2 b + \cos^2 c + 2 \cos a \cos b \cos c \\ = 4 \cos s \cos (s-a) \cos (s-b) \cos (s-c).$

856  $4 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2} \\ = \cos s + \cos (s-a) + \cos (s-b) + \cos (s-c).$

857  $4 \sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2} \\ = -\sin s + \sin (s-a) + \sin (s-b) + \sin (s-c).$

858  $\pi^2 = 6 \left( 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right) = 8 \left( 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right).$

PROOF.—Equate coefficients of  $\theta^2$  in the expansion of  $\frac{\sin \theta}{\theta}$  by (764) and (815) or of  $\cos \theta$  by (765) and (816).

**859**      *Examples of the Solutions of Triangles.*

Ex. 1: CASE II. (724).—Two sides of a triangle  $b, c$ , being 900 and 700 feet, and the included angle  $47^\circ 25'$ , to find the remaining angles.

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{1}{8} \cot 23^\circ 42' 30'';$$

therefore  $\log \tan \frac{1}{2} (B-C) = \log \cot \frac{A}{2} - \log 8;$

therefore  $L \tan \frac{1}{2} (B-C) = L \cot 23^\circ 42' 30'' - 3 \log 2,$

10 being added to each side of the equation.

$$\begin{aligned} \therefore L \cot 23^\circ 42' 30'' &= 10.3573942^* \\ 3 \log 2 &= .9030900 \end{aligned} \left\{ \begin{array}{l} \therefore \frac{1}{2} (B-C) = 15^\circ 53' 19.55''^* \\ \text{and } \frac{1}{2} (B+C) = 66^\circ 17' 30'' \end{array} \right.$$

$$\therefore L \tan \frac{1}{2} (B-C) = 9.4543042 \quad \therefore B = 82^\circ 10' 49.55''$$

And, by subtraction,  $C = 50^\circ 24' 10.45''.$

Ex. 2: CASE III. (732).—Given the sides  $a, b, c = 7, 8, 9$  respectively, to find the angles.

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{4.3}{12.5}} = \sqrt{\frac{2}{10}};$$

therefore  $L \tan \frac{A}{2} = 10 + \frac{1}{2} (\log 2 - 1) = 9.650515;$

therefore  $\frac{1}{2} A = 24^\circ 5' 41.43''.^*$

$\frac{1}{2} B$  is found in a similar manner, and  $C = 180^\circ - A - B.$

Ex. 3.—In a right-angled triangle, given the hypotenuse  $c = 6953$  and a side  $b = 3$ , to find the remaining angles.

Here  $\cos A = \frac{3}{6953}$ . But, since  $A$  is nearly a right angle, it cannot be determined accurately from  $\log \cos A$ . Therefore take

$$\sin \frac{A}{2} = \sqrt{\frac{1 - \cos A}{2}} = \sqrt{\frac{3475}{6953}};$$

therefore  $L \sin \frac{A}{2} = 10 + \frac{1}{2} (\log 3475 - \log 6953) = 9.8493913;$

therefore  $\frac{A}{2} = 44^\circ 59' 15.52'',^*$

therefore  $A = 89^\circ 58' 31.04''$  and  $B = 0^\circ 1' 28.96''.$

\* See Chambers's Mathematical Tables for a concise explanation of the method of obtaining these figures.



# SPHERICAL TRIGONOMETRY.

## INTRODUCTORY THEOREMS.

**870 Definitions.**—Planes through the centre of a sphere intersect the surface in *great circles*; other planes intersect it in *small circles*. Unless otherwise stated, all arcs are measured on great circles.

The *poles* of a great circle are the extremities of the diameter perpendicular to its plane.

The sides  $a, b, c$  of a spherical triangle are the arcs of great circles  $BC, CA, AB$  on a sphere of radius unity; and the angles  $A, B, C$  are the angles between the tangents to the sides at the vertices, or the angles between the planes of the great circles. The centre of the sphere will be denoted by  $O$ .

The *polar triangle* of a spherical triangle  $ABC$  has for its angular points  $A', B', C'$ , the poles of the sides  $BC, CA, AB$  of the *primitive triangle* in the directions of  $A, B, C$  respectively (since each great circle has two poles). The sides of  $A'B'C'$  are denoted by  $a', b', c'$ .

**871** The sides and angles of the polar triangle are respectively the supplements of the angles and sides of the primitive triangle; that is,

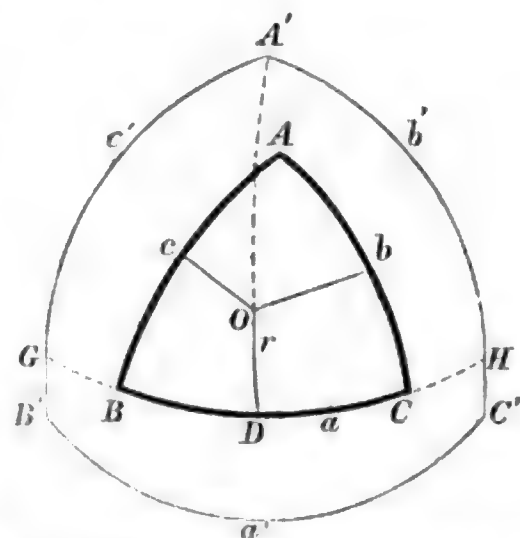
$$a' + A = b' + B = c' + C = \pi,$$

$$a + A' = b + B' = c + C' = \pi.$$

Let  $BC$  produced cut the sides  $A'B', C'A'$  in  $G, H$ .  $B$  is the pole of  $A'C'$ , therefore  $BH = \frac{\pi}{2}$ . Similarly  $CG = \frac{\pi}{2}$ ,

therefore, by addition,  $a + GH = \pi$  and  $GH = A'$ , because  $A'$  is the pole of  $BC$ .

The polar diagram of a spherical polygon is formed in the same way, and the same relations subsist between the sides and angles of the two figures.



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**RULE.**—Hence, any equation between the sides and angles of a spherical triangle produces a supplementary equation by changing  $a$  into  $\pi - A$  and  $A$  into  $\pi - a$ , &c.

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**872** The centre of the inscribed circle, radius  $r$ , is also the centre of the circumscribed circle, radius  $R'$ , of the polar triangle, and  $r + R' = \frac{1}{2}\pi$ .

**PROOF.**—In the last figure, let  $O$  be the centre of the inscribed circle of  $ABC$ ; then  $OD$ , the perpendicular on  $BC$ , passes through  $A'$ , the pole of  $BC$ . Also,  $OD = r$ ; therefore  $OA' = \frac{1}{2}\pi - r$ . Similarly  $OB' = OC' = \frac{1}{2}\pi - r$ ; therefore  $O$  is the centre of the circumscribed circle of  $A'B'C'$ , and  $r + R' = \frac{1}{2}\pi$ .

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**873** The sine of the arc joining a point on the circumference of a small circle with the pole of a parallel great circle, is equal to the ratio of the circumferences or corresponding arcs of the two circles.

For it is equal to the radius of the small circle divided by the radius of the sphere; that is, by the radius of the great circle.

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**874** Two sides of a triangle are greater than the third.  
[By XI. 20.]

**875** The sides of a triangle are together less than the circumference of a great circle.  
[By XI. 21.]

**876** The angles of a triangle are together greater than two right angles.

For  $\pi - A + \pi - B + \pi - C$  is  $< 2\pi$ , by (875) and the polar triangle.

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**877** If two sides of a triangle are equal, the opposite angles are equal.  
[By the geometrical proof in (894).]

**878** If two angles of a triangle are equal, the opposite sides are equal.  
[By the polar triangle and (877).]

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**879** The greater angle of a triangle has the greater side opposite to it.

**PROOF.**—If  $B$  be  $> A$ , draw the arc  $BD$  meeting  $AC$  in  $D$ , and make  $\angle ABD = A$ , therefore  $BD = AD$ ; but  $BD + DC > BC$ , therefore  $AC > BC$ .

**880** The greater side of a triangle has the greater angle opposite to it.  
[By the polar triangle and (879).]

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## RIGHT-ANGLED TRIANGLES.

**881 Napier's Rules.**—In the triangle  $ABC$  let  $C$  be a right angle, then  $a$ ,  $(\frac{1}{2}\pi - B)$ ,  $(\frac{1}{2}\pi - c)$ ,  $(\frac{1}{2}\pi - A)$ , and  $b$ , are called the five *circular parts*. Taking any part for middle part, Napier's rules are—

I. *sine of middle part = product of tangents of adjacent parts.*

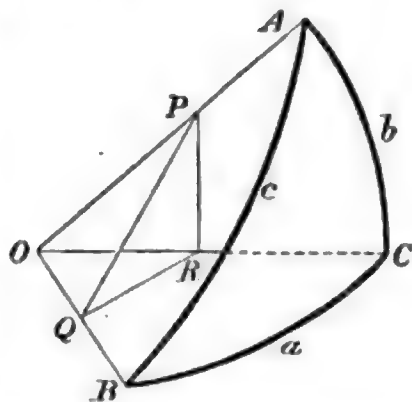
II. *sine of middle part = product of cosines of opposite parts.*

In applying the rules we can take  $A$ ,  $B$ ,  $c$  instead of their complements, and change sine into cos, or *vice versa*, for those parts at once. Thus, taking  $b$  for the middle part,

$$\sin b = \tan a \cot A = \sin B \sin c.$$

Ten equations in all are given by the rules.

**PROOF.**—From any point  $P$  in  $OA$ , draw  $PR$  perpendicular to  $OC$ , and  $RQ$  to  $OB$ ; therefore  $PRQ$  is a right angle; therefore  $OB$  is perpendicular to  $PR$  and  $QR$ , and therefore to  $PQ$ . Then prove any formula by proportion from the triangles of the tetrahedron  $OPQR$ , which are all right-angled. Otherwise, prove by the formulæ for oblique-angled triangles.



## OBLIQUE-ANGLED TRIANGLES.

**882**  $\cos a = \cos b \cos c + \sin b \sin c \cos A.$

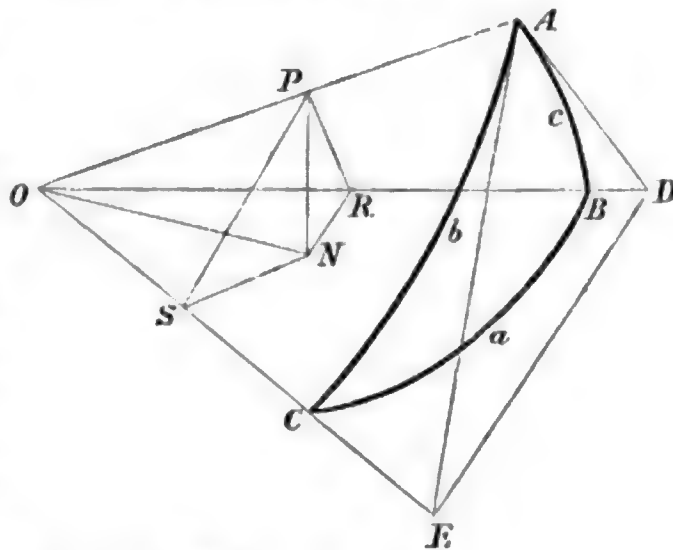
**PROOF.**—Draw tangents at  $A$  to the sides  $c$ ,  $b$  to meet  $OB$ ,  $OC$  in  $D$  and  $E$ . Express  $DE^2$  by (702) applied to each of the triangles  $DAE$  and  $DOE$ , and subtract.

If  $AB$  and  $AC$  are both  $> \frac{\pi}{2}$ , produce them to meet in  $A'$ , the pole of  $A$ , and employ the triangle  $A'BC$ .

If  $AB$  alone be  $> \frac{\pi}{2}$ , produce  $BA$  to meet  $BC$ .

The supplementary formula, by (871), is

**883**  $\cos A = -\cos B \cos C + \sin B \sin C \cos a.$



$$884 \quad \sin \frac{A}{2} = \sqrt{\frac{\sin(s-b) \sin(s-c)}{\sin b \sin c}}.$$

$$885 \quad \cos \frac{A}{2} = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}.$$

$$886 \quad \tan \frac{A}{2} = \sqrt{\frac{\sin(s-b) \sin(s-c)}{\sin s \sin(s-a)}} \text{ where } s = \frac{1}{2}(a+b+c).$$

PROOF.— $\sin^2 \frac{A}{2} = \frac{1}{2}(1 - \cos A)$ . Substitute for  $\cos A$  from (872), and throw the numerator of the whole expression into factors by (673). Similarly for  $\cos \frac{A}{2}$ .

The supplementary formulæ are obtained in a similar way, or by the rule in (871). They are

$$887 \quad \cos \frac{a}{2} = \sqrt{\frac{\cos(S-B) \cos(S-C)}{\sin B \sin C}}.$$

$$888 \quad \sin \frac{a}{2} = \sqrt{\frac{-\cos S \cos(S-A)}{\sin B \sin C}}.$$

$$889 \quad \tan \frac{a}{2} = \sqrt{\frac{-\cos S \cos(S-A)}{\cos(S-B) \cos(S-C)}} \text{ where } S = \frac{1}{2}(A+B+C).$$

$$890 \quad \text{Let } \sigma = \sqrt{\sin s \sin(s-a) \sin(s-b) \sin(s-c)} \\ = \frac{1}{2} \sqrt{1 + 2 \cos a \cos b \cos c - \cos^2 a - \cos^2 b - \cos^2 c}.$$

Then the supplementary form, by (871), is

$$891 \quad \Sigma = \sqrt{-\cos S \cos(S-A) \cos(S-B) \cos(S-C)} \\ = \frac{1}{2} \sqrt{1 - 2 \cos A \cos B \cos C - \cos^2 A - \cos^2 B - \cos^2 C}.$$

$$892 \quad \sin A = \frac{2\sigma}{\sin b \sin c}, \quad \sin a = \frac{2\Sigma}{\sin B \sin C}.$$

[By  $\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$  and (884, 885), &c.]

893 The following rules will produce the ten formulæ (884 to 892)—

I. Write *sin* before each factor in the *s* values of  $\sin \frac{A}{2}$ ,

$\cos \frac{A}{2}$ ,  $\tan \frac{A}{2}$ ,  $\sin A$ , and  $\Delta$ , in *Plane Trigonometry* (704–707), to obtain the corresponding formulæ in *Spherical Trigonometry*.

II. To obtain the supplementary forms of the five results, transpose large and small letters everywhere, and transpose  $\sin$  and  $\cos$  everywhere but in the denominators, and write minus before  $\cos S$ .

$$894 \quad \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

PROOF.—By (882). Otherwise, in the figure of 882, draw  $PN$  perpendicular to  $BOC$ , and  $NR$ ,  $NS$  to  $OB$ ,  $OC$ . Prove  $PRO$  and  $PSO$  right angles by I. 47, and therefore  $PN = OP \sin c \sin B = OP \sin b \sin C$ .

$$895 \quad \cos b \cos C = \cot a \sin b - \cot A \sin C.$$

To remember this formula, take any four consecutive angles and sides (as  $a$ ,  $C$ ,  $b$ ,  $A$ ), and, calling the first and fourth the extremes, and the second and third the middle parts, employ the following rule:—

RULE.—Product of cosines of middle parts = cot extreme side  $\times$  sin middle side — cot extreme angle  $\times$  sin middle angle.

PROOF.—In the formula for  $\cos a$  (882) substitute a similar value for  $\cos c$ , and for  $\sin c$  put  $\sin C \frac{\sin a}{\sin A}$ .

$$896 \quad \text{NAPIER'S FORMULÆ.}$$

$$(1) \quad \tan \frac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \frac{C}{2}.$$

$$(2) \quad \tan \frac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \frac{C}{2}.$$

$$(3) \quad \tan \frac{1}{2}(a - b) = \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \tan \frac{c}{2}.$$

$$(4) \quad \tan \frac{1}{2}(a + b) = \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} \tan \frac{c}{2}.$$

RULE.—In the value of  $\tan \frac{1}{2}(A - B)$  change  $\sin$  to  $\cos$  to obtain  $\tan \frac{1}{2}(A + B)$ . To obtain (3) and (4) from (1) and (2), transpose sides and angles, and change  $\cot$  to  $\tan$ .

PROOF.—In the values of  $\cos A$  and  $\cos B$ , by (883), put  $m \sin a$  and  $m \sin b$  for  $\sin A$  and  $\sin B$ , and add the two equations. Then put  $m = \frac{\sin A \pm \sin B}{\sin a \pm \sin b}$ , and transform by (670–672).

897

GAUSS'S FORMULÆ.

$$(1) \quad \frac{\sin \frac{1}{2}(A+B)}{\cos \frac{1}{2}C} = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}c}.$$

$$(2) \quad \frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2}C} = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}c}.$$

$$(3) \quad \frac{\cos \frac{1}{2}(A+B)}{\sin \frac{1}{2}C} = \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}c}.$$

$$(4) \quad \frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2}C} = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}c}.$$

From any of these formulæ the others may be obtained by the following rule:—

RULE.—Change the sign of the letter *B* (large or small) on one side of the equation, and write *sin* for *cos* and *cos* for *sin* on the other side.

PROOF.—Take  $\sin \frac{1}{2}(A+B) = \sin \frac{1}{2}A \cos \frac{1}{2}B + \cos \frac{1}{2}A \sin \frac{1}{2}B$ , substitute the *s* values by (884, 885), and reduce.

## SPHERICAL TRIANGLE AND CIRCLE.

**898** Let *r* be the radius of the inscribed circle of *ABC*; *r<sub>a</sub>* the radius of the escribed circle touching the side *a*, and *R*, *R<sub>a</sub>* the radii of the circumscribed circles; then

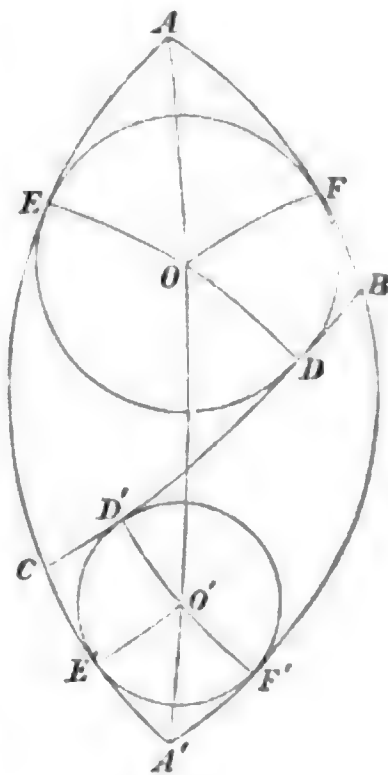
$$(1) \quad \tan r = \tan \frac{1}{2}A \sin(s-a) = \frac{\sigma}{\sin s}$$

$$(3) \quad = \frac{2 \sin a}{\sin A} \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$$

$$(4) \quad = \frac{\Sigma}{2 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C}$$

$$= \frac{2\Sigma}{\cos S + \cos(S-A) + \cos(S-B) + \&c.}$$

PROOF.—The first value is found from the right-angled triangle *OAF*, in which *AF* = *s* − *a*. The other values by (884–892).





$$899 \quad (1) \quad \tan r_a = \tan \frac{1}{2}A \sin s = \frac{\sigma}{\sin(s-a)}$$

$$(3) \quad = \frac{2 \sin a}{\sin A} \sin \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$$

$$(4) \quad = \frac{\Sigma}{2 \cos \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C}$$

$$= \frac{2\Sigma}{-\cos S - \cos(S-A) + \cos(S-B) + \cos(S-C)}.$$

PROOF.—From the right-angled triangle  $O'AF'$ , in which  $AF' = s$ .

NOTE.—The first two values of  $\tan r_a$  may be obtained from those of  $\tan r$  by interchanging  $s$  and  $s-a$ .

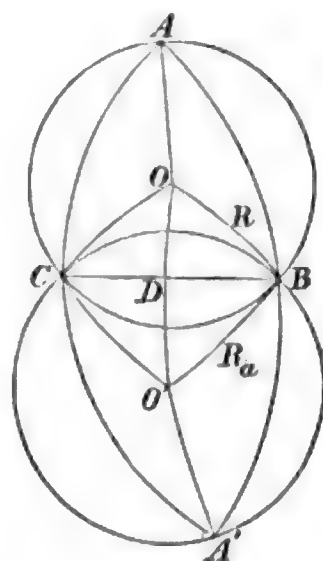
$$900 \quad (1) \quad \tan R = \frac{\tan \frac{1}{2}a}{\cos(S-A)} = \frac{-\cos S}{\Sigma}$$

$$(3) \quad = \frac{\sin \frac{1}{2}a}{\sin A \cos \frac{1}{2}b \cos \frac{1}{2}c}$$

$$(4) \quad = \frac{2 \sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c}{\sigma}$$

$$= \frac{-\sin s + \sin(s-a) + \sin(s-b) + \sin(s-c)}{2\sigma}.$$

PROOF.—The first value from the right-angled triangle  $OBD$ , in which  $\angle OBD = S-A$ . The other values by the formulæ (887-892).



$$901 \quad (1) \quad \tan R_a = \frac{\tan \frac{1}{2}a}{-\cos S} = \frac{\cos(S-A)}{\Sigma}$$

$$(3) \quad = \frac{\sin \frac{1}{2}a}{\sin A \sin \frac{1}{2}b \sin \frac{1}{2}c}$$

$$(4) \quad = \frac{2 \sin \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c}{\sigma}$$

$$(5) \quad = \frac{\sin s - \sin(s-a) + \sin(s-b) + \sin(s-c)}{2\sigma}.$$

PROOF.—From the right-angled triangle  $O'BD$ , in which  $\angle O'BD = \pi - S$ .



## SPHERICAL AREAS.

**902** area of  $ABC = (A + B + C - \pi) r^2 = Er^2$ ,

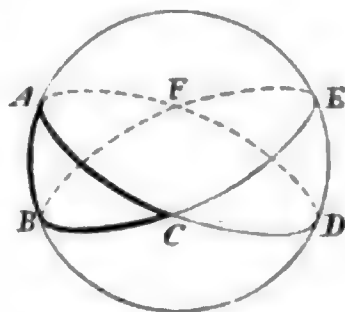
where  $E = A + B + C - \pi$ , the *spherical excess*.

PROOF.—By adding the three lunes

$ABDC$ ,  $BCEA$ ,  $CAFB$ ,

and observing that  $ABF = CDE$ ,

we get  $\left(\frac{A}{\pi} + \frac{B}{\pi} + \frac{C}{\pi}\right) 2\pi r^2 = 2\pi r^2 + 2ABC$ .



**903**

## AREA OF SPHERICAL POLYGON.

$n$  being the number of sides,

$$\begin{aligned} \text{Area} &= \{\text{Interior Angles} - (n-2)\pi\} r^2 \\ &= \{2\pi - \text{Exterior Angles}\} r^2 \\ &= \{2\pi - \text{sides of Polar Diagram}\} r^2. \end{aligned}$$

The last value holds for a curvilinear area in the limit.

PROOF.—By joining the vertices with an interior point, and adding the areas of the spherical triangles so formed.

**904**

*Cagnoli's Theorem.*

$$\sin \frac{1}{2}E = \frac{\sqrt{\{\sin s \sin (s-a) \sin (s-b) \sin (s-c)\}}}{2 \cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c}$$

PROOF.—Expand  $\sin \left[ \frac{1}{2}(A+B) - \frac{1}{2}(\pi-C) \right]$  by (628), and transform by Gauss's equations (897 i., iii.) and (669, 890).

**905**

*Llhuillier's Theorem.*

$$\tan \frac{1}{4}E = \sqrt{\tan \frac{1}{2}s \tan \frac{1}{2}(s-a) \tan \frac{1}{2}(s-b) \tan \frac{1}{2}(s-c)}.$$

PROOF.—Multiply numerator and denominator of the left side by  $2 \cos \frac{1}{4}(A+B-C+\pi)$  and reduce by (667, 668), then eliminate  $\frac{1}{2}(A+B)$  by Gauss's formulæ (897 i., iii.) Transform by (672, 673), and substitute from (886).

POLYHEDRONS.

Let the number of faces, solid angles, and edges, of any polyhedron be  $F$ ,  $S$ , and  $E$ ; then

906 
$$F + S = E + 2.$$

PROOF.—Project the polyhedron upon an internal sphere. Let  $m$  = number of sides, and  $s$  = sum of angles of one of the spherical polygons so formed. Then its area =  $\{s - (m - 2)\pi\}r^2$ , by (903). Sum this for all the polygons, and equate to  $4\pi r^2$ .

THE FIVE REGULAR SOLIDS.

Let  $m$  be the number of sides in each face,  $n$  the number of plane angles in each solid angle; therefore

907 
$$mF = nS = 2E.$$

From these equations and (906), find  $F$ ,  $S$ , and  $E$  in terms of  $m$  and  $n$ , thus,

$$\frac{1}{F} = \frac{m}{2} \left( \frac{1}{m} + \frac{1}{n} - \frac{1}{2} \right), \quad \frac{1}{S} = \frac{n}{2} \left( \frac{1}{m} + \frac{1}{n} - \frac{1}{2} \right), \quad \frac{1}{E} = \frac{1}{m} + \frac{1}{n} - \frac{1}{2}.$$

In order that  $F$ ,  $S$ , and  $E$  may be positive, we must have  $\frac{1}{m} + \frac{1}{n} > \frac{1}{2}$ , a relation which admits of five solutions in whole numbers, corresponding to the five regular solids. The values of  $m$ ,  $n$ ,  $F$ ,  $S$ , and  $E$  for the five regular solids are exhibited in the following table:—

	$m$	$n$	$F$	$S$	$E$
Tetrahedron ... ..	3	3	4	4	6
Hexahedron ... ..	4	3	6	8	12
Octahedron ... ..	3	4	8	6	12
Dodecahedron ... ..	5	3	12	20	30
Icosahedron .. ..	3	5	20	12	30

908 The sum of all the plane angles of any polyhedron  

$$= 2\pi(S - 2);$$

Or, Four right angles for every vertex less eight right angles.  
 2 c

**909** If  $I$  be the angle between two adjacent faces of a regular polyhedron,

$$\sin \frac{1}{2}I = \cos \frac{\pi}{n} \div \sin \frac{\pi}{m}.$$

PROOF.—Let  $PQ = a$  be the edge, and  $S$  the centre of a face,  $T$  the middle point of  $PQ$ ,  $O$  the centre of the inscribed and circumscribed spheres,  $ABC$  the projection of  $PST$  upon a concentric sphere. In this spherical triangle,

$$C = \frac{\pi}{2}, \quad A = \frac{\pi}{n}, \quad \text{and} \quad B = \frac{\pi}{m} = PST.$$

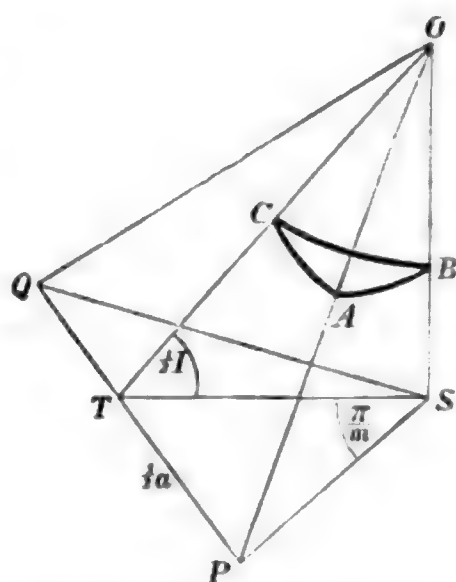
Also  $STO = \frac{1}{2}I$ .

Now, by (881, ii.),

$$\cos A = \sin B \cos BC;$$

that is,  $\cos \frac{\pi}{n} = \sin \frac{\pi}{m} \sin \frac{1}{2}I$ .

Q. e. d.



If  $r, R$  be the radii of the inscribed and circumscribed spheres of a regular polyhedron,

$$\mathbf{910} \quad r = \frac{a}{2} \tan \frac{1}{2}I \cot \frac{\pi}{m}, \quad R = \frac{a}{2} \tan \frac{1}{2}I \tan \frac{\pi}{n}.$$

PROOF.—In the above figure,  $OS = r$ ,  $OP = R$ ,  $PT = \frac{a}{2}$ ; and  $OS = PT \cot \frac{\pi}{m} \tan \frac{1}{2}I$ . Also  $OP = PT \operatorname{cosec} AC$ , and by (881, i.),

$$\sin AC = \tan BC \cot A = \cot \frac{1}{2}I \cot \frac{\pi}{n}; \text{ therefore, \&c.}$$

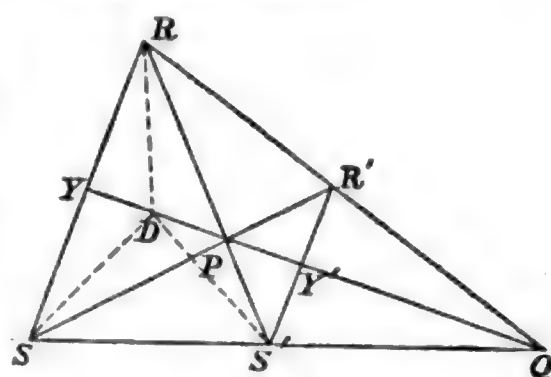
# ELEMENTARY GEOMETRY.

## MISCELLANEOUS PROPOSITIONS.

**920** To find the point in a given line  $QY$ , the sum of whose distances from two fixed points  $S, S'$  is a minimum.

Draw  $SYR$  at right angles to  $QY$ , making  $YR = YS$ . Join  $RS'$ , cutting  $QY$  in  $P$ . Then  $P$  will be the required point.

**PROOF.**—For, if  $D$  be any other point on the line,  $SD = DR$  and  $SP = PR$ . But  $RD + DS'$  is  $> RS'$ ; therefore, &c.  $R$  is called the reflection of the point  $S$ , and  $SPS'$  is the path of a ray of light reflected at the line  $QY$ .



If  $S, S'$  and  $QY$  are not in the same plane, make  $SY, YR$  equal perpendiculars as before, but the last in the plane of  $S'$  and  $QY$ .

Similarly, the point  $Q$  in the given line, the difference of whose distances from the fixed points  $S$  and  $R'$  is a maximum, is found by a like construction.

The minimum sum of distances from  $S, S'$  is given by

$$(SP + S'P)^2 = SS'^2 + 4SY \cdot S'Y'.$$

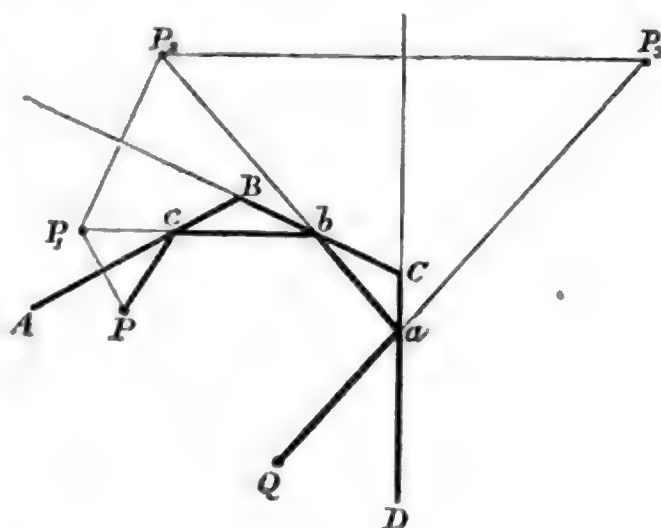
And the maximum difference from  $S$  and  $R'$  is given by

$$(SQ - R'Q)^2 = (SR')^2 - 4SY \cdot R'Y'.$$

Proved by VI. D., since  $SRR'S'$  can be inscribed in a circle.

**921** Hence, to find the shortest distance from  $P$  to  $Q$  *en route* of the lines  $AB, BC, CD$ ; in other words, the path of the ray reflected at the successive surfaces  $AB, BC, CD$ .

Find  $P_1$ , the reflection of  $P$  at the first surface; then  $P_2$ , the reflection of  $P_1$  at the second surface; next  $P_3$ , the reflection of  $P_2$  at the third surface; and so on if



there be more surfaces. Lastly, join  $Q$  with  $P_3$ , the last reflection, cutting  $CD$  in  $a$ . Join  $aP_2$ , cutting  $BO$  in  $b$ . Join  $bP_1$ , cutting  $AB$  in  $c$ . Join  $cP$ .  $PcbaQ$  is the path required.

The same construction will give the path when the surfaces are not, as in the case considered, all perpendicular to the same plane.

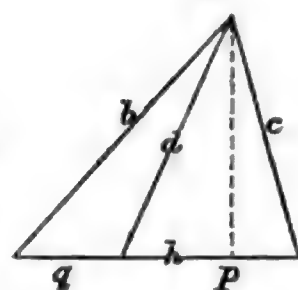
**922** If the straight line  $d$  from the vertex of a triangle divide the base into segments  $p, q$ , and if  $h$  be the distance from the point of section to the foot of the perpendicular from the vertex on the base, then

$$b^2 + c^2 = p^2 + q^2 + 2d^2 + 2h(p - q). \quad [\text{II. 12, 13.}]$$

The following cases are important:—

(i.) When  $p = q$ ,  $b^2 + c^2 = 2q^2 + 2d^2$ ;

*i.e., the sum of the squares of two sides of a triangle is equal to twice the square of half the base, together with twice the square of the bisecting line drawn from the vertex.*



(ii.) When  $p = 2q$ ,  $2b^2 + c^2 = 6q^2 + 3d^2$ . (II. 12 or 13)

(iii.) When the triangle is isosceles,

$$b^2 = c^2 = pq + d^2.$$

**923** If  $O$  be the centre of an equilateral triangle  $ABC$  and  $P$  any point in space. Then

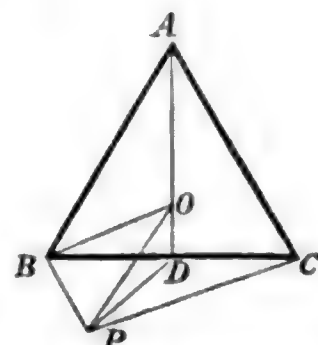
$$PA^2 + PB^2 + PC^2 = 3(PO^2 + OA^2).$$

PROOF.—  $PB^2 + PC^2 = 2PD^2 + 2BD^2$ . (922, i.)

Also  $PA^2 + 2PD^2 = 6OD^2 + 3PO^2$ , (922, ii.)

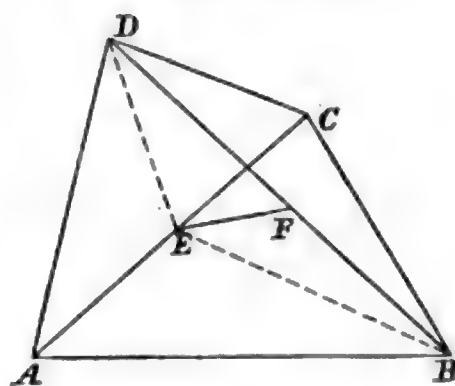
and  $BO = 2OD$ ;

therefore, &c.



COR.—Hence, if  $P$  be any point on the surface of a sphere, centre  $O$ , the sum of the squares of its distances from  $A, B, C$  is constant. And if  $r$ , the radius of the sphere, be equal to  $OA$ , the sum of the same squares is equal to  $6r^2$ .

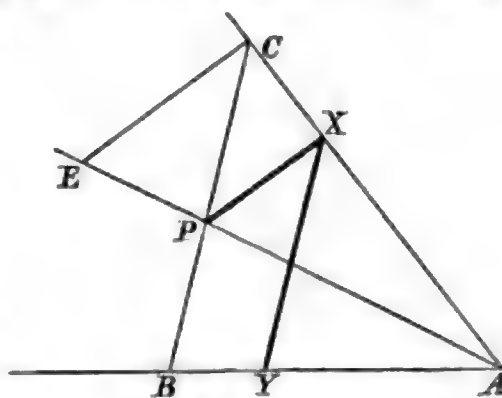
**924** The sum of the squares of the sides of a quadrilateral is equal to the sum of the squares of the diagonals plus four times the square of the line joining the middle points of the diagonals. (922, i.)



**925** Cor.—The sum of the squares of the sides of a parallelogram is equal to the sum of the squares of the diagonals.

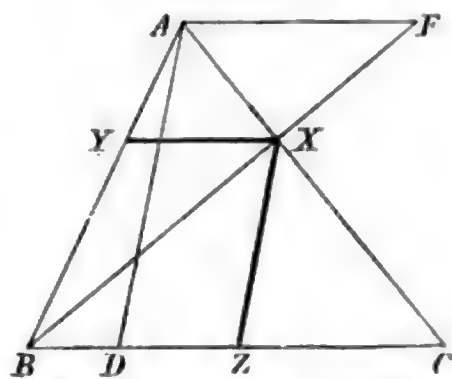
**926** In a given line  $AC$ , to find a point  $X$  whose distance from a point  $P$  shall have a given ratio to its distance in a given direction from a line  $AB$ .

Through  $P$  draw  $BPC$  parallel to the given direction. Produce  $AP$ , and make  $CE$  in the given ratio to  $CB$ . Draw  $PX$  parallel to  $EC$ , and  $XY$  to  $CB$ . There are two solutions when  $CE$  cuts  $AP$  in two points. [PROOF.—By (VI. 2).]



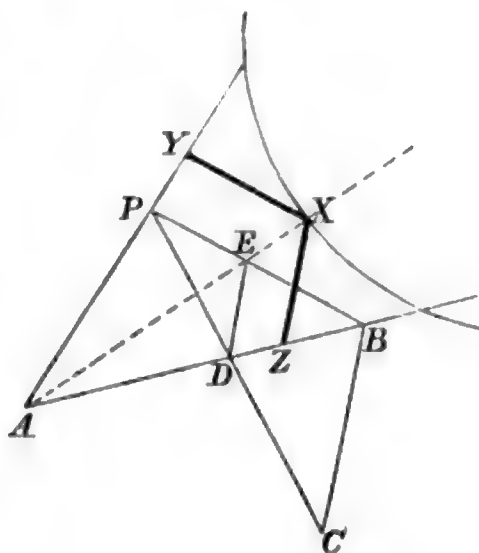
**927** To find a point  $X$  in  $AC$ , whose distance  $XY$  from  $AB$  parallel to  $BC$  shall have a given ratio to its distance  $XZ$  from  $BC$  parallel to  $AD$ .

Draw  $AE$  parallel to  $BC$ , and having to  $AD$  the given ratio. Join  $BE$  cutting  $AC$  in  $X$ , the point required. [Proved by (VI. 2).]



**928** To find a point  $X$  on any line, straight or curved, whose distances  $XY$ ,  $XZ$ , in given directions from two given lines  $AP$ ,  $AB$ , shall be in a given ratio.

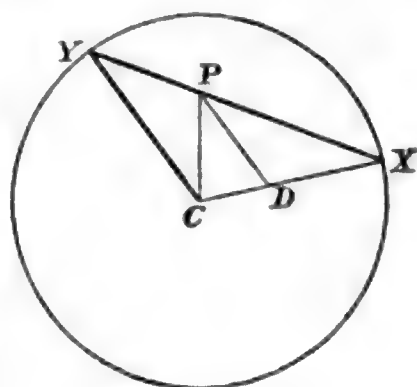
Take  $P$  any point in the first line. Draw  $PB$  parallel to the direction of  $XY$ , and  $BC$  parallel to that of  $XZ$ , making  $PB$  have to  $BC$  the given ratio. Join  $PC$ , cutting  $AB$  in  $D$ . Draw  $DE$  parallel to  $CB$ . Then  $AE$  produced cuts the line in  $X$ , the point required, and is the locus of such points. [PROOF.—By (VI. 2).]



**929** To draw a line  $XY$  through a given point  $P$  so that the segments  $XP$ ,  $PY$ , intercepted by a given circle, shall be in a given ratio.

Divide the radius of the circle in that ratio, and, with the parts for sides, construct a triangle  $PDC$  upon  $PC$  as base. Produce  $CD$  to cut the circle in  $X$ . Draw  $XPY$  and  $CY$ .

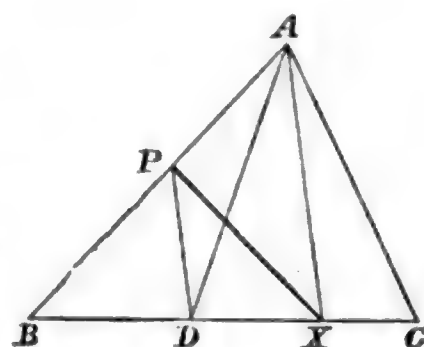
Then  $PD + DC = \text{radius}$ ;  
 therefore  $PD = DX$ ;  
 But  $CY = CX$ ;  
 therefore  $PD$  is parallel to  $CY$  (I. 5, 28); therefore  
 &c., by (VI. 2).



**930** From a given point  $P$  in the side of a triangle, to draw a line  $PX$  which shall divide the area of the triangle in a given ratio.

Divide  $BC$  in  $D$  in the given ratio, and draw  $AX$  parallel to  $PD$ .  $PX$  will be the line required.

$ABD : ADC = \text{the given ratio}$  (VI. 1), and  
 $APD = XPD$  (I. 37); therefore, &c.



**931** To divide the triangle  $ABC$  in a given ratio by a line  $XY$  drawn parallel to any given line  $AE$ .

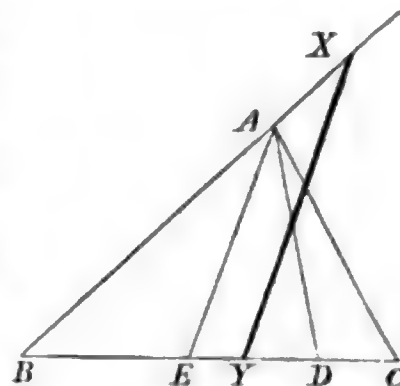
Make  $BD$  to  $DC$  in the given ratio. Then make  $BY$  a mean proportional to  $BE$  and  $BD$ , and draw  $YX$  parallel to  $EA$ .

PROOF.— $AD$  divides  $ABO$  in the given ratio (VI. 1). Now

$$ABE : XBY :: BE : BD, \quad (\text{VI. 19})$$

or  $ABE : ABD;$

therefore  $XBY = ABD.$



**932** If the interior and exterior vertical angles at  $P$  of the triangle  $APB$  be bisected by straight lines which cut the base in  $C$  and  $D$ , then the circle circumscribing  $CPD$  gives the locus of the vertices of all triangles on the base  $AB$  whose sides  $AP$ ,  $BP$  are in a constant ratio.

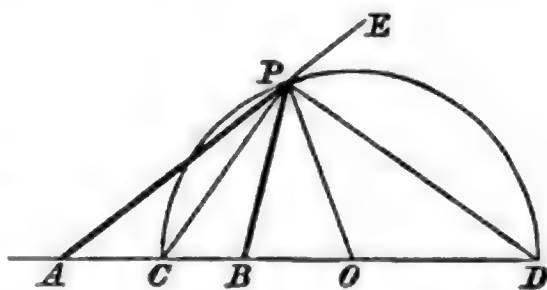


PROOF.—

The  $\angle CPD = \frac{1}{2} (APB + BPE)$   
 $=$  a right angle;

therefore  $P$  lies on the circumference of the circle, diameter  $CD$  (III. 31). Also

$AP : PB :: AC : CB :: AD : DB$   
 (VI. 3, and A.), a fixed ratio.



**933**  $AD$  is divided harmonically in  $B$  and  $C$ ; i.e.,  $AD : DB :: AC : CB$ ; or, the whole line is to one extreme part as the other extreme part is to the middle part. If we put  $a, b, c$  for the lengths  $AD, BD, CD$ , the proportion is expressed algebraically by  $a : b :: a - c : c - b$ , which is equivalent to

$$\frac{1}{a} + \frac{1}{b} = \frac{2}{c}.$$

**934** Also  $AP : BP = OA : OC = OC : OB$

and  $AP^2 : BP^2 = OA : OB$ , (VI. 19)  
 $AP^2 - AC^2 : CP^2 : BP^2 - BC^2$ . (VI. 3, & B.)

**935** If  $Q$  be the centre of the inscribed circle of the triangle  $ABC$ , and if  $AQ$  produced meet the circumscribed circle, radius  $R$ , in  $F$ ; and if  $FOG$  be a diameter, and  $AD$  perpendicular to  $BC$ ; then

(i.)  $FC = FQ = FB = 2R \sin \frac{A}{2}$ .

(ii.)  $\angle FAD = FAO = \frac{1}{2} (B - C)$ ,  
 and  $\angle CAG = \frac{1}{2} (B + C)$ .

PROOF OF (i.)—

$$\angle FQC = QOA + QAO.$$

But  $QAC = QAB = BCF$ ; (III. 21)

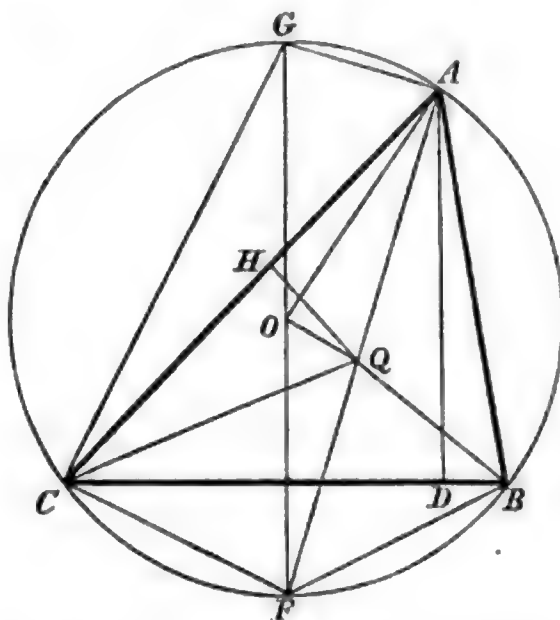
$$\therefore FQC = FCQ; \quad \therefore FC = FQ.$$

Similarly  $FB = FQ$ .

Also  $\angle GCF$  is a right angle, and

$$FGC = FAC = \frac{1}{2} A; \quad (\text{III. 21})$$

$$\therefore FC = 2R \sin \frac{A}{2}.$$



**936** If  $R, r$  be the radii of the circumscribed and inscribed circles of the triangle  $ABC$  (see last figure), and  $O, Q$  the centres; then  $OQ^2 = R^2 - 2Rr$ .

PROOF.—Draw  $QH$  perpendicular to  $AO$ ; then  $QH = r$ . By the isosceles triangle  $AOH$ ,  $OQ^2 = R^2 - AQ \cdot QF$  (922, iii.), and  $QF = FC$  (935, i.), and by similar triangles  $GFC, AQH$ ,  $AQ : QH :: GF : FC$ ;

therefore  $AQ \cdot FC = GF \cdot QH = 2Rr$ .





**944** COR.—The tangents from  $O$  to all circles which touch the given circles, either both externally or both internally, are equal.

For the square of the tangent is always equal to  $OK \cdot ON$  or  $OL \cdot OM$ .

**945** The solutions for the cases of three given straight lines or three given points are to be found in Euc. IV., Props. 4, 5.

**946** In the remaining cases of the tangencies, straight lines and circles alone are given. By drawing a circle concentric with the required one through the centre of the least given circle, the problem can always be made to depend upon one of the preceding cases; the centre of the least circle becoming one of the *given* points.

**947** DEFINITION.—A *centre of similarity* of two plane curves is a point such that, any straight line being drawn through it to cut the curves, the segments of the line intercepted between the point and the curves are in a constant ratio.

**948** If  $AB, AC$  touch a circle at  $B$  and  $C$ , then any straight line  $AEDF$ , cutting the circle, is divided harmonically by the circumference and the chord of contact  $BC$ .

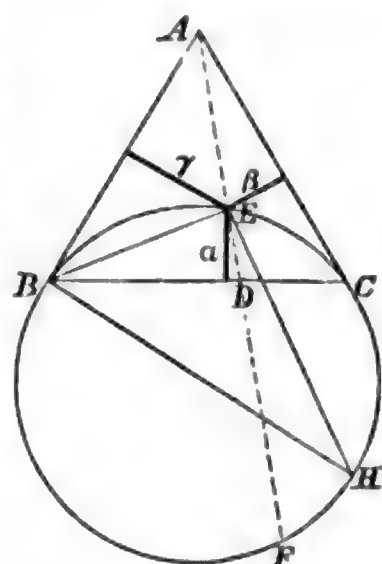
Proof from  $AE \cdot AF = AB^2$ . (III. 36)

$AB^2 = BD \cdot DC + AD^2$ , (923)

and  $BD \cdot DC = ED \cdot DF$ . (III. 35)

**949** If  $\alpha, \beta, \gamma$ , in the same figure, be the perpendiculars to the sides of  $ABC$  from any point  $E$  on the circumference of the circle, then  $\beta\gamma = \alpha^2$ .

PROOF.—Draw the diameter  $BH \equiv d$ ; then  $EB^2 = \beta d$ , because  $BEH$  is a right angle. Similarly  $EC^2 = \gamma d$ . But  $EB \cdot EC = \alpha d$  (VI. D.), therefore &c.



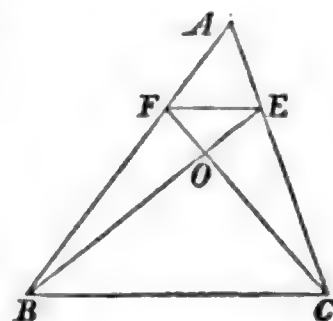
**950** If  $FE$  be drawn parallel to the base  $BC$  of a triangle, and if  $EB, FC$  intersect in  $O$ , then

$$AE : AC :: EO : OB :: FO : OC.$$

By VI. 2. Since each ratio =  $FE : BC$ .

COR.—If  $AC = n \cdot AE$ , then

$$BE = (n+1) OE.$$



**951** The three lines drawn from the angles of a triangle to the middle points of the opposite sides, intersect in the same point, and divide each other in the ratio of two to one.

For, by the last theorem, any one of these lines is divided by each of the others in the ratio of two to one, measuring from the same extremity, and must therefore be intersected by them in the same point.

This point will be referred to as the *centroid* of the triangle.

**952** The perpendiculars from the angles upon the opposite sides of a triangle intersect in the same point.

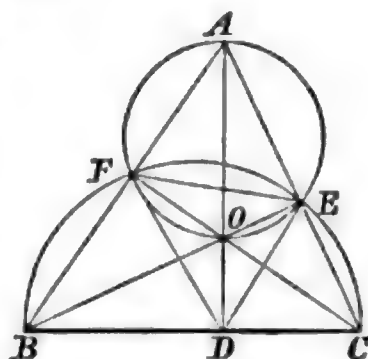
Draw  $BE$ ,  $CF$  perpendicular to the sides, and let them intersect in  $O$ . Let  $AO$  meet  $BC$  in  $D$ . Circles will circumscribe  $AEOF$  and  $BFEC$ , by (III. 31);

therefore  $\angle FAO = FEO = FCB$ ; (III. 21)

therefore  $\angle BDA = BFC = \text{a right angle}$ ;

i.e.,  $AO$  is perpendicular to  $BC$ , and therefore the perpendicular from  $A$  on  $BC$  passes through  $O$ .

$O$  is called the *orthocentre* of the triangle  $ABC$ .



COR.—The perpendiculars on the sides bisect the angles of the triangle  $DEF$ , and the point  $O$  is therefore the centre of the inscribed circle of that triangle.

PROOF.—From (III. 21), and the circles circumscribing  $OFAE$  and  $OECB$ .

**953** If the inscribed circle of a triangle  $ABC$  touches the sides  $a$ ,  $b$ ,  $c$  in the points  $D$ ,  $E$ ,  $F$ ; and if the escribed circle to the side  $a$  touches  $a$  and  $b$ ,  $c$  produced in  $D'$ ,  $E'$ ,  $F'$ ; and if

$$s = \frac{1}{2}(a+b+c);$$

then

$$BF' = BD' = CD = s - c,$$

$$\text{and } AE' = AF' = s;$$

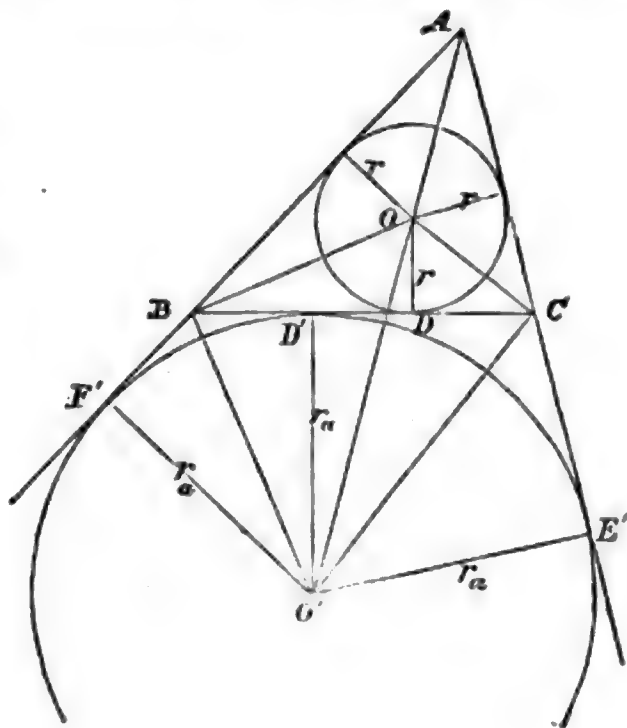
and similarly with respect to the other segments.

PROOF.—The two tangents from any vertex to either circle being equal, it follows that  $CD + c = \text{half the perimeter of } ABC$ , which is made up of three pairs of equal segments; therefore  $CD = s - c$ .

Also

$$AE' + AF' = AC + CD' + AB + BD' = 2s;$$

$$\text{therefore } AE' = AF' = s.$$





### The Nine-Point Circle.

**954** The *Nine-point* circle is the circle described through  $D, E, F$ , the feet of the perpendiculars on the sides of the triangle  $ABC$ . It also passes through the middle points of the sides of  $ABC$  and the middle points of  $OA, OB, OC$ ; in all, through nine points.

**PROOF.**—Let the circle cut the sides of  $ABC$  again in  $G, H, K$ ; and  $OA, OB, OC$  in  $L, M, N$ .  
 $\angle EMF = EDF$  (III. 21)  
 $= 2ODF$  (952, Cor.);  
 therefore, since  $OB$  is the diameter of the circle circumscribing  $OFBD$  (III. 31),  $M$  is the centre of that circle (III. 20), and therefore bisects  $OB$ .

Similarly  $OC$  and  $OA$  are bisected at  $N$  and  $L$ .

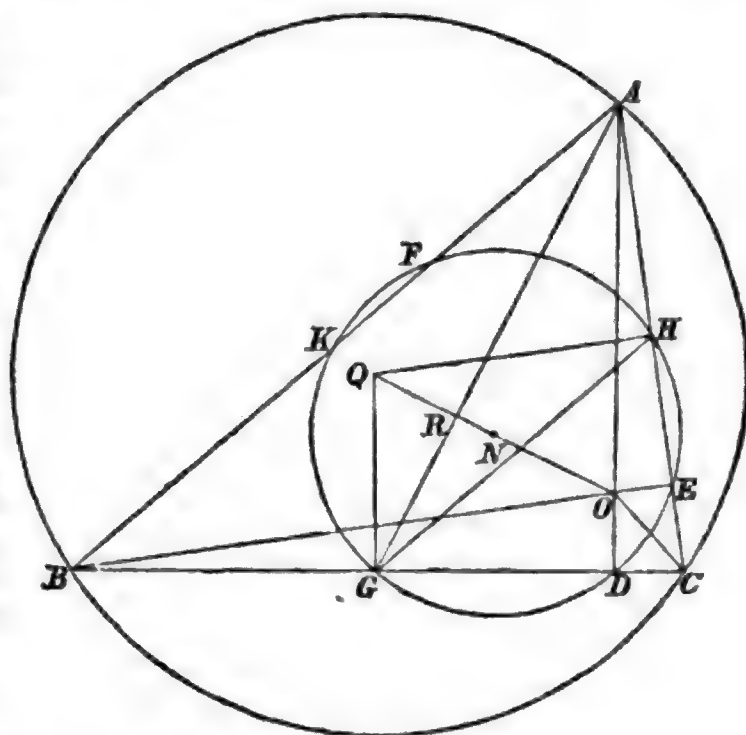
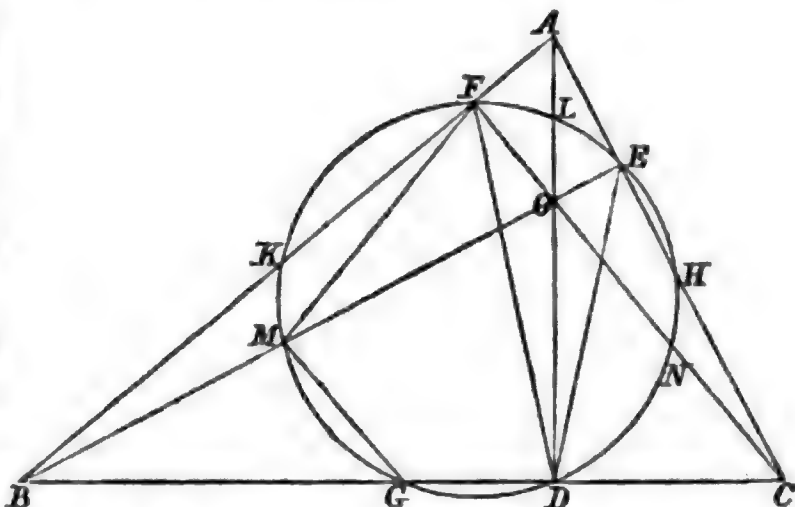
Again,  $\angle MGB = MED$  (III. 22)  $= \angle OCD$ , (III. 21), by the circle circumscribing  $OECD$ . Therefore  $MG$  is parallel to  $OC$ , and therefore bisects  $BC$ . Similarly  $H$  and  $K$  bisect  $CA$  and  $AB$ .

**955** The centre of the nine-point circle is the middle point of  $OQ$ , the line joining the orthocentre and the centre of the circumscribing circle of the triangle  $ABC$ .

For the centre of the N. P. circle is the intersection of the perpendicular bisectors of the chords  $DG, EH, FK$ , and these perpendiculars bisect  $OQ$  in the same point  $N$ , by (VI. 2).

**956** The centroid of the triangle  $ABC$  also lies on the line  $OQ$  and divides it in  $R$  so that  $OR = 2RQ$ .

**PROOF.**—The triangles  $QHG, OAB$  are similar, and  $AB = 2HG$ ; therefore  $AO = 2GQ$ ; therefore  $OR = 2RQ$ ; and  $AR = 2RG$ ; therefore  $R$  is the centroid, and it divides  $OQ$  as stated (951).



**957** Hence the line joining the centres of the circumscribed and nine-point circles is divided harmonically in the ratio of 2 : 1 by the centroid and the orthocentre of the triangle.

These two points are therefore centres of similitude of the circumscribed and nine-point circles; and any line drawn through either of the points is divided by the circumferences in the ratio of 2 : 1. See (1037.)

**958** The lines  $DE$ ,  $EF$ ,  $FD$  intersect the sides of  $ABC$  in the radical axis of the two circles.

For, if  $EF$  meets  $BC$  in  $P$ , then by the circle circumscribing  $BCEF$ ,  $PE \cdot PF = PC \cdot PB$ ; therefore (III. 36) the tangents from  $P$  to the circles are equal (985).

**959** The nine-point circle touches the inscribed and escribed circles of the triangle.

PROOF.—Let  $O$  be the orthocentre, and  $I$ ,  $Q$  the centres of the inscribed and circumscribed circles. Produce  $AI$  to bisect the arc  $BC$  in  $T$ . Bisect  $AO$  in  $L$ , and join  $GL$ , cutting  $AT$  in  $S$ .

The N. P. circle passes through  $G$ ,  $D$ , and  $L$  (954), and  $D$  is a right angle. Therefore  $GL$  is a diameter, and is therefore  $= R = QA$  (957). Therefore  $GL$  and  $QA$  are parallel. But  $QA = QT$ , therefore

$$GS = GT = CT \sin \frac{A}{2} = 2R \sin^2 \frac{A}{2}. \quad (935, i.)$$

Also  $ST = 2GS \cos \theta$   
( $\theta$  being the angle  $GST = GTS$ ).

$N$  being the centre of the N. P. circle, its radius  $= NG = \frac{1}{2}R$ ; and  $r$  being the radius of the inscribed circle, it is required to shew that

$$NI = NG - r.$$

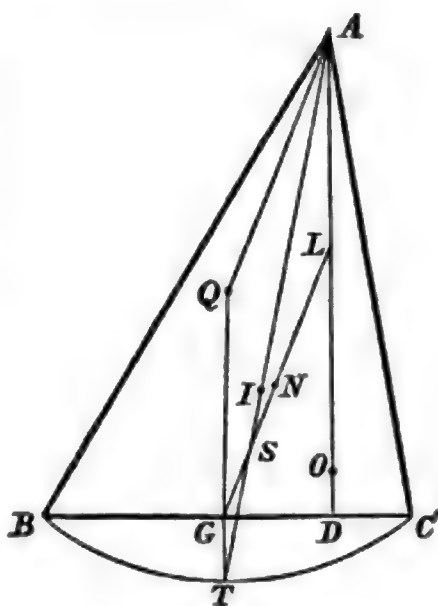
Now  $NI^2 = SN^2 + SI^2 - 2SN \cdot SI \cos \theta.$

Substitute  $SN = \frac{1}{2}R - GS;$

$$SI = TI - ST = 2R \sin \frac{A}{2} - 2GS \cos \theta;$$

and  $GS = 2R \sin^2 \frac{A}{2}$ , to prove the proposition.

If  $J$  be the centre of the escribed circle touching  $BC$ , and  $r_s$  its radius, it is shewn in a similar way that  $NJ = NG + r_s$ .



(702)

*To construct a triangle from certain data.*

**960** When amongst the data we have the sum or difference of the two sides  $AB$ ,  $AC$ ; or the sum of the segments of the base made by  $AG$ , the bisector of the exterior vertical angle; or the difference of the segments made by  $AF$ , the bisector of





equal to a mean proportional between  $DB$  and  $DC$ , describe a circle. It will be the required locus.

PROOF.—Suppose  $P$  to be a point on the required locus. Join  $P$  with  $A, B, C$ , and  $D$ .

Describe a circle about  $PBC$  cutting  $AP$  in  $F$ , and another about  $ABF$  cutting  $PB$  in  $G$ , and join  $AG$  and  $BF$ . Then

$$PK^2 = AP^2 - AK^2 = AI^2 - BA \cdot AC \text{ (by constr.)} = AP^2 - PA \cdot AF \text{ (III. 36)} \\ = AP \cdot PF \text{ (II. 2)} = GP \cdot PB \text{ (III. 36)}.$$

Therefore, by hypothesis,

$$p^2 : q^2 = GP \cdot PB : PB^2 = GP : PB = AD : DB \text{ (by constr.)};$$

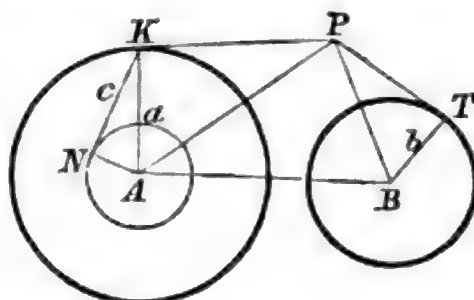
therefore  $\angle DPG = PGA$  (VI. 2)  $= PFB$  (III. 22)  $= PCB$  (III. 21).

Therefore the triangles  $DPB, DCP$  are similar; therefore  $DP$  is a mean proportional to  $DB$  and  $DC$ . Hence the construction.

**964** COR.—If  $p = q$  the locus becomes the perpendicular bisector of  $BC$ , as is otherwise shown in (1003).

**965** To find the locus of a point  $P$ , the tangents from which to two given circles shall have a given ratio. (See also 1036.)

Let  $A, B$  be the centres,  $a, b$  the radii ( $a > b$ ), and  $p : q$  the given ratio. Take  $c$ , so that  $c : b = p : q$ , and describe a circle with centre  $A$  and radius  $AN = \sqrt{a^2 - c^2}$ . Find the locus of  $P$  by the last proposition, so that the tangent from  $P$  to this circle may have the given ratio to  $PB$ . It will be the required locus.



PROOF.—By hypothesis and construction,

$$\frac{p^2}{q^2} = \frac{PK^2}{PT^2} = \frac{c^2}{b^2} = \frac{PK^2 + c^2}{PT^2 + b^2} = \frac{AP^2 - a^2 + c^2}{BP^2} = \frac{AP^2 - AN^2}{BP^2}.$$

COR.—Hence the point can be found on any curve from which the tangents to two circles shall have a given ratio.

**966** To find the locus of the point from which the tangents to two given circles are equal.

Since, in (965), we have now  $p = q$ , and therefore  $c = b$ , the construction simplifies to the following:

Take  $AN = \sqrt{a^2 - b^2}$ , and in  $AB$  take  $AB : AN : AC$ . The perpendicular bisector of  $BC$  is the required locus. But, if the circles intersect, then their common chord is at once the line required. See Radical Axis (985).

*Collinear and Concurrent systems of points and lines.*

**967** DEFINITIONS.—Points lying in the same straight line are *collinear*. Straight lines passing through the same point are *concurrent*, and the point is called the *focus* of the pencil of lines.

**Theorem.**—If the sides of the triangle  $ABC$ , or the sides produced, be cut by any straight line in the points  $a, b, c$  respectively, the line is called a *transversal*, and the segments of the sides are connected by the equation

**968**  $(Ab : bC)(Ca : aB)(Bc : cA) = 1.$

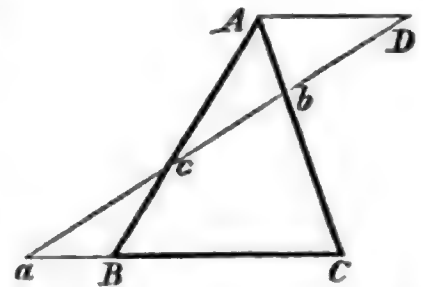
Conversely, if this relation holds, the points  $a, b, c$  will be collinear.

PROOF.—Through any vertex  $A$  draw  $AD$  parallel to the opposite side  $BC$ , to meet the transversal in  $D$ , then

$$Ab : bC = AD : Ca \text{ and } Bc : cA = aB : AD$$

(VI. 4), which proves the theorem.

NOTE.—In the formula the segments of the sides are estimated positive, independently of direction, the sequence of the letters being preserved the better to assist the memory. A point may be supposed to travel from  $A$  over the segments  $Ab, bC$ , &c. *continuously*, until it reaches  $A$  again.



**969** By the aid of (701) the above relation may be put in the form

$$(\sin ABb : \sin bBC)(\sin CAa : \sin aAB)(\sin BCc : \sin cCA) = 1$$

**970** If  $O$  be any focus in the plane of the triangle  $ABC$ , and if  $AO, BO, CO$  meet the sides in  $a, b, c$ ; then, as before,

$$(Ab : bC)(Ca : aB)(Bc : cA) = 1.$$

Conversely, if this relation holds, the lines  $Aa, Bb, Cc$  will be concurrent.

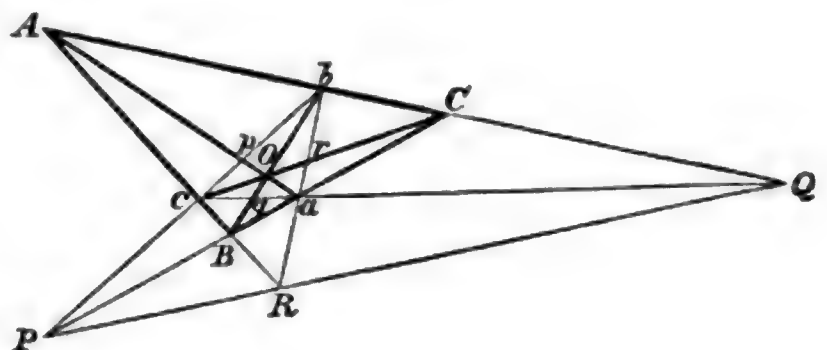
PROOF.—By the transversal  $Bb$  to the triangle  $AaC$ , we have (968)

$$(Ab : bC)(CB : Ba) \times (aO : OA) = 1.$$

And, by the transversal  $Cc$  to the triangle  $AaB$ ,

$$(Bc : cA)(AO : Oa) \times (aC : CB) = 1.$$

Multiply these equations together.



**971** If  $bc, ca, ab$ , in the last figure, be produced to meet the sides of  $ABC$  in  $P, Q, R$ , then each of the nine lines in the figure will be divided harmonically, and the points  $P, Q, R$  will be collinear.

PROOF.—(i.) Take  $bP$  a transversal to  $ABC$ ; therefore, by (968),

$$(CP : PB) (Bc : cA) (Ab : bC) = 1;$$

therefore, by (970),  $CP : PB = Ca : aB$ .

(ii.) Take  $CP$  a transversal to  $Abc$ , therefore

$$(AB : Bc) (cP : Pb) (bC : CA) = 1.$$

But, by (970), taking  $O$  for focus to  $Abc$ ,

$$(AB : Bc) (cp : pb) (bC : CA) = 1;$$

therefore  $cP : Pb = cp : pb$ .

(iii.) Take  $PC$  a transversal to  $AOc$ , and  $b$  a focus to  $AOc$ ; therefore, by (968 & 970),

$$(Aa : aO) (OC : Cc) (cB : BA) = 1,$$

and  $(Ap : pO) (OC : Cc) (cB : BA) = 1;$

therefore  $Aa : aO = Ap : pO$ .

Thus all the lines are divided harmonically.

(iv.) In the equation of (970) put  $Ab : bC = AQ : QC$  the harmonic ratio, and similarly for each ratio, and the result proves that  $P, Q, R$  are collinear, by (968).

COR.—If in the same figure  $qr, rp, pq$  be joined, the three lines will pass through  $P, Q, R$  respectively.

PROOF.—Take  $O$  as a focus to the triangle  $abc$ , and employ (970) and the harmonic division of  $bc$  to show that the transversal  $rq$  cuts  $bc$  in  $P$ .

**972** If a transversal intersects the sides  $AB, BC, CD$ , &c. of any polygon in the points  $a, b, c$ , &c. in order, then

$$(Aa : aB) (Bb : bC) (Cc : cD) (Dd : dE) \dots \&c. = 1.$$

PROOF.—Divide the polygon into triangles by lines drawn from one of the angles, and, applying (968) to each triangle, combine the results.

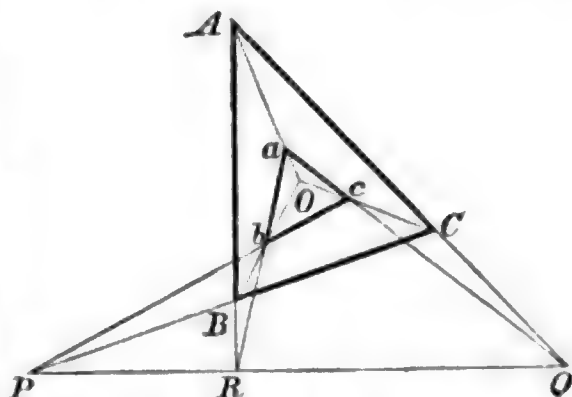
**973** Let any transversal cut the sides of a triangle and their three intersectors  $AO, BO, CO$  (see figure of 970) in the points  $A', B', C', a', b', c'$ , respectively; then, as before,

$$(A'b' : b'C') (C'a' : a'B') (B'c' : c'A') = 1.$$

PROOF.—Each side forms a triangle with its intersector and the transversal. Take the four remaining lines in succession for transversals to each triangle, applying (968) symmetrically, and combine the twelve equations.

**974** If the lines joining corresponding vertices of two triangles  $ABC$ ,  $abc$  are concurrent, the points of intersection of the pairs of corresponding sides are collinear, and conversely.

PROOF.—Let the concurrent lines  $Aa$ ,  $Bb$ ,  $Cc$  meet in  $O$ . Take  $bc$ ,  $ca$ ,  $ab$  transversals respectively to the triangles  $OBC$ ,  $OCA$ ,  $OAB$ , applying (968), and the product of the three equations shows that  $P$ ,  $R$ ,  $Q$  lie on a transversal to  $ABC$ .



**975** Hence it follows that, if the lines joining each pair of corresponding vertices of any two rectilinear figures are concurrent, the pairs of corresponding sides intersect in points which are collinear.

The figures in this case are said to be *in perspective*, or *in homology*, with each other. The point of concurrence and the line of collinearity are called respectively the *centre* and *axis* of perspective or homology. See (1083).

**976 Theorem.**—When three perpendiculars to the sides of a triangle  $ABC$ , intersecting them in the points  $a$ ,  $b$ ,  $c$  respectively, are *concurrent*, the following relation is satisfied; and conversely, if the relation be satisfied, the perpendiculars are concurrent.

$$Ab^2 - bC^2 + Ca^2 - aB^2 + Bc^2 - cA^2 = 0.$$

PROOF.—If the perpendiculars meet in  $O$ , then  $Ab^2 - bC^2 = AO^2 - OC^2$ , &c. (I. 47).

EXAMPLES.—By the application of this theorem, the concurrence of the three perpendiculars is readily established in the following cases:—

- (1) When the perpendiculars bisect the sides of the triangle
- (2) When they pass through the vertices. (By employing I. 47.)
- (3) The three radii of the escribed circles of a triangle at the points of contact between the vertices are concurrent. So also are the radii of the inscribed circle at the point of contact with one side, and the radii of the two escribed circles of the remaining sides at the points of contact beyond the included angle.

In these cases employ the values of the segments given in (953).

- (4) The perpendiculars equidistant from the vertices with three concurrent perpendiculars are also concurrent.

- (5) When the three perpendiculars from the vertices of one triangle upon the sides of the other are concurrent, then the perpendiculars from the vertices of the second triangle upon the sides of the first are also concurrent.

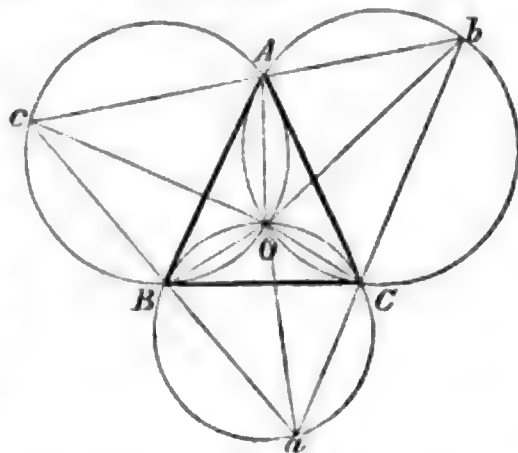
PROOF.—If  $A$ ,  $B$ ,  $C$  and  $A'$ ,  $B'$ ,  $C'$  are corresponding vertices of the triangles, join  $AB'$ ,  $AC'$ ,  $BC'$ ,  $BA'$ ,  $CA'$ ,  $CB'$ , and apply the theorem in conjunction with (I. 47).



*Triangles of constant species circumscribed to a triangle.*

**977** Let  $ABC$  be any triangle, and  $O$  any point; and let circles circumscribe  $AOB$ ,  $BOC$ ,  $COA$ . The circumferences will be the loci of the vertices of a triangle of constant form whose sides pass through the points  $A$ ,  $B$ ,  $C$ .

PROOF.—Draw any line  $bAc$  from circle to circle, and produce  $bC$ ,  $cB$  to meet in  $a$ . The angles  $AOB$ ,  $COA$  are supplements of the angles  $c$  and  $b$  (III. 22); therefore  $BOC$  is the supplement of  $a$  (I. 32); therefore  $a$  lies on the circle  $OBC$ . Also, the angles at  $O$  being constant, the angles  $a$ ,  $b$ ,  $c$  are constant.



**978** The triangle  $abc$  is a maximum when its sides are perpendicular to  $OA$ ,  $OB$ ,  $OC$ .

PROOF.—The triangle is greatest when its sides are greatest. But the sides vary as  $Oa$ ,  $Ob$ ,  $Oc$ , which are greatest when they are diameters of the circles; therefore &c., by (III. 31).

**979** To construct a triangle of given species and of given limited magnitude which shall have its sides passing through three given points  $A$ ,  $B$ ,  $C$ .

Determine  $O$  by describing circles on the sides of  $ABC$  to contain angles equal to the supplements of the angles of the specified triangle. Construct the figure  $abcO$  independently from the known sides of  $abc$ , and the now known angles  $ObC = OAC$ ,  $OaC = OBC$ , &c. Thus the lengths  $Oa$ ,  $Ob$ ,  $Oc$  are found, and therefore the points  $a$ ,  $b$ ,  $c$ , on the circles, can be determined.

The demonstrations of the following propositions will now be obvious.

*Triangles of constant species inscribed to a triangle.*

**980** Let  $abc$ , in the last figure, be a fixed triangle, and  $O$  any point. Take any point  $A$  on  $bc$ , and let the circles circumscribing  $OAc$ ,  $OAb$  cut the other sides in  $B$ ,  $C$ . Then  $ABC$  will be a triangle of constant form, and its angles will have the values  $A = Oba + Oca$ , &c. (III. 21.)

**981** The triangle  $ABC$  will evidently be a minimum when  $OA$ ,  $OB$ ,  $OC$  are drawn perpendicular to the sides of  $abc$ .

**982** To construct a triangle of given form and of given limited magnitude having its vertices upon three fixed lines  $bc$ ,  $ca$ ,  $ab$ .



Construct the figure  $ABCO$  independently from the known sides of  $ABC$  and the angles at  $O$ , which are equal to the supplements of the given angles  $a, b, c$ . Thus the angles  $OAC$ , &c are found, and therefore the angles  $ObC$ , &c., equal to them (III. 21), are known. From these last angles the point  $O$  can be determined, and the lengths  $OA, OB, OC$  being known from the independent figure, the points  $A, B, C$  can be found.

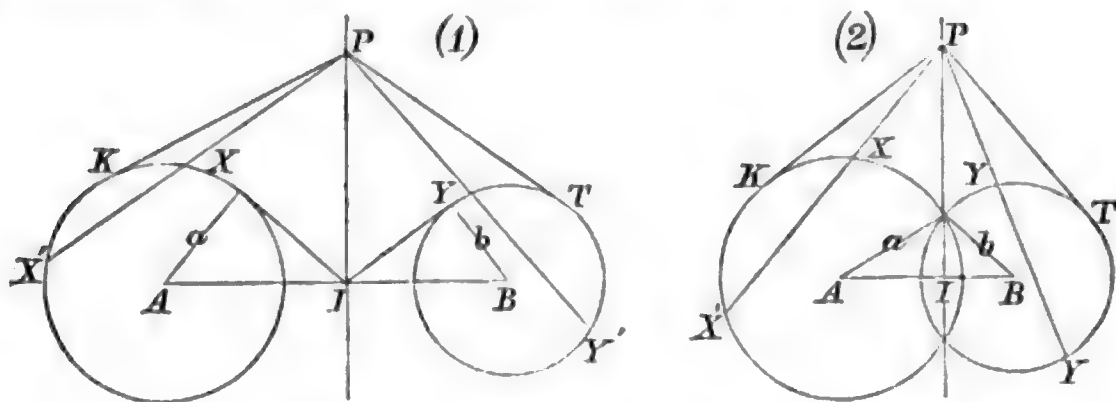
Observe that, wherever the point  $O$  may be taken, the angles  $AOB, BOC, COA$  are in all cases either the supplements of, or equal to, the angles  $c, a, b$  respectively; while the angles  $aOb, bOc, cOa$  are in all cases equal to  $C \pm c, A \pm a, B \pm b$ .

**983** NOTE.—In general problems, like the foregoing, which admit of different cases, it is advisable to choose for reference a standard figure which has all its elements of the same affection or sign. In adapting the figure to other cases, all that is necessary is to follow the same construction, letter for letter, observing the convention respecting positive and negative, which applies both to the lengths of lines and to the magnitudes of angles, as explained in (607—609).

### Radical Axis.

**984** DEFINITION.—The *radical axis* of two circles is that perpendicular to the line of centres which divides the distance between the centres into segments, the difference of whose squares is equal to the difference of the squares of the radii.

Thus,  $A, B$  being the centres,  $a, b$  the radii, and  $IP$  the the radical axis,  $AI^2 - BI^2 = a^2 - b^2$ .



**985** It follows that, if the circles intersect, the radical axis is their common chord; and that, if they do not intersect, the radical axis cuts the line of centres in a point the tangents from which to the circles are equal (I. 47).

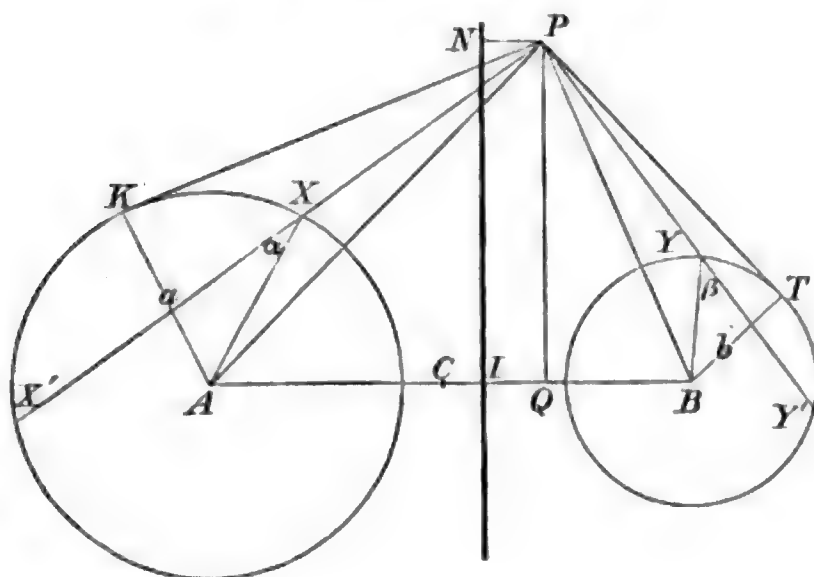
To draw the axis in this case, see (966).

Otherwise: let the two circles cut the line of centres in  $C, D$  and  $C', D'$  respectively. Describe any circle through  $C$  and  $D$ , and another through  $C'$  and  $D'$ , intersecting the former in  $E$  and  $F$ . Their common chord  $EF$  will cut the central axis in the required point  $I$ .

PROOF. —  $IC \cdot ID = IE \cdot IF = IC' \cdot ID'$  (III. 36); therefore the tangents from  $I$  to the circles are equal.

**986 Theorem.**—The difference of the squares of tangents from any point  $P$  to two circles is equal to twice the rectangle under the distance between their centres and the distance of the point from their radical axis, or

$$PK^2 - PT^2 = 2AB \cdot PN.$$



PROOF.

$PK^2 - PT^2 = (AP^2 - BP^2) - (a^2 - b^2) = (AQ^2 - BQ^2) - (AI^2 - BI^2)$ ,  
by (I. 47) & (984). Bisect  $AB$  in  $C$ , and substitute for each difference of squares, by (II. 12).

**987 COR. 1.**—If  $P$  be on the circle whose centre is  $B$ , then  
 $PK^2 = 2AB \cdot PN.$

**988 COR. 2.**—If two chords be drawn through  $P$  to cut the circles in  $X, X', Y, Y'$ ; then, by (III. 36),

$$PX \cdot PX' - PY \cdot PY' = 2AB \cdot PN.$$

**989** If a variable circle intersect two given circles at constant angles  $\alpha$  and  $\beta$ , it will intersect their radical axis at a constant angle; and its radius will bear a constant ratio to the distance of its centre from the radical axis. Or

$$PN : PX = a \cos \alpha - b \cos \beta : AB.$$

PROOF.—In the same figure, if  $P$  be the centre of the variable circle, and if  $PX = PY$  be its radius; then, by (988),

$$PX (XX' - YY') = 2AB \cdot PN.$$

But  $XX' = 2a \cos \alpha$  and  $YY' = 2b \cos \beta$ ;

therefore  $PN : PX = a \cos \alpha - b \cos \beta : AB$ ,

which is a constant ratio if the angles  $\alpha, \beta$  are constant.

**990** Also  $PX : PN =$  the cosine of the angle at which the circle of radius  $PX$  cuts the radical axis. This angle is therefore constant.

**991** COR.—A circle which touches two fixed circles has its radius in a constant ratio to the distance of its centre from their radical axis.

This follows from the proposition by making  $\alpha = \beta = 0$  or  $2\pi$ .

If  $P$  be on the radical axis; then (see Figs. 1 and 2 of 984)

**992** (i.) The tangents from  $P$  to the two circles are equal,  
or  $PK = PT.$  (986)

**993** (ii.) The rectangles under the segments of chords through  $P$  are equal, or  $PX \cdot PX' = PY \cdot PY'.$  (988)

**994** (iii.) Therefore the four points  $X, X', Y, Y'$  are concyclic (III. 36); and, conversely, if they are concyclic, the chords  $XX', YY'$  intersect in the radical axis.

**995** DEFINITION.—Points which lie on the circumference of a circle are termed *conconcyclic*.

**996** (iv.) If  $P$  be the centre, and if  $PX = PY$  be the radius of a circle intersecting the two circles in the figure at angles  $\alpha$  and  $\beta$ ; then, by (993),  $XX' = YY'$ , or  $a \cos \alpha = b \cos \beta$ ; that is, *The cosines of the angles of intersection are inversely as the radii of the fixed circles.*

**997** The radical axes of three circles (Fig. 1046), taken two and two together, intersect at a point called their *radical centre*.

PROOF.—Let  $A, B, C$  be the centres,  $a, b, c$  the radii, and  $X, Y, Z$  the points in which the radical axes cut  $BC, CA, AB$ . Write the equation of the definition (984) for each pair of circles. Add the results, and apply (976).

**998** A circle whose centre is the radical centre of three other circles intersects them in angles whose cosines are inversely as their radii (996).

Hence, if this fourth circle cuts one of the others orthogonally, it cuts them all orthogonally.

**999** The circle which intersects at angles  $\alpha, \beta, \gamma$  three fixed circles, whose centres are  $A, B, C$  and radii  $a, b, c$ , has its centre at distances from the radical axes of the fixed circles proportional to

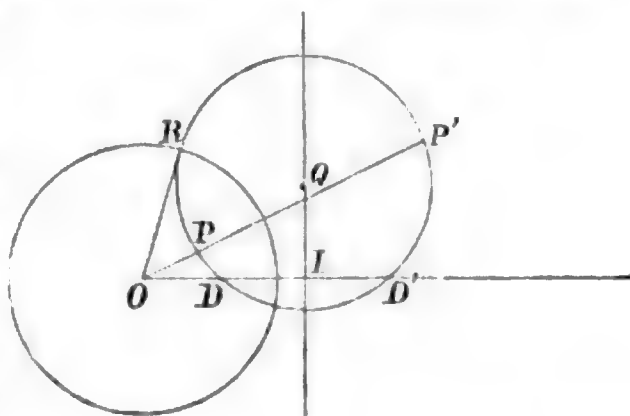
$$\frac{b \cos \beta - c \cos \gamma}{BC}, \quad \frac{c \cos \gamma - a \cos \alpha}{CA}, \quad \frac{a \cos \alpha - b \cos \beta}{AB}.$$

And therefore the locus of its centre will be a straight line passing through the radical centre and inclined to the three radical axes at angles whose sines are proportional to these fractions.

**PROOF.**—The result is obtained immediately by writing out equation (989) for each pair of fixed circles.

### *The Method of Inversion.*

**1000** DEFINITIONS. — Any two points  $P, P'$ , situated on a diameter of a fixed circle whose centre is  $O$  and radius  $k$ , so that  $OP \cdot OP' = k^2$ , are called *inverse points* with respect to the circle, and either point is said to be the *inverse* of the other. The circle and its centre are called the *circle and centre of inversion*, and  $k$  the *constant of inversion*.



**1001** If every point of a plane figure be inverted with respect to a circle, or every point of a figure in space with respect to a sphere, the resulting figure is called the *inverse* or *image* of the original one.

Since  $OP : k : OP'$ , therefore

$$\mathbf{1002} \quad OP : OP' = OP^2 : k^2 = k^2 : OP'^2.$$

**1003** Let  $D, D'$ , in the same figure, be a pair of inverse points on the diameter  $OD'$ . In the perpendicular bisector of  $DD'$ , take any point  $Q$  as the centre of a circle passing through  $D, D'$ , cutting the circle of inversion in  $R$ , and any straight line through  $O$  in the points  $P, P'$ . Then, by (III. 36),  $OP \cdot OP' = OD \cdot OD' = OR^2$  (1000). Hence

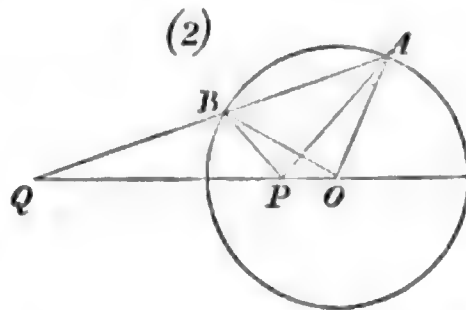
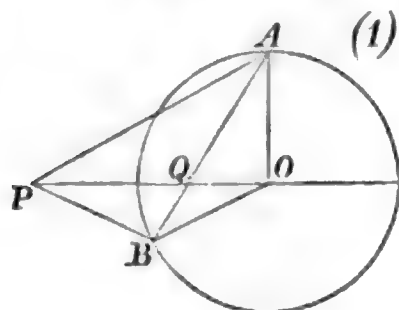
**1004** (i.)  $P, P'$  are inverse points; and, conversely, any two pairs of inverse points lie on a circle.

**1005** (ii.) The circle cuts orthogonally the circle of inversion (III. 37); and, conversely, every circle cutting another orthogonally intersects each of its diameters in a pair of inverse points.

**1006** (iii.) The line  $IQ$  is the locus of a point the tangent from which to a given circle is equal to its distance from a given point  $D$ .

**1007** DEF.—The line  $IQ$  is called the *axis of reflexion* for the two inverse points  $D, D'$ , because there is another circle of inversion, the reflexion of the former, to the right of  $IQ$ , having also  $D, D'$  for inverse points.

**1008** The straight lines drawn from any point  $P$ , within or without a circle (Figs. 1 and 2), to the extremities of any chord  $AB$  passing through the inverse point  $Q$ , make equal angles with the diameter through  $PQ$ . Also, the four points  $O, A, B, P$  are concyclic, and  $QA \cdot QB = QO \cdot QP$ .



PROOF.—In either figure  $OP : OA : OQ$  and  $OP : OB : OQ$  (1000), therefore, by similar triangles,  $\angle OPA = \angle OAB$  and  $\angle OPB = \angle OBA$  in figure (1) and the supplement of it in figure (2). But  $OAB = OBA$  (I. 5), therefore, &c.

Also, because  $\angle OPA = \angle OBA$ , the four points  $O, A, B, P$  lie on a circle in each case (III. 21), and therefore  $QA \cdot QB = QO \cdot QP$  (III. 35, 36).

**1009** The inverse of a circle is a circle, and the centre of inversion is the centre of similitude of the two figures. See also (1037).

PROOF.—In the figure of (1043), let  $O$  be the point where the common tangent  $RT$  of the two circles, centres  $A$  and  $B$ , cuts the central axis, and let any other line through  $O$  cut the circles in  $P, Q, P', Q'$ . Then, in the demonstration of (942), it is shown that  $OP \cdot OQ' = OQ \cdot OP' = k^2$ , a constant quantity. Therefore either circle is the inverse of the other,  $k$  being the radius of the circle of inversion.



**1010** To make the inversions of two given circles equal circles.

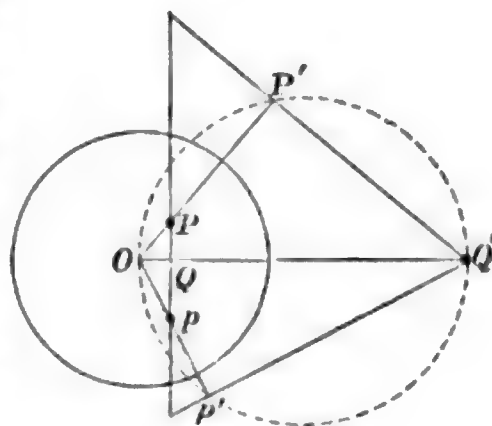
**RULE.**—Take the centre of inversion so that the squares of the tangents from it to the given circles may be proportional to their radii (965).

**PROOF.**—(Fig. 1043)  $AT : BR = OT : OR = OT^2 : k^2$ , since  $OT : k : OR$ . Therefore  $OT^2 : AT = k^2 : BR$ , therefore  $BR$  remains constant if  $OT^2 \propto AT$ .

**1011** Hence three circles may be inverted into equal circles, for the required centre of inversion is the intersection of two circles that can be drawn by (965).

**1012** The inverse of a straight line is a circle passing through the centre of inversion.

**PROOF.**—Draw  $OQ$  perpendicular to the line, and take  $P$  any other point on it. Let  $Q, P'$  be the inverse points. Then  $OP \cdot OP' = OQ \cdot OQ'$ ; therefore, by similar triangles,  $\angle OP'Q = \angle OQP$ , a right angle; and  $OQ'$  is constant, therefore the locus of  $P'$  is the circle whose diameter is  $OQ'$ .



**1013** **EXAMPLE.**—The inversion of a polygon produces a figure bounded by circular arcs which intersect in angles equal to the corresponding angles of the polygon, the complete circles intersecting in the centre of inversion.

**1014** If the extremities of a straight line  $P'Q'$  in the last figure are the inversions of the extremities of  $PQ$ , then

$$PQ : P'Q = \sqrt{OP \cdot OQ} : \sqrt{OP' \cdot OQ'}.$$

**PROOF.**—By similar triangles,  $PQ : P'Q = OP : OQ'$  and  $PQ : P'Q = OQ : OP'$ . Compound these ratios.

**1015** From the above it follows that any homogeneous equation between the lengths of lines joining pairs of points in space, such as  $PQ \cdot RS \cdot TU = PR \cdot QT \cdot SU$ , the same points appearing on both sides of the equation, will be true for the figure obtained by joining the corresponding pairs of inverse points.

For the ratio of each side of the equation to the corresponding side of the equation for the inverted points will be the same, namely,

$$\sqrt{(OP \cdot OQ \cdot OR \dots)} : \sqrt{(OP' \cdot OQ' \cdot OR' \dots)}.$$

2 F



### *Pole and Polar.*

**1016** DEFINITION.—The *polar* of any point  $P$  with respect to a circle is the perpendicular to the diameter  $OP$  (Fig. 1012) drawn through the inverse point  $P'$ .

**1017** It follows that the polar of a point exterior to the circle is the *chord of contact* of the tangents from the point; that is, the line joining their points of contact.

**1018** Also,  $P'Q'$  is the polar of  $P$  with respect to the circle, centre  $O$ , and  $PQ$  is the polar of  $Q'$ . In other words, *any point  $P$  lying on the polar of a point  $Q'$ , has its own polar always passing through  $Q'$ .*

**1019** The line joining any two points  $P, p$  is the polar of  $Q'$ , the point of intersection of their polars.

PROOF.—The point  $Q'$  lies on both the lines  $P'Q'$ ,  $p'Q'$ , and therefore has its polar passing through the pole of each line, by the last theorem.

**1020** The polars of any two points  $P, p$ , and the line joining the points form a *self-reciprocal* triangle with respect to the circle, the three vertices being the poles of the opposite sides. The centre  $O$  of the circle is evidently the orthocentre of the triangle (952). The circle and its centre are called the *polar circle* and *polar centre* of the triangle.

If the radii of the polar and circumscribed circles of a triangle  $ABC$  be  $r$  and  $R$ , then

$$r^2 = 4R^2 \cos A \cos B \cos C.$$

PROOF.—In Fig. (952),  $O$  is the centre of the polar circle, and the circles described round  $ABC$ ,  $BOC$ ,  $COA$ ,  $AOB$  are all equal; because the angle  $BOC$  is the supplement of  $A$ ; &c. Therefore  $2R \cdot OD = OB \cdot OC$  (VI. C) and  $r^2 = OA \cdot OD = OA \cdot OB \cdot OC \div 2R$ . Also,  $OA = 2R \cos A$  by a diameter through  $B$ , and (III. 21).

### *Coaxal Circles.*

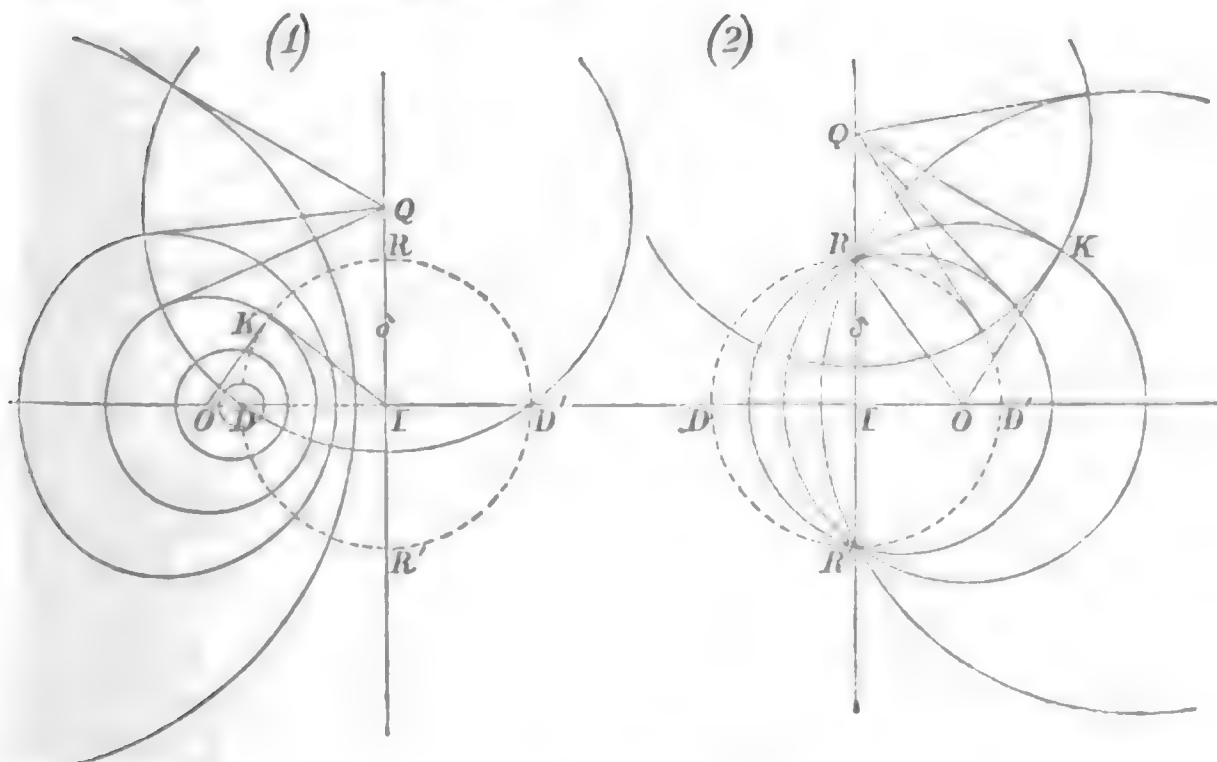
**1021** DEFINITION.—A system of circles having a common line of centres called the *central axis*, and a common radical axis, is termed a *coaxal* system.

**1022** If  $O$  be the variable centre of one of the circles, and

OK its radius, the whole system is included in the equation

$$OI^2 - OK^2 = \pm \delta^2,$$

where  $\delta$  is a constant length.



**1023** In the first species (Fig. 1),

$$OI^2 - OK^2 = \delta^2,$$

and  $\delta$  is the length of the tangent from  $I$  to any circle of the system (985). Let a circle, centre  $I$  and radius  $\delta$ , cut the central axis in  $D, D'$ . When  $O$  is at  $D$  or  $D'$ , the circle whose radius is  $OK$  vanishes. When  $O$  is at an infinite distance, the circle develops into the radical axis itself and into a line at infinity.

The points  $D, D'$  are called the *limiting points*.

**1024** In the second species (Fig. 2),

$$OR^2 - OI^2 = \delta^2,$$

and  $\delta$  is half the chord  $RR'$  common to all the circles of the system. These circles vary between the circle with centre  $I$  and radius  $\delta$ , and the circle with its centre at infinity as described above. The points  $R, R'$  are the *common points* of all circles of this system. The two systems are therefore distinguished as the *limiting points species* and the *common points species* of coaxal circles.

**1025** There is a conjugate system of circles having  $R, R'$  for limiting points, and  $D, D'$  for common points, and the circles of one species intersect all the circles of the conjugate system of the other species orthogonally (1005).

Thus, in figures (1) and (2),  $Q$  is the centre of a circle of the opposite species intersecting the other circles orthogonally.

**1026** In the first species of coaxal circles, the limiting points  $D, D'$  are *inverse* points for every circle of the system, the radical axis being the axis of reflexion for the system.

PROOF.—(Fig. 1)  $OI^2 - \delta^2 = OK^2$ ,  
therefore  $OD \cdot OD' = OK^2$ , (II. 13)  
therefore  $D, D'$  are inverse points (1000).

**1027** Also, the points in which any circle of the system cuts the central axis are inverse points for the circle whose centre is  $I$  and radius  $\delta$ . [PROOF.—Similar to the last.

**1028** PROBLEM.—Given two circles of a coaxal system, to describe a circle of the same system—(i.) to pass through a given point; or (ii.) to touch a given circle; or (iii.) to cut a given circle orthogonally.

**1029** I. If the system be of the common points species, then, since the required circle always passes through two known points, the first and second cases fall under the Tangencies. See (941).

**1030** To solve the third case, describe a circle through the given common points, and through the inverse of either of them with respect to the given circle, which will then be cut orthogonally, by (1005).

**1031** II. If the system be of the limiting points species, the problem is solved in each case by the aid of a circle of the conjugate system. Such a circle always passes through the known limiting points, and may be called a conjugate circle of the limiting points system. Thus,

**1032** To solve case (i.)—Draw a conjugate circle through the given point, and the tangent to it at that point will be the radius of the required circle.

**1033** To solve case (ii.)—Draw a conjugate circle through the inverse of either limiting point with respect to the given circle, which will thus be cut orthogonally, and the tangent to the cutting circle at either point of intersection will be the radius of the required circle.

**1034** To solve case (iii.)—Draw a conjugate circle to touch the given one, and the common tangent of the two will be the radius of the required circle.

**1035** Thus, according as we wish to make a circle of the system *touch*, or *cut orthogonally*, the given circle, we must draw a conjugate circle to *cut orthogonally*, or *touch it*.

**1036** If three circles be coaxal, the squares of the tangents drawn to any two of them from a point on the third are in the ratio of the distances of the centre of the third circle from the centres of the other two.

PROOF.—Let  $A, B, C$  be the centres of the circles;  $PK, PT$  the tangents from a point  $P$  on the circle, centre  $C$ , to the other two;  $PN$  the perpendicular on the radical axis. By (986),

$$PK^2 = 2AC \cdot PN \quad \text{and} \quad PT^2 = 2BC \cdot PN,$$

therefore

$$PK^2 : PT^2 = AC : BC.$$

### *Centres and axes of similitude.*

**1037** DEFINITIONS.—Let  $OO'$  be the centres of similitude (Def. 947) of the two circles in the figure below, and let any line through  $O$  cut the circles in  $P, Q, P', Q'$ . Then the constant ratio  $OP : OP' = OQ : OQ'$  is called the ratio of *similitude* of the two figures; and the constant product  $OP \cdot OQ' = OQ \cdot OP'$  is called the product of *anti-similitude*. See (942), (1009), and (1043).

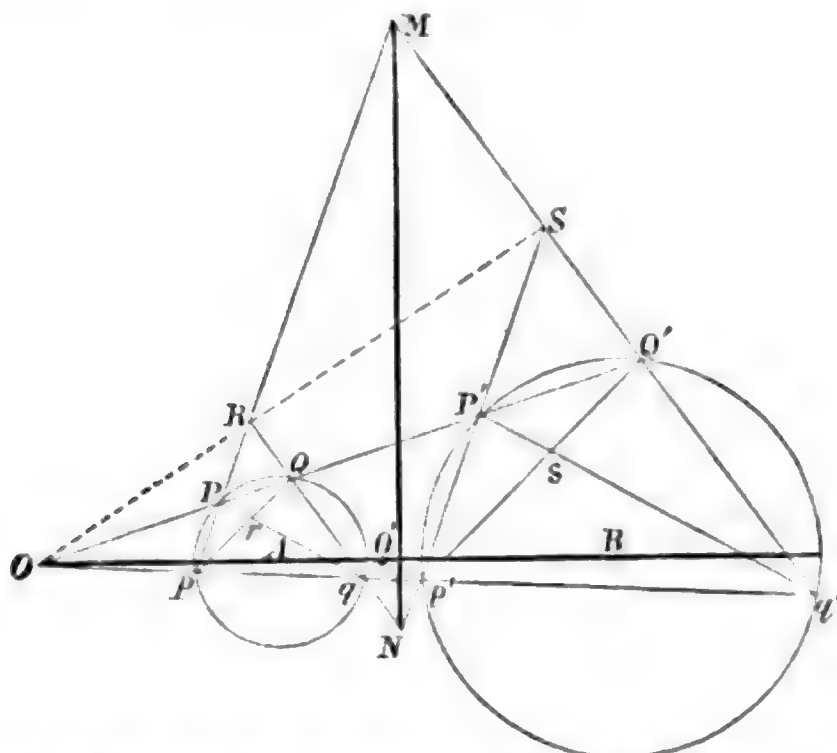
The corresponding points  $P, P'$  or  $Q, Q'$  on the same straight line through  $O$  are termed *homologous*, and  $P, Q'$  or  $Q, P'$  are termed *anti-homologous*.

**1038** Let any other line  $Opqp'q'$  be drawn through  $O$ . Then, if any two points  $P, p$  on the one figure be joined, and if  $P', p'$ , homologous to  $P, p$  on the other figure, be also joined, the lines so formed are termed *homologous*. But if the points which are joined on the second figure are anti-homologous to those on the first, the two lines are termed *anti-homologous*. Thus,  $Pq, Qp'$  are anti-homologous lines.

**1039** The circle whose centre is  $O$ , and whose radius is equal to the square root of the product of anti-similitude, is called the *circle of anti-similitude*.

**1040** The four pairs of homologous chords  $Pp$  and  $P'p'$ ,  $Qq$  and  $Q'q'$ ,  $Pq$  and  $P'q'$ ,  $Qp$  and  $Q'p'$  of the two circles in the figure are parallel. And in all similar and similarly situated figures homologous lines are parallel.

PROOF.—By (VI. 2) and the definition (947).



**1041** The four pairs of anti-homologous chords,  $Pp$  and  $Qq$ ,  $Qq$  and  $P'p'$ ,  $Pq$  and  $Q'p'$ ,  $Qp$  and  $P'q'$ , of the two circles meet on their radical axis.

PROOF.—

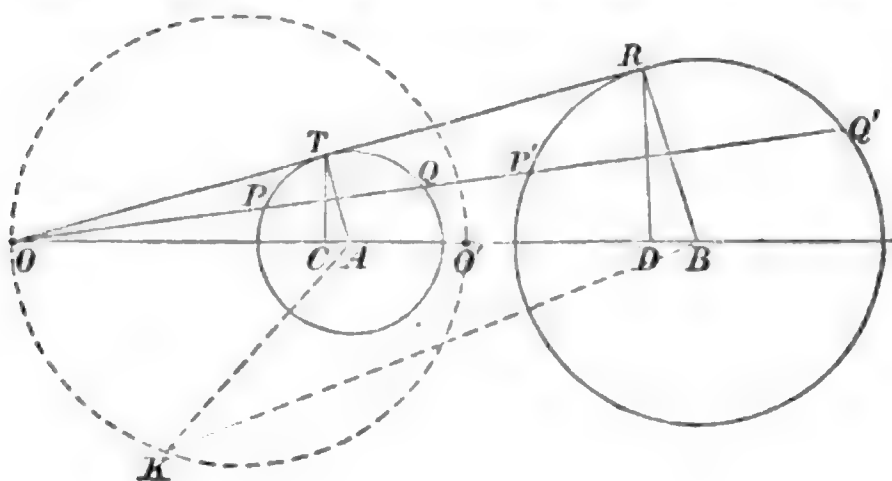
$$OP \cdot OQ' = Op \cdot Oq' = k^2,$$

where  $k$  is the constant of inversion; therefore  $P, p, Q', q'$  are concyclic; therefore  $Pp$  and  $Q'q'$  meet on the radical axis. Similarly for any other pair of anti-homologous chords.

**1042** COR.—From this and the preceding proposition it follows that the tangents at homologous points are parallel; and that the tangents at anti-homologous points meet on the radical axis. For these tangents are the limiting positions of homologous or anti-homologous chords. (1160)

**1043** Let  $C, D$  be the inverse points of  $O$  with respect to two circles, centres  $A$  and  $B$ ; then the constant product of anti-similitude

$$OP \cdot OQ' \text{ or } OQ \cdot OP' = OA \cdot OD \text{ or } OB \cdot OC.$$



PROOF.—By similar right-angled triangles,

$$OA : OT : OC \text{ and } OB : OR : OD;$$

$$\text{therefore} \quad OA \cdot OD = OB \cdot OC \dots\dots\dots (1),$$

$$\text{and also} \quad OA \cdot OC = OT^2 = OP \cdot OQ, \quad (\text{III. 36})$$

$$\text{and} \quad OB \cdot OD = OR^2 = OP' \cdot OQ';$$

$$\text{therefore} \quad OA \cdot OB \cdot OC \cdot OD = OP \cdot OQ \cdot OP' \cdot OQ',$$

therefore &c., by (1).

**1044** The foregoing definitions and properties (1037 to 1043), which have respect to the external centre of similitude  $O$ , hold good for the internal centre of similitude  $O'$ , with the usual convention of positive and negative for distances measured from  $O'$  upon lines passing through it.

**1045** Two circles will subtend equal angles at any point on the circumference of the circle whose diameter is  $OO'$ , where  $O, O'$  are the centres of similitude (Fig. 1043). This circle is also coaxal with the given circles, and has been called the *circle of similitude*.

PROOF.—Let  $A, B$  be the centres,  $a, b$  the radii, and  $K$  any point on the circle, diameter  $OO'$ . Then, by (932),

$$KA : KB = AO : BO = AO' : BO' = a : b,$$

by the definition (943);

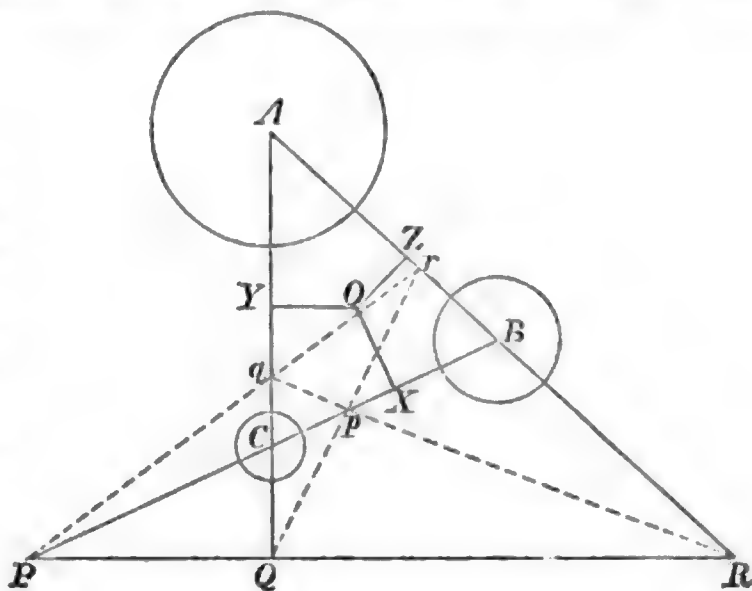
$$\text{therefore} \quad a : KA = b : KB;$$

that is, the sines of the halves of the angles in question are equal, which proves the first part. Also, because the tangents from  $K$  are in the constant ratio of the radii  $a, b$ , this circle is coaxal with the given ones, by (1036, 934).

**1046** The six centres of similitude  $P, p, Q, q, R, r$  of three circles lie three and three on four straight lines  $PQR, Pqr, Qpr, Rpq$ , called *axes of similitude*.

PROOF. — Taking any three of the sets of points named, say  $P, q, r$ , they are shewn at once to be collinear by the transversal theorem (968) applied to the triangle  $ABC$ .

For the segments of its sides made by the points  $P, q, r$  are in the ratios of the radii of the circles.





**1047** From the investigation in (942), it appears that one circle touches two others in a pair of anti-homologous points, and that the following rule obtains:—

**RULE.**—*The right line joining the points of contact passes through the external or internal centre of similitude of the two circles according as the contacts are of the same or of different kinds.*

**1048** **DEFINITION.**—Contact of curves is either *internal* or *external* according as the curvatures at the point of contact are in the same or opposite directions.

**1049** *Gergonne's method of describing the circles which touch three given circles.*

Take  $Pqr$ , one of the four axes of similitude, and find its poles  $\alpha, \beta, \gamma$  with respect to the given circles, centres  $A, B, C$  (1016). From  $O$ , the radical centre, draw lines through  $\alpha, \beta, \gamma$ , cutting the circles in  $a, a', b, b', c, c'$ . Then  $a, b, c$  and  $a', b', c'$  will be the points of contact of two of the required circles.

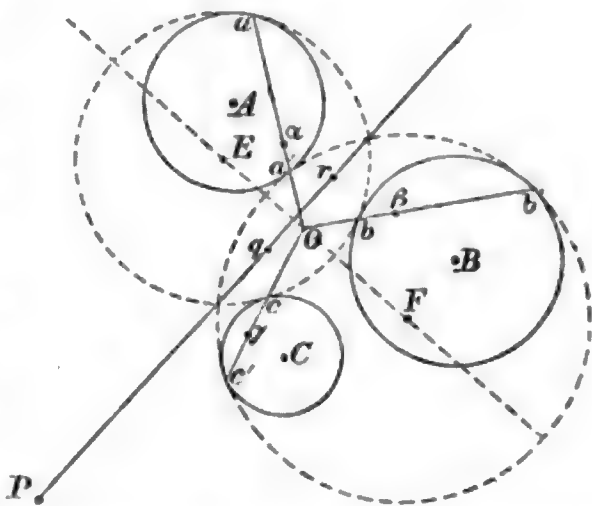
**PROOF.**—*Analysis.*—Let the circles  $E, F$  touch the circles  $A, B, C$  in  $a, b, c, a', b', c'$ . Let  $bc, b'c'$  meet in  $P$ ;  $ca, c'a'$  in  $q$ ; and  $ab, a'b'$  in  $r$ .

Regarding  $E$  and  $F$  as touched by  $A, B, C$  in turn, Rule (1047) shews that  $aa', bb', cc'$  meet in  $O$ , the centre of similitude of  $E$  and  $F$ ; and (1041) shews that  $P, q$ , and  $r$  lie on the radical axis of  $E$  and  $F$ .

Regarding  $B$  and  $C$ , or  $C$  and  $A$ , or  $A$  and  $B$ , as touched by  $E$  and  $F$  in turn, Rule (1047) shews that  $P, q, r$  are the centres of similitude of  $B$  and  $C$ ,  $C$  and  $A$ ,  $A$  and  $B$  respectively;

and (1041) shews that  $O$  is on the radical axis of each pair, and is therefore the radical centre of  $A, B$ , and  $C$ .

Again, because the tangents to  $E$  and  $F$ , at the anti-homologous points  $a, a'$ , meet on  $Pqr$ , the radical axis of  $E$  and  $F$  (1042); therefore the point of meeting is the pole of  $aa'$  with respect to the circle  $A$  (1017). Therefore  $aa'$  passes through the pole of the line  $Pqr$  (1018). Similarly,  $bb'$  and  $cc'$  pass through the poles of the same line  $Pqr$  with respect to  $B$  and  $C$ . Hence the construction.



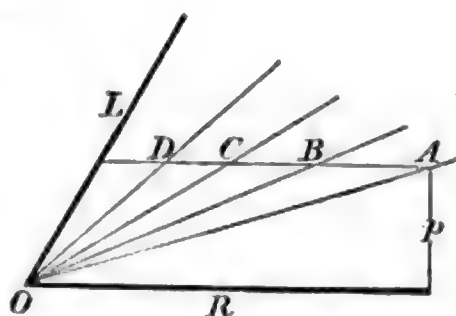
**1050** In the given configuration of the circles  $A, B, C$ , the demonstration shews that each of the three internal axes of similitude  $Pqr, Qrp, Rpq$  (Fig. 1046) is a radical axis and common chord of two of the eight osculating circles which can be drawn. The external axis of similitude  $PQR$  is the

radical axis of the two remaining circles which touch  $A$ ,  $B$ , and  $C$  either all externally or all internally.

**1051** The radical centre  $O$  of the three given circles is also the common internal centre of similitude of the four pairs of osculating circles. Therefore the central axis of each pair passes through  $O$ , and is perpendicular to the radical axis. Thus, in the figure,  $EF$  passes through  $O$ , and is perpendicular to  $Pqr$ .

### Anharmonic Ratio.

**1052** DEFINITION.—Let a pencil of four lines through a point  $O$  be cut by a transversal in the points  $A$ ,  $B$ ,  $C$ ,  $D$ . The anharmonic ratio of the pencil is any one of the three fractions



$$\frac{AB \cdot CD}{AD \cdot BC} \quad \text{or} \quad \frac{AB \cdot CD}{AC \cdot BD} \quad \text{or} \quad \frac{AD \cdot BC}{AC \cdot BD}.$$

**1053** The relation between these three different ratios is obtained from the equation

$$AB \cdot CD + AD \cdot BC = AC \cdot BD.$$

Denoting the terms on the left side by  $p$  and  $q$ , the three anharmonic ratios may be expressed by

$$p : q, \quad p : p + q, \quad q : p + q.$$

The ratios are therefore mutually dependent. Hence, if the identity merely of the anharmonic ratio in any two systems is to be established, it is immaterial which of the three ratios is selected.

**1054** In future, when the ratio of an anharmonic pencil  $\{O, ABCD\}$  is mentioned, the form  $AB \cdot CD : AD \cdot BC$  will be the one intended, whatever the actual order of the points  $A$ ,  $B$ ,  $C$ ,  $D$  may be. For, it should be observed that, by making the line  $OD$  revolve about  $O$ , the ratio takes in turn each of the forms given above. This ratio is shortly expressed by the notation  $\{O, ABCD\}$ , or simply  $\{ABCD\}$ .

**1055** If the transversal be drawn parallel to one of the lines, for instance  $OD$ , the two factors containing  $D$  become infinite, and their ratio becomes unity. They may therefore be omitted. The anharmonic ratio then reduces to  $AB : BC$ . Thus, when  $D$  is at infinity, we may write

$$\{O, ABC \infty\} = AB : BC.$$

**1056** The anharmonic ratio

$$\frac{AB \cdot CD}{AD \cdot BC} = \frac{\sin AOB \sin COD}{\sin AOD \sin BOC},$$

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and its value is therefore the same for all transversals of the pencil.

PROOF.—Draw  $OR$  parallel to the transversal, and let  $p$  be the perpendicular from  $A$  upon  $OR$ . Multiply each factor in the fraction by  $p$ . Then substitute  $p \cdot AB = OA \cdot OB \sin AOB$ , &c. (707).

**1057** The anharmonic ratio (1056) becomes harmonic when its value is unity. See (933). The harmonic relation there defined may also be stated thus: four points divide a line harmonically when the product of the extreme segments is equal to the product of the whole line and the middle segment.

### *Homographic Systems of Points.*

**1058** DEFINITION.—If  $x, a, b, c$  be the distances of one variable point and three fixed points on a straight line from a point  $O$  on the same; and if  $x', a', b', c'$  be the distances of similar points on another line through  $O$ ; then the variable points on the two lines will form two *homographic systems* when they are connected by the anharmonic relation

$$\frac{(x-a)(b-c)}{(x-c)(a-b)} = \frac{(x'-a')(b'-c')}{(x'-c')(a'-b')}.$$

Expanding, and writing  $A, B, C, D$  for the constant coefficients, the equation becomes

$$\mathbf{1060} \quad Axx' + Bx + Cx' + D = 0.$$

From which

$$\mathbf{1061} \quad x = -\frac{Cx' + D}{Ax' + B}, \text{ and } x' = -\frac{Bx + D}{Ax + C}.$$

**1062** THEOREM.—Any four arbitrary points  $x_1, x_2, x_3, x_4$  on one of the lines will have four corresponding points  $x'_1, x'_2, x'_3, x'_4$  on the other determined by the last equation, and the two sets of points will have equal anharmonic ratios.

PROOF.—This may be shown by actual substitution of the value of each  $x$  in terms of  $x'$ , by (1061), in the harmonic ratio  $\{x_1, x_2, x_3, x_4\}$ .

**1063** If the distances of four points on a right line from a point  $O$  upon it, in order, are  $a, a', \beta, \beta'$ , where  $a, \beta; a', \beta'$  are the respective roots of the two quadratic equations

$$ax^2 + 2hx + b = 0, \quad a'x^2 + 2h'x + b' = 0;$$

the condition that the two pairs of points may be *harmonically conjugate* is

$$\mathbf{1064} \quad ab' + a'b = 2hh'.$$

PROOF.—The harmonic relation, by (1057), is

$$(\alpha - \alpha')(\beta - \beta') = (\alpha - \beta)(\alpha' - \beta').$$

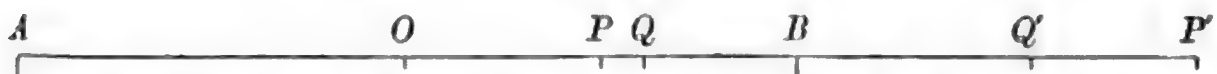
Multiply out, and substitute for the sums and products of the roots of the quadratics above in terms of their coefficients by (51, 52).

**1065** If  $u_1, u_2$  be the quadratic expressions in (1063) for two pairs of points, and if  $u$  represent a third pair harmonically conjugate with  $u_1$  and  $u_2$ , then the pair of points  $u$  will also be harmonically conjugate with every pair given by the equation  $u_1 + \lambda u_2 = 0$ , where  $\lambda$  is any constant. For the condition (1064) applied to the last equation will be identically satisfied.

### Involution.

**1066** DEFINITIONS.—Pairs of *inverse points*  $PP', QQ', \&c.$ , on the same right line, form a system *in involution*, and the relation between them, by (1000), is

$$OP \cdot OP' = OQ \cdot OQ' = \&c. = k^2.$$



The radius of the circle of inversion is  $k$ , and the centre  $O$  is called the *centre of the system*. Inverse points are also termed *conjugate points*.

When two inverse points coincide, the point is called a *focus*.

**1067** The equation  $OP^2 = k^2$  shows that there are two foci  $A, B$  at the distance  $k$  from the centre, and on opposite sides of it, real or imaginary according as any two inverse points lie on the same side or on opposite sides of the centre.

**1068** If the two homographic systems of points in (1058) be on the *same line*, they will constitute a system *in involution* when  $B = C$ .

PROOF.—Equation (1060) may now be written

$$Axx' + H(x + x') + B = 0,$$

or 
$$\left(x + \frac{H}{A}\right) \left(x' + \frac{H}{A}\right) = \frac{H^2}{A^2} - \frac{B}{A} = k^2,$$

a constant. Therefore  $-\frac{H}{A}$  is the distance of the origin  $O$  from the centre of inversion. Measuring from this centre, the equation becomes  $\xi\xi' = k^2$ , representing a system in involution.

**1069** Any four points whatever of a system in involution on a right line have their anharmonic ratio equal to that of their four conjugates.

PROOF.—Let  $p, p'; q, q'; r, r'; s, s'$  be the distances of the pairs of inverse points from the centre.

In the anharmonic ratio of *any* four of the points, for instance  $\{pqrs\}$ , substitute  $p = k^2 \div p', q = k^2 \div q', \&c.$ , and the result is the anharmonic ratio  $\{p'q'r's'\}$ .

**1070** Any two inverse points  $P, P'$  are in harmonic relation with the foci  $A, B$ .



PROOF.—Let  $p, p'$  be the distances of  $P, P'$  from the centre  $O$ ; then  $pp' = k^2$ , therefore  $\frac{p'}{k} = \frac{k}{p}$ , therefore  $\frac{p'+k}{p'-k} = \frac{k+p}{k-p}$ ;

that is,  $\frac{AP'}{BP'} = \frac{AP}{BP}$ . (933)

**1071** If a system of points in involution be given, as in (1068), by the equation

$$Axx' + H(x+x') + B = 0 \dots\dots\dots (1);$$

and a pair of conjugate points by the equation

$$ax^2 + 2hx + b = 0 \dots\dots\dots (2);$$

the necessary relation between  $a, h$ , and  $b$  is

$$\mathbf{1072} \quad Ab + Ba = 2Hh.$$

PROOF.—The roots of equation (2) must be simultaneous values of  $x, x'$  in (1); therefore substitute in (1)

$$x + x' = -\frac{2h}{a} \quad \text{and} \quad xx' = \frac{b}{a}. \quad (51)$$

**1073** COR.—A system in involution may be determined from two given pairs of corresponding points.

Let the equations for these points be

$$ax^2 + 2hx + b = 0 \quad \text{and} \quad a'x^2 + 2hx + b' = 0.$$

Then there are two conditions (1072),

$$Ab + Ba = 2Hh \quad \text{and} \quad Ab' + Ba' = 2Hh',$$

from which  $A, H, B$  can be found.

A geometrical solution is given in (985).  $C, D; C', D'$  are, in that construction, pairs of inverse points, and  $I$  is the centre of a system in involution defined by a series of coaxal circles (1022). Each circle intersects the central axis in a pair of inverse points with respect to the circle whose centre is  $O$  and radius  $\delta$ .

**1074** The relations which have been established for a system of *collinear* points may be transferred to a system of concurrent lines by the method of (1056), in which the distance between two points corresponds to the sine of the angle between two lines passing through those points.



### The Method of Projection.

**1075** DEFINITIONS.—The *projection* of any point  $P$  in space (Fig. of 1079) is the point  $p$  in which a right line  $OP$ , drawn from a fixed point  $O$  called the *vertex*, intersects a fixed plane called the *plane of projection*.

If all the points of any figure, plane or solid, be thus projected, the figure obtained is called the *projection* of the original figure.

**1076** *Projective Properties*.—The projection of a right line is a right line. The projections of parallel lines are parallel. The projections of a curve, and of the tangent at any point of it, are another curve and the tangent at the corresponding point.

**1077** The anharmonic ratio of the segments of a right line is not altered by projection; for the line and its projection are but two transversals of the same anharmonic pencil. (1056)

**1078** Also, any relation between the segments of a line similar to that in (1015), in which each letter occurs in every term, is a *projective property*. [Proof as in (1056).]

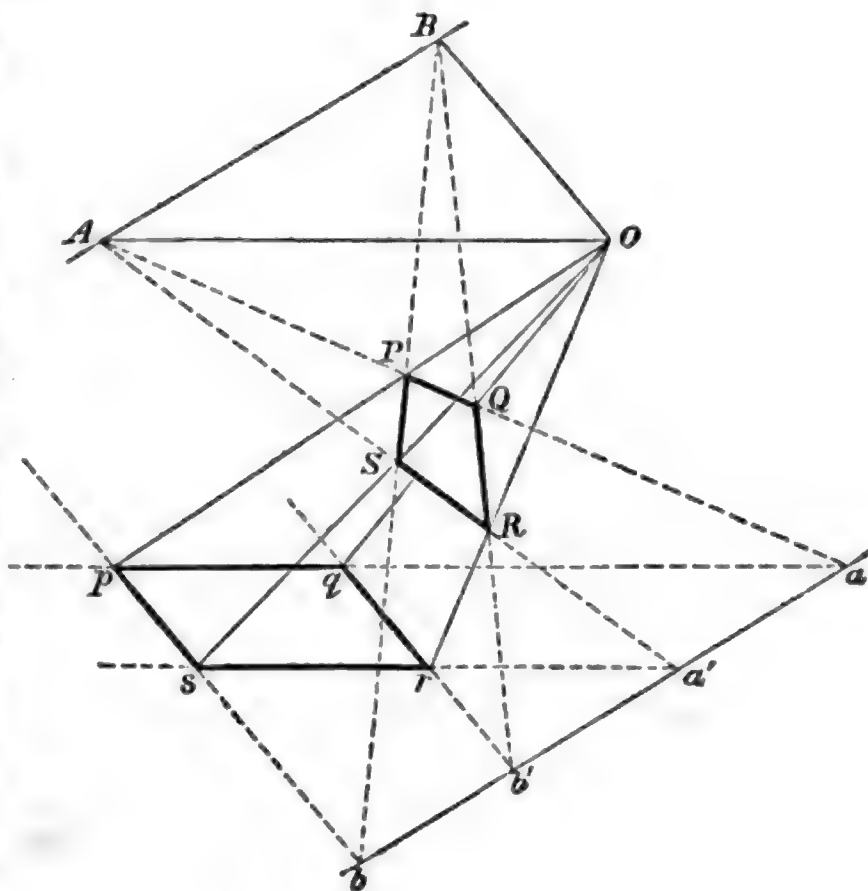
**1079** *Theorem*.—Any quadrilateral  $PQRS$  may be projected into a parallelogram.

CONSTRUCTION. —

Produce  $PQ$ ,  $SR$  to meet in  $A$ , and  $PS$ ,  $QR$  to meet in  $B$ .

Then, with any point  $O$  for vertex, project the quadrilateral upon any plane  $pab$  parallel to  $OAB$ . The projected figure  $pqrs$  will be a parallelogram.

PROOF. — The planes  $OPQ$ ,  $ORS$  intersect in  $OA$ , and they intersect the plane of projection which is parallel to  $OA$  in the lines  $pq$ ,  $rs$ . Therefore  $pq$  and  $rs$  are parallel to  $OA$ , and therefore to each other. Similarly,  $ps$ ,  $qr$  are parallel to  $OB$ .





**1080** COR. 1.—The opposite sides of the parallelogram  $pqrs$  meet in two points at infinity, which are the projections of the points  $A, B$ ; and  $AB$  itself, which is the third diagonal of the complete quadrilateral  $PQRS$ , is projected into a line at infinity.

**1081** Hence, to project any figure so that a certain line in it may pass to infinity—Take the plane of projection parallel to the plane which contains the given line and the vertex.

**1082** COR. 2.—To make the projection of the quadrilateral a rectangle, it is only necessary to make  $AOB$  a right angle.

### On Perspective Drawing.

**1083** Taking the parallelogram  $pqrs$ , in (1079), for the original figure, the quadrilateral  $PQRS$  is its projection on the plane  $ABab$ . Suppose this plane to be the plane of the paper. Let the planes  $OAB, pab$ , while remaining parallel to each other, be turned respectively about the fixed parallel lines  $AB, ab$ . In every position of the planes, the lines  $Op, Oq, Or, Os$  will intersect the dotted lines in the same points  $P, Q, R, S$ . When the planes coincide with that of the paper,  $pqrs$  becomes a ground plan of the parallelogram, and  $PQRS$  is the representation of it in perspective.

$AB$  is then called the *horizontal line*,  $ab$  the *picture line*, and the plane of both the *picture plane*.

**1084** To find the projection of any point  $p$  in the *ground plan*.

RULE.—Draw  $pb$  to any point  $b$  in the picture line, and draw  $OB$  parallel to  $pb$ , to meet the horizontal line in  $B$ . Join  $Op, Bb$ , and they will intersect in  $P$ , the point required.

In practice,  $pb$  is drawn perpendicular to  $ab$ , and  $OB$  therefore perpendicular to  $AB$ . The point  $B$  is then called the *point of sight*, or *centre of vision*, and  $O$  the *station point*.

**1085** To find the projection of a point in the *ground plan*, not in the original plane, but at a perpendicular distance  $c$  above it.

RULE.—Take a new picture line parallel to the former, and at a distance above it  $= c \operatorname{cosec} a$ , where  $a$  is the angle between the original plane and the plane of projection. For a plane through the given point, parallel to the original plane, will intersect the plane of projection in the new picture line so constructed.

Thus, every point of a figure in the *ground plan* is transferred to the drawing.

**1086** The whole theory of perspective drawing is virtually included in the foregoing propositions. The original plane is commonly horizontal, and the plane of projection vertical. In this case,  $\operatorname{cosec} a = 1$ , and the height of the picture line for any point is equal to the height of the point itself above the original plane.

The distance  $BO$ , when  $B$  is the *point of sight*, may be measured along  $AB$ , and  $bp$  along  $ab$ , in the opposite direction; for the line  $Bb$  will continue to intersect  $Op$  in the point  $P$ .

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### *Orthogonal Projection.*

**1087** DEFINITION. — In orthogonal projection the lines of projection are parallel to each other, and perpendicular to the plane of projection. The vertex in this case may be considered to be at infinity.

**1088** The projections of parallel lines are parallel, and the projected segments are in a constant ratio to the original segments.

**1089** Areas are in a constant ratio to their projections.

For, lines parallel to the intersection of the original plane and the plane of projection are unaltered in length, and lines at right angles to the former are altered in a constant ratio. This ratio is the ratio of the areas, and is the cosine of the angle between the two planes.

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### *Projections of the Sphere.*

**1090** In *Stereographic projection*, the vertex is on the surface of the sphere, and the diameter through the vertex is perpendicular to the plane of projection which passes through the other extremity of the diameter. The projection is therefore the *inversion* of the surface of the sphere (1012), and the diameter is the constant  $k$ .

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**1091** In *Globular projection*, the vertex is taken at a distance from the sphere equal to the radius  $\div \sqrt{2}$ , and the diameter through the vertex is perpendicular to the plane of projection.

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**1092** In *Gnomonic projection*, which is used in the construction of sun-dials, the vertex is at the centre of the sphere.

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**1093** *Mercator's projection*, which is employed in navigation, and sometimes in maps of the world, is not a projection at all as defined in (1075). Meridian circles of the sphere are represented on a plane by parallel right lines at intervals equal to the intervals on the equator. The parallels of latitude are represented by right lines perpendicular to the meridians, and at increasing intervals, so as to preserve the actual ratio between the increments of longitude and latitude at every point.

With  $r$  for the radius of the sphere, the distance, on the chart, from the equator of a point whose latitude is  $\lambda$ , is  $= r \log \tan (45^\circ + \frac{1}{2}\lambda)$ .

*Additional Theorems.*

**1094** The sum of the squares of the distances of any point  $P$  from  $n$  equidistant points on a circle whose centre is  $O$  and radius  $r$   

$$= n(r^2 + OP^2).$$

PROOF.—Sum the values of  $PB^2, PC^2, \&c.$ , given in (819), and apply (803). This theorem is the generalization of (923).

**1095** In the same figure, if  $P$  be on the circle, the sum of the squares of the perpendiculars from  $P$  on the radii  $OB, OC, \&c.$ , is equal to  $\frac{1}{2}nr^2$ .

PROOF.—Describe a circle upon the radius through  $P$  as diameter, and apply the foregoing theorem to this circle.

**1096** COR. 1.—The sum of the squares of the intercepts on the radii between the perpendiculars and the centre is also equal to  $\frac{1}{2}nr^2$ . (I. 47)

**1097** COR. 2.—The sum of the squares of the perpendiculars from the equidistant points on the circle to any right line passing through the centre is also equal to  $\frac{1}{2}nr^2$ .

Because the perpendiculars from two points on a circle to the diameters drawn through the points are equal.

**1098** COR. 3.—The sum of the squares of the intercepts on the same right line between the centre of the circle and the perpendiculars is also equal to  $\frac{1}{2}nr^2$ . (I. 47)

If the radii of the inscribed and circumscribed circles of a regular polygon of  $n$  sides be  $r, R$ , and the centre  $O$ ; then,

**1099** I. The sum of the perpendiculars from any point  $P$  upon the sides is equal to  $nr$ .

**1100** II. If  $p$  be the perpendicular from  $O$  upon any right line, the sum of the perpendiculars from the vertices upon the same line is equal to  $np$ .

**1101** III. The sum of the squares of the perpendiculars from  $P$  on the sides is  $= n(r^2 + \frac{1}{2}OP^2)$ .

**1102** IV. The sum of the squares of the perpendiculars from the vertices upon the right line is  $= n(p^2 + \frac{1}{2}R^2)$ .

PROOF.—In theorem I., the values of the perpendiculars are given by  $r - OP \cos\left(\theta + \frac{2m\pi}{n}\right)$ , with successive integers for  $m$ . Add together the  $n$  values, and apply (803).

Similarly, to prove II.; take for the perpendiculars the values

$$p - R \cos\left(\theta + \frac{2m\pi}{n}\right).$$

To prove III. and IV., take the same expressions for the perpendiculars; square each value; add the results, and apply (803, 804).

For additional propositions in the subjects of this section, see the section entitled *Plane Coordinate Geometry*.



# GEOMETRICAL CONICS.

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## THE SECTIONS OF THE CONE.

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**1150** DEFINITIONS.—A *Conic Section* or *Conic* is the curve  $AP$  in which any plane intersects the surface of a right cone.

A *right cone* is the solid generated by the revolution of one straight line about another which it intersects in a fixed point at a constant angle.

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Let the axis of the cone, in Fig. (1) or Fig. (2), be in the plane of the paper, and let the cutting plane  $PMXN$  be perpendicular to the paper. (*Read either the accented or unaccented letters throughout.*) Let a sphere be inscribed in the cone, touching it in the circle  $EQF$  and touching the cutting plane in the point  $S$ , and let the cutting plane and the plane of the circle  $EQF$  intersect in  $XM$ . The following theorem may be regarded as the *defining property* of the curve of section.

**1151** *Theorem*.—The distance of any point  $P$  on the conic from the point  $S$ , called the *focus*, is in a constant ratio to  $PM$ , its distance from the line  $XM$ , called the *directrix*, or

$$PS : PM = PS' : PM' = AS : AX = e, \text{ the eccentricity.}$$

[See next page for the *Proof*.]

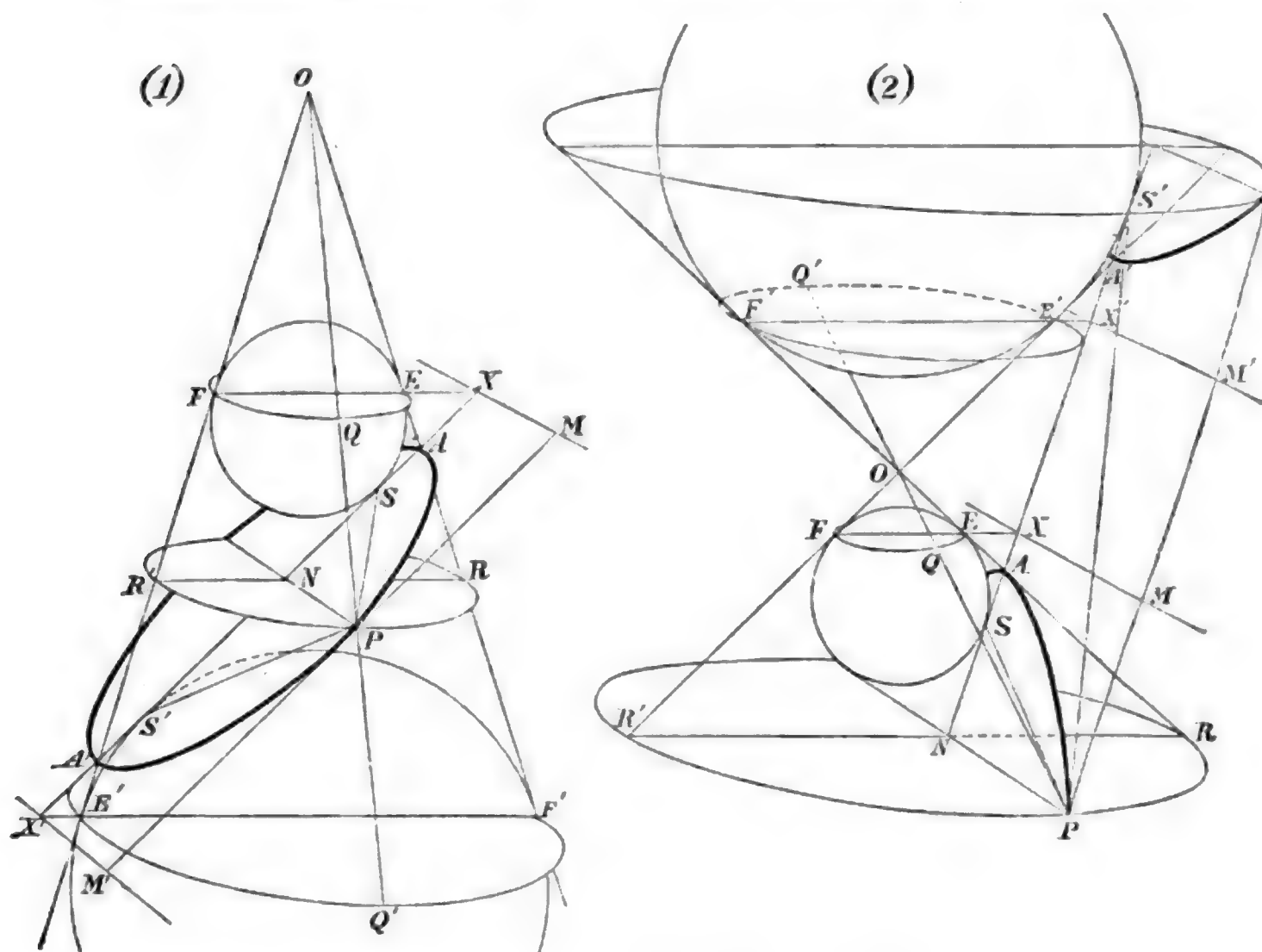
**1152** COR.—The conic may be generated in a plane from either focus  $S, S'$ , and either directrix  $XM, X'M'$ , by the law just proved.

**1153** The conic is an *Ellipse*, a *Parabola*, or an *Hyperbola*, according as  $e$  is less than, equal to, or greater than unity. That is, according as the cutting plane emerges on both sides of the lower cone, or is parallel to a side of the cone, or intersects both the upper and lower cones.

**1154** All sections made by parallel planes are similar; for the inclination of the cutting plane determines the ratio  $AE : AX$ .

**1155** The limiting forms of the curve are respectively—a circle when  $e$  vanishes, and two coincident right lines when  $e$  becomes infinite.

PROOF OF THEOREM 1151.—Join  $P, S$  and  $P, O$ , cutting the circular section in  $Q$ , and draw  $PM$  parallel to  $NX$ . Because all tangents from the same point,  $O$  or  $A$ , to either sphere are equal, therefore  $RE = PQ = PS$  and  $AE = AS$ . Now, by (VI. 2),  $RE : NX = AE : AX$  and  $NX = PM$ ; therefore  $PS : PM = AS : AX$ , a constant ratio denoted by  $e$  and called the *eccentricity* of the conic.



Referring the letters either to the ellipse or the hyperbola in the subjoined figure, let  $C$  be the middle point of  $AA'$  and  $N$  any other point on it. Let  $DD', RR'$  be the two circular sections of the cone whose planes pass through  $C$  and  $N$ ;  $BCB'$  and  $PN$  the intersections with the plane of the conic. In the ellipse,  $BC$  is the common ordinate of the ellipse and circle; but, in the hyperbola,  $BC$  is to be taken equal to the tangent from  $C$  to the circle  $DD'$ .

**1156** The fundamental equation of the ellipse or hyperbola is

$$PN^2 : AN \cdot NA' = BC^2 : AC^2.$$

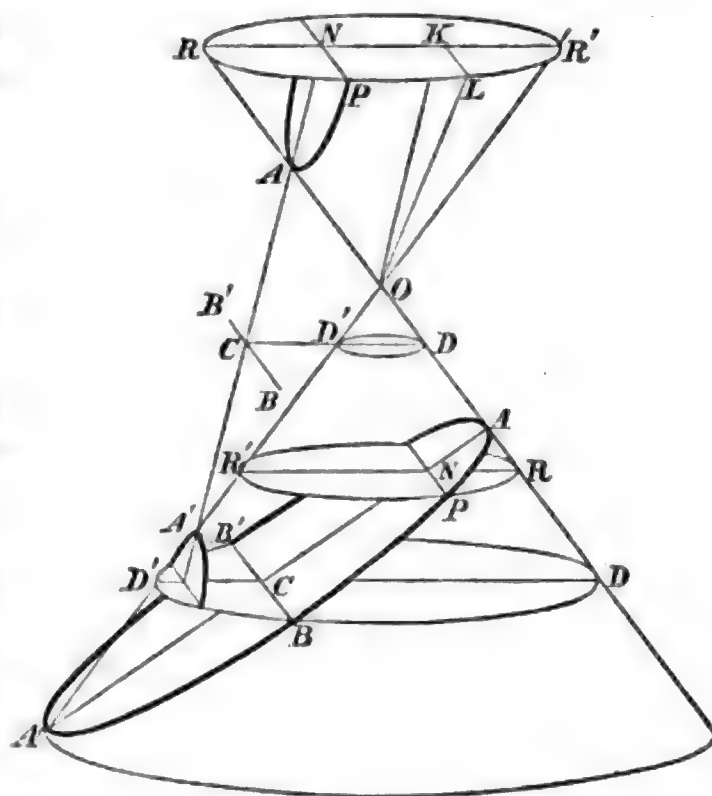
PROOF.— $PN^2 = NR \cdot NR'$  and  $BC^2 = CD \cdot CD'$  (III. 35, 36). Also, by similar triangles (VI. 2, 6),  $NR : CD = AN : AC$  and  $NR' : CD' = A'N : A'C$ . Multiply the last equations together.



**1157** COR. 1.— $PN$  has equal values at two points equi-distant from  $AA'$ . Hence the curve is symmetrical with respect to  $AA'$  and  $BB'$ .

These two lines are called the *major* and *minor axes*, otherwise the *transverse* and *conjugate axes* of the conic.

When the axes are equal, or  $BC = AC$ , the ellipse becomes a circle, and the hyperbola becomes *rectangular* or *equilateral*.



**1158** Any ellipse or hyperbola is the orthogonal projection of a circle or rectangular hyperbola respectively.

PROOF.—Along the ordinate  $NP$ , measure  $NP' = AN \cdot NA'$ ; therefore by the theorem  $PN : P'N = BC : AC$ . Therefore a circle or rectangular hyperbola, having  $AA'$  for one axis, and having its plane inclined to that of the cone at an angle whose cosine  $= BC \div AC$ , projects orthogonally into the ellipse or hyperbola in question, by (1089). See Note to (1201).

**1159** Hence any *projective property* (1076–78), which is known to belong to the circle or rectangular hyperbola, will also be universally true for the ellipse and hyperbola respectively.

## THE ELLIPSE AND HYPERBOLA.

### *Joint properties of the Ellipse and Hyperbola.*

**1160** DEFINITIONS.—The *tangent* to a curve at a point  $P$  (Fig. 1166) is the right line  $PQ$ , drawn through an adjacent point  $Q$ , in its ultimate position when  $Q$  is made to coincide with  $P$ .

The *normal* is the perpendicular to the tangent through the point of contact.



In (Fig. 1171), referred to rectangular axes through the centre  $O$  (see Coordinate Geometry); the length  $CN$  is called the *abscissa*;  $PN$  the *ordinate*;  $PT$  the *tangent*;  $PG$  the *normal*;  $NT$  the *subtangent*; and  $NG$  the *subnormal*.  $S, S'$  are the *foci*;  $XM, X'M'$  the *directrices*;  $PS, PS'$  the *focal distances*, and a double ordinate through  $S$  the *Latus Rectum*.

The *auxiliary circle* (Fig. 1173) is described upon  $AA'$  as diameter.

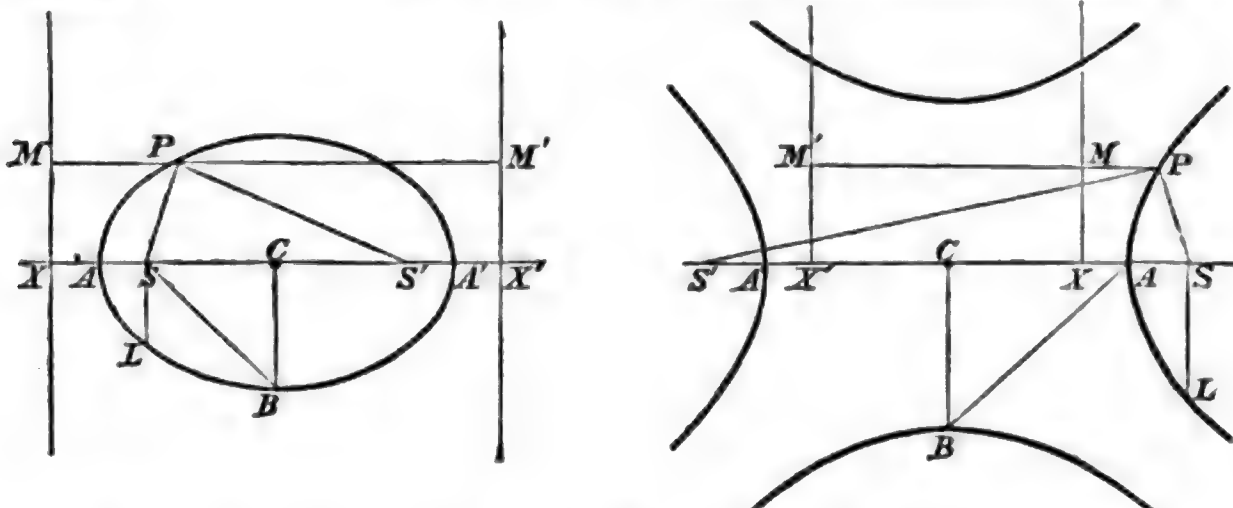
A diameter parallel to the tangent at the extremity of another diameter is termed a *conjugate diameter* with respect to the other.

The *conjugate hyperbola* has  $BC$  for its major, and  $AC$  for its minor axis (1157).

**1161** The following theorems (1162) to (1181) are deduced from the property  $PS : PM = e$  obtained in (1151).

The propositions and demonstrations are nearly identical for the ellipse and the hyperbola, any difference in the application being specified.

**1162**  $CS : CA : CX$ , and the common ratio is  $e$ .



PROOF.—By (1151),  $e = \frac{AS}{AX} = \frac{A'S}{A'X} = \frac{\frac{1}{2}(A'S \pm AS)}{\frac{1}{2}(A'X \pm AX)} = \frac{CS}{CA}$  or  $\frac{CA}{CX}$ .

**1163** In the ellipse the sum, and in the hyperbola the difference, of the focal distances of  $P$  is equal to the major axis, or

$$PS' \pm PS = AA'.$$

PROOF.—With the same figures we have, in the ellipse, by (1151),

$$e = \frac{PS + PS'}{PM + PM'} = \frac{PS + PS'}{XX'}, \text{ and also } e = \frac{AS + A'S}{AX + A'X} = \frac{AA'}{XX'}, \text{ therefore \&c.}$$

For the hyperbola take *difference* instead of *sum*.

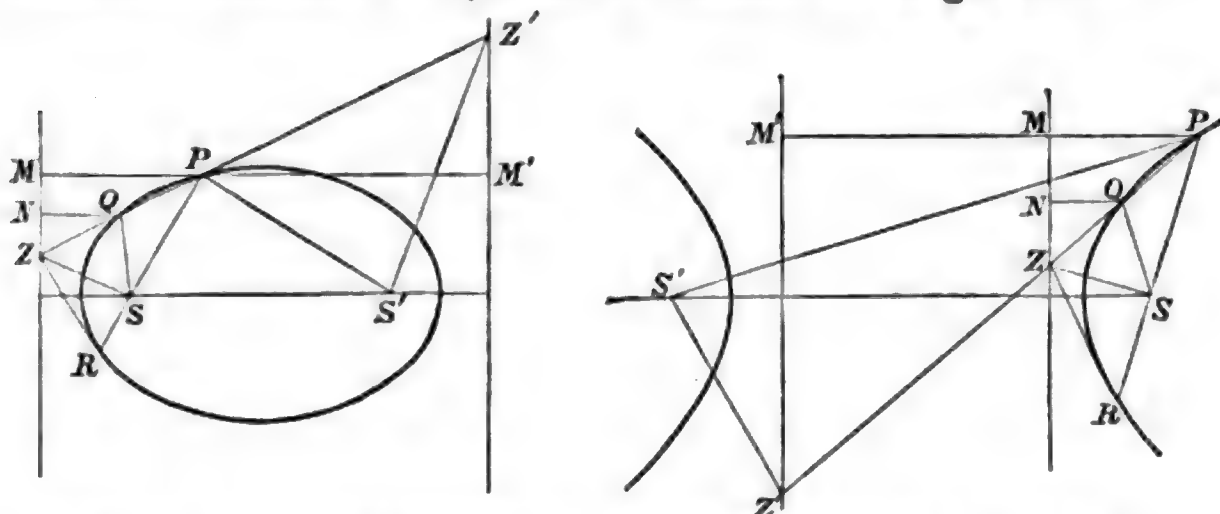
**1164**  $CS^2 = AC^2 - BC^2$  in the ellipse.  
[For  $BS = AC$ , by (1163).

$CS^2 = AC^2 + BC^2$  in the hyperbola.  
[By assuming  $BC$ . See (1176).

**1165**  $BC^2 = SL \cdot AC$ .

PROOF.—(Figs. of 1162)  $SL : SX = CS : CA$ , (1151, 1162)  
 $\therefore SL \cdot AC = CS \cdot SX = CS(CX \sim CS) = CA^2 \sim CS^2$  (1162) =  $BC^2$  (1164).

**1166** If a right line through  $P, Q$ , two points on the conic, meets the directrix in  $Z$ , then  $SZ$  bisects the angle  $QSR$ .



PROOF.—By similar triangles,  $ZP : ZQ = MP : NQ = SP : SQ$  (1151), therefore by (VI. A.)

**1167** If  $PZ$  be a tangent at  $P$ , then  $PSZ$  and  $PS'Z'$  are right angles.

PROOF.—Make  $Q$  coincide with  $P$  in the last theorem.

**1168** The tangent makes equal angles with the focal distances.

PROOF.—In (1166),  $PS : PS' = PM : PM' = PZ : PZ'$ ; therefore, when  $PQ$  becomes the tangent at  $P$ ,  $\angle SPZ = \angle S'PZ'$ , by (1167) and (VI. 7).

**1169** The tangents at the extremities of a focal chord intersect in the directrix.

PROOF.—(Figs. of 1166). Join  $ZR$ ; then, if  $ZP$  is a tangent,  $ZR$  is also, for (1167) proves  $RSZ$  to be a right angle.

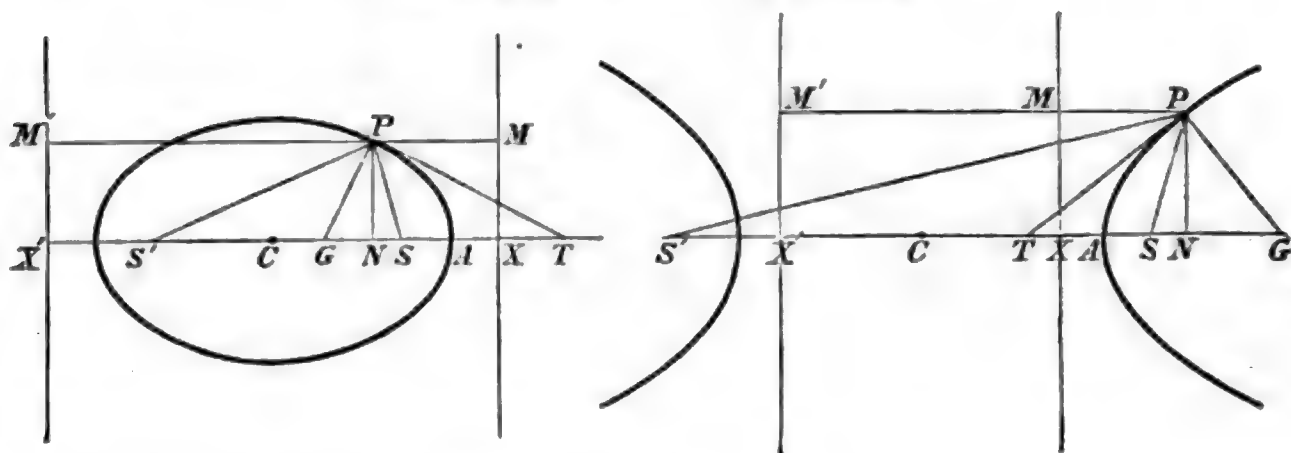
**1170**  $CN \cdot CT = AC^2$ .

PROOF.—(Figs. 1171.)  $\frac{TS'}{TS} = \frac{PS'}{PS}$  (VI. 3, A.) =  $\frac{PM'}{PM}$  (1151) =  $\frac{NX'}{NX}$ ,

therefore  $\frac{TS' + TS}{TS' - TS} = \frac{NX' + NX}{NX' - NX}$  or  $\frac{2CT}{2CS} = \frac{2CX}{2CN}$ ,

therefore  $CN \cdot CT = CS \cdot CX = AC^2$ . (1162)

**1171** If  $PG$  be the normal,  
 $GS : PS = GS' : PS' = e.$



PROOF.—By (1168) and (VI. 3, A.),

$$\frac{GS}{PS} = \frac{GS'}{PS'} = \frac{GS' + GS}{PS' + PS} = \frac{2CS}{2CA} = e. \quad (1162)$$

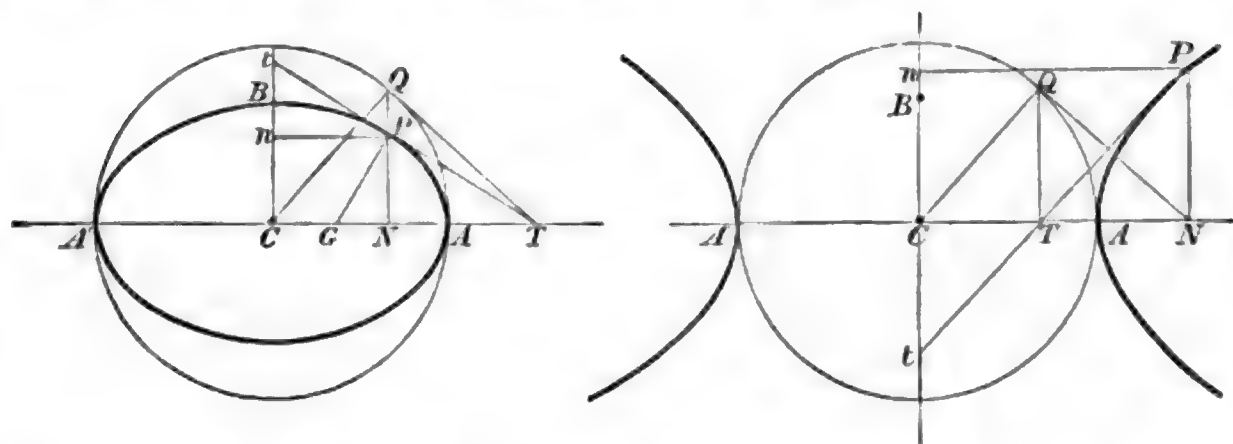
But, for the hyperbola, change *plus* to *minus*.

**1172** The subnormal and the abscissa are as the squares of the axes, or  $NG : NC = BC^2 : AC^2.$

PROOF.—(Figs. 1171.) Exactly as in (1170), taking the normal instead of the tangent, we obtain  $\frac{CG}{CS} = \frac{CN}{CX}$ ,  $\therefore \frac{CN}{CG} = \frac{CX}{CS} = \frac{CA^2}{CS^2}$  (1162),

$$\therefore \frac{CN \sim CG}{CN} = \frac{CA^2 \sim CS^2}{CA^2}, \text{ or } \frac{NG}{NC} = \frac{BC^2}{AC^2} \quad (1164).$$

**1173** The tangents at  $P$  and  $Q$ , the corresponding points on the ellipse and auxiliary circle, meet the axis in the same point  $T$ . But in the hyperbola, the ordinate  $TQ$  of the circle being drawn, the tangent at  $Q$  cuts the axis in  $N$ .



PROOF.—For the ellipse: Join  $TQ$ . Then  $CN \cdot CT = CQ^2$  (1170); therefore  $CQT$  is a right angle (VI. 8); therefore  $QT$  is a tangent.

For the hyperbola: Interchange  $N$  and  $T$ .

**1174**  $PN : QN = BC : AC.$ 

PROOF.—(Figs. 1173).  $NG \cdot NT = PN^2$ , and  $CN \cdot NT = QN^2$ . (VI. 8) Therefore  $NG : NC = PN^2 : QN^2$ ; therefore, by (1172).

This proposition is equivalent to (1158), and shows that an ellipse is the orthogonal projection of a circle equal to the auxiliary circle.

**1175** COR.—The area of the ellipse is to that of the auxiliary circle as  $BC : AC$  (1089).

**1176**  $PN^2 : AN \cdot NA' = BC^2 : AC^2.$ 

PROOF.—By (1174), since  $QN^2 = AN \cdot NA'$  (III. 35, 36). An independent proof of this theorem is given in (1156). The construction for  $BC$  in the hyperbola in (1164) is thus verified.

**1177**  $Cn \cdot Ct = BC^2.$ 

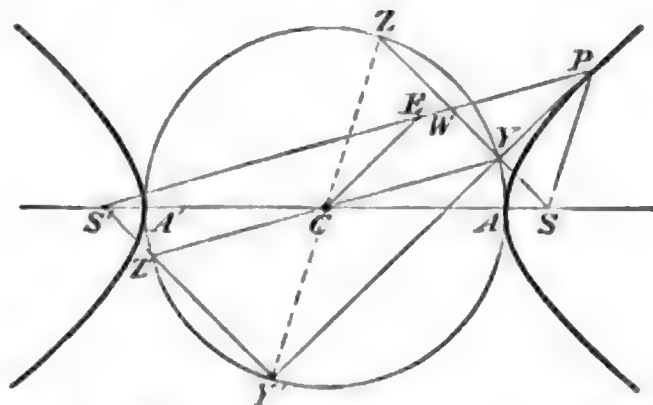
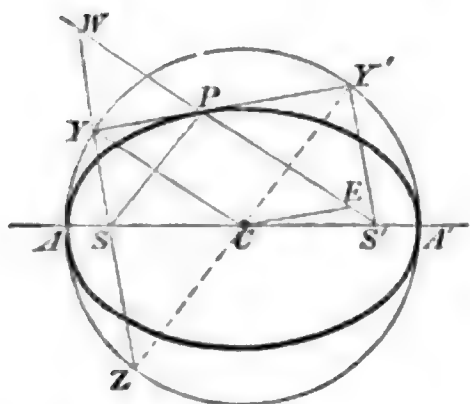
PROOF.—(Figs. 1173.)

$$\frac{Ct}{CT} = \frac{PN}{NT}; \therefore \frac{Cn \cdot Ct}{CN \cdot CT} = \frac{PN^2}{CN \cdot NT} = \frac{PN^2}{QN^2} \text{ (VI. 8)} = \frac{PN^2}{AN \cdot NA'} \text{ (III. 35, 36).}$$

Therefore, by (1170) and (1176),  $Cn \cdot Ct : AC^2 = BC^2 : AC^2$ .

**1178** If  $SY, S'Y'$  are the perpendiculars on the tangent, then  $Y, Y'$  are points on the auxiliary circle, and

$$SY \cdot S'Y' = BC^2.$$



PROOF.—Let  $PS$  meet  $SY$  in  $W$ . Then  $PS = PW$  (1168). Therefore  $S'W = AA'$  (1163). Also,  $SY = YW$ , and  $SC = CS'$ . Therefore  $CY = \frac{1}{2}SW = AC$ . Similarly  $CY' = AC$ . Therefore  $Y, Y'$  are on the circle.

Hence  $ZY'$  is a diameter (III. 31), and therefore  $SZ = S'Y'$ , by similar triangles; therefore  $SY \cdot SZ = SA \cdot SA'$  (III. 35, 36)  $= CS^2 \sim CA^2$  (II. 5)  $= BC^2$  (1164).

**1179** COR.—If  $CE$  be drawn parallel to the tangent at  $P$ , then  $PE = CY = AC$ .

**1180** PROBLEM.—To draw tangents from any point  $O$  to an ellipse or hyperbola.

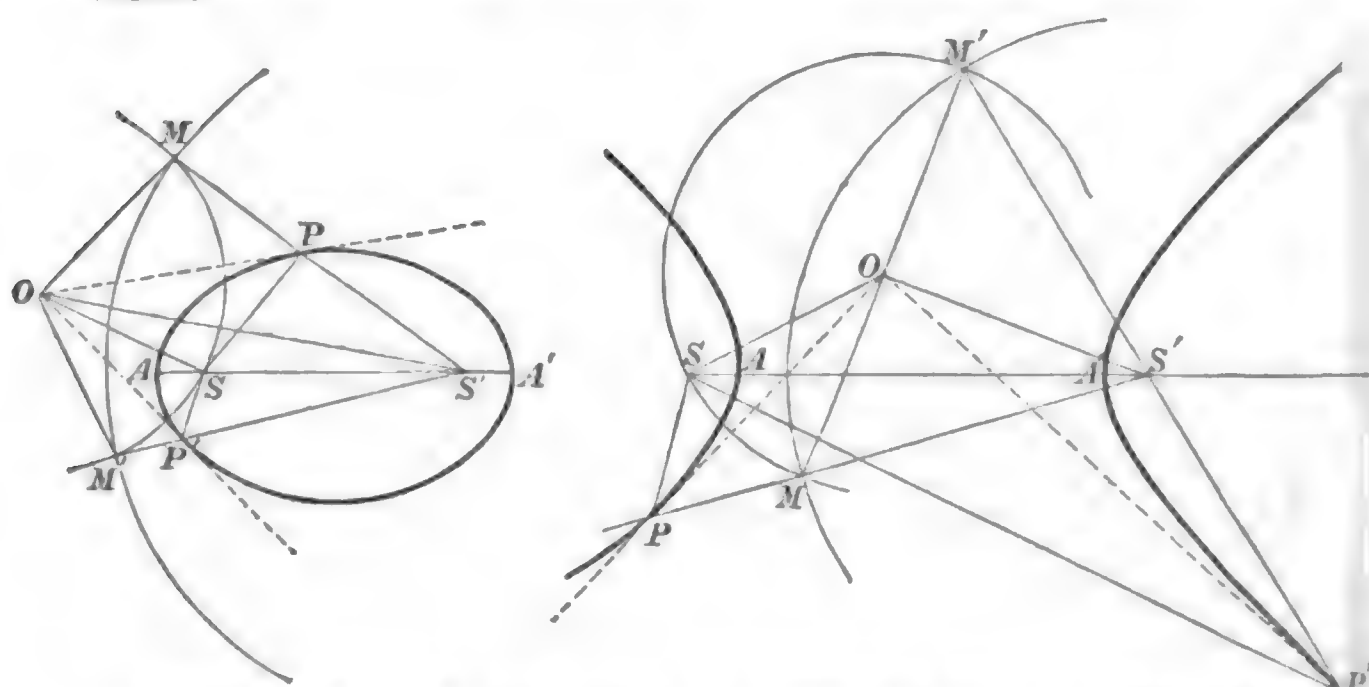
CONSTRUCTION.—(Figs. 1181.) Describe two circles, one with centre  $O$  and radius  $OS$ , and another with centre  $S'$  and radius  $= AA'$ , intersecting in  $M$ ,

*M*. Join  $MS'$ ,  $M'S$ . These lines will intersect the curve in  $P$ ,  $P'$ , the points of contact. For another method see (1204).

PROOF.—By (1163),  $PS \pm PS' = AA' = S'M$  by construction. Therefore  $PS = PM$ , therefore  $\angle OPS = OPM$  (I. 8), therefore  $OP$  is a tangent by (1168).

Similarly  $P'S = P'M'$ , and  $OP'$  is a tangent.

**1181** The tangents  $OP$ ,  $OP'$  subtend equal angles at either focus.

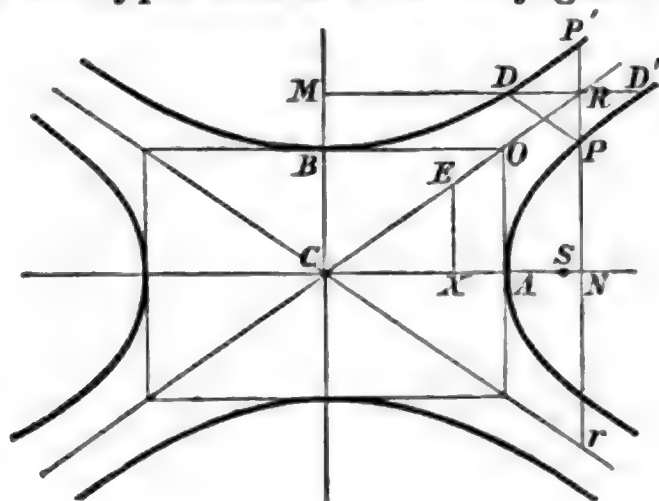


PROOF.—The angles  $OSP$ ,  $OSP'$  are respectively equal to  $OMP$ ,  $OM'P'$ , by (I. 8), as above; and these last angles are equal, by the triangles  $OSM$ ,  $OS'M'$ , and (I. 8). Similarly at the other focus.

### *Asymptotic Properties of the Hyperbola.*

**1182** DEF.—The *asymptotes* of the hyperbola are the diagonals of the rectangle formed by tangents at the vertices  $A$ ,  $A'$ ,  $B$ ,  $B'$ .

**1183** If the ordinates  $RN$ ,  $RM$  from any point  $R$  on an asymptote cut the hyperbola and its conjugate in  $P$ ,  $P'$ ,  $D$ ,  $D'$ ,



then either of the following pairs of equations will define both the branches of each curve—

$$RN^2 - PN^2 = BC^2 = P'N^2 - RN^2 \dots\dots\dots (1),$$

$$RM^2 - DM^2 = AC^2 = D'M^2 - RM^2 \dots\dots\dots (2).$$

PROOF.—Firstly, to prove (1): By proportion from the similar triangles  $RNC$ ,  $OAC$ , we have  $\frac{RN^2}{CN^2} = \frac{BC^2}{AC^2} = \frac{PN^2}{CN^2 - AC^2}$ ;

by (1176), since  $AN \cdot NA' = CN^2 - AC^2$ . By (II. 6)

Therefore  $\frac{RN^2 - PN^2}{AC^2} = \frac{BC^2}{AC^2}$ , by the theorem (69);

therefore  $RN^2 - PN^2 = BC^2$ .

Also, by (1176), applied to the conjugate hyperbola, the axes being now reversed,

$$\frac{CN^2}{P'N^2 - BC^2} = \frac{AC^2}{BC^2} = \frac{CN^2}{RN^2} \quad \text{by similar triangles;}$$

therefore  $P'N^2 - BC^2 = RN^2$  or  $P'N^2 - RN^2 = BC^2$ .

Secondly, to prove (2): By proportion from the triangles  $RMC$ ,  $OBC$ , we have  $\frac{RM^2}{CM^2} = \frac{AC^2}{BC^2} = \frac{DM^2}{CM^2 - BC^2}$ ,

by (1176), applied to the conjugate hyperbola, for in this case we should have  $BM \cdot MB' = CM^2 - BC^2$ .

Therefore  $\frac{RM^2 - DM^2}{BC^2} = \frac{AC^2}{BC^2}$ ; therefore  $RM^2 - DM^2 = AC^2$ .

Also, by (1176), since  $CM$ ,  $DM$  are equal to the coordinates of  $D$ ,

$$\frac{CM^2}{D'M^2 - AC^2} = \frac{BC^2}{AC^2} = \frac{CM^2}{RM^2}, \quad \text{by similar triangles;}$$

therefore  $D'M^2 - AC^2 = RM^2$  or  $D'M^2 - RM^2 = AC^2$ .

**1184** COR. 1.—If the same ordinates  $RN$ ,  $RM$  meet the other asymptote in  $r$  and  $r'$ , then

$$PR \cdot Pr = BC^2 \quad \text{and} \quad DR \cdot Dr' = AC^2. \quad (\text{II. 5})$$

**1185** COR. 2.—As  $R$  recedes from  $C$ ,  $PR$  and  $DR$  continually diminish. Hence the curves continually approach the asymptote.

**1186** If  $XE$  be the directrix,  $CE = AC$ .

PROOF.—  $CE : CO = CX : CA = CA : CS$  and  $CS = CO$ . (1164)

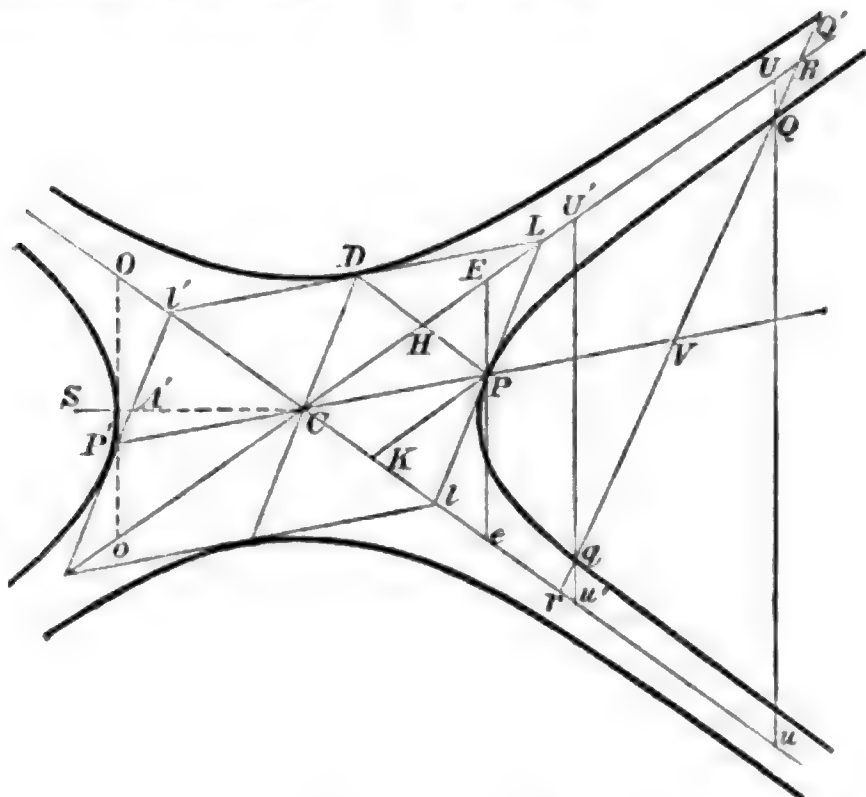
**1187**  $PD$  is parallel to the asymptote.

PROOF.—  $\frac{RN^2}{RM^2} = \frac{BC^2}{AC^2} = \frac{RN^2 - PN^2}{RM^2 - DM^2}$  (1183)  $= \frac{PN^2}{DM^2}$  (69).

Therefore  $RN : PN = RM : DM$ ; therefore, by (VI. 2).



**1188** The segments of any right line between the curve and the asymptote are equal, or  $QR = qr$ .



PROOF.—  $QR : QU = qR : qU' \}$  Compound the ratios,  
and  $Qr : Qu = qr : qu' \}$  and employ (1184).

**1189** COR. 1.— $PL = Pl$  and  $QV = qV$ .

**1190** COR. 2.— $CH = HL$ . Because  $PD$  is parallel to  $lC$ .  
(1187)

**1191**  $QR \cdot Qr = PL^2 = RV^2 - QV^2 = Q'V^2 - RV^2$ .

PROOF.—  $QR : QU = PL : PE \}$  Compound the ratios. Therefore, by (1184),  
and  $Qr : Qu = Pl : Pe \}$   $QR \cdot Qr = PL \cdot Pl = PL^2$  (1189).

**1192**  $4PH \cdot PK = CS^2$ .

PROOF.—  $PH : PE = CO : Oo \}$   $\therefore PH \cdot PK : PE \cdot Pe = CO^2 : Oo^2$   
and  $PK : Pe = Co : Oo \}$   $= CS^2 : 4EC^2$ ; therefore, by (1184).

### *Joint Properties of the Ellipse and Hyperbola resumed.*

If  $PCP'$  be a diameter, and  $QV$  an ordinate parallel to the conjugate diameter  $CD$  (Figs. 1195 and 1188).

**1193**  $QV^2 : PV \cdot VP' = CD^2 : CP^2$ .

This is the fundamental equation of the conic, equation (1176) being the most important form of it.

Otherwise :

In the ellipse,  $QV^2 : CP^2 - CV^2 = CD^2 : CP^2$ .

In the hyperbola,  $QV^2 : CV^2 - CP^2 = CD^2 : CP^2$ ;

and  $Q'V^2 : CV^2 + CP^2 = CD^2 : CP^2$ .

PROOF.—(*Ellipse*. Fig. 1195.)—By orthogonal projection from a circle. If  $C, P, P', D, Q, V$  are the projections of  $c, p, p', d, q, v$  on the circle;  $qv^2 = pv \cdot vp'$  and  $cd^2 = cp^2$ . The proportion is therefore true in the case of the circle. Therefore &c., by (1088).

(*Hyperbola*. Fig. 1188.)—

$$\frac{CI^2}{CF^2} = \frac{RV^2}{CV^2} = \frac{PL^2}{CP^2} = \frac{RV^2 \pm PL^2}{CV^2 \pm CP^2} = \frac{QV^2}{CV^2 - CP^2} \text{ or } \frac{Q'V^2}{CV^2 + CP^2}. \quad (1191)$$

**1194** The parallelogram formed by tangents at the extremities of conjugate diameters is of constant area, and therefore,  $PF$  being perpendicular to  $CD$  (Figs. 1195),

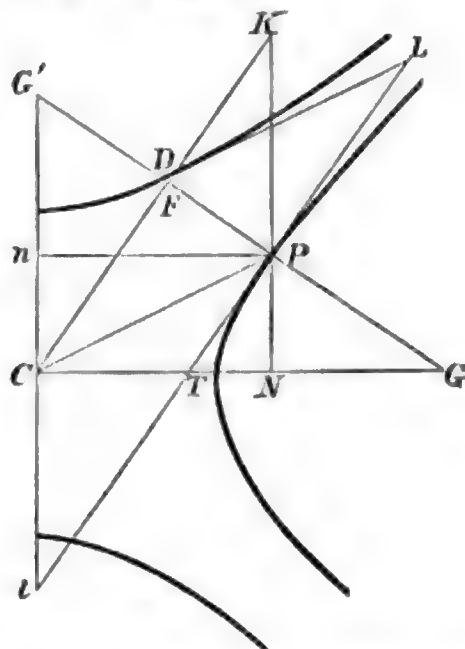
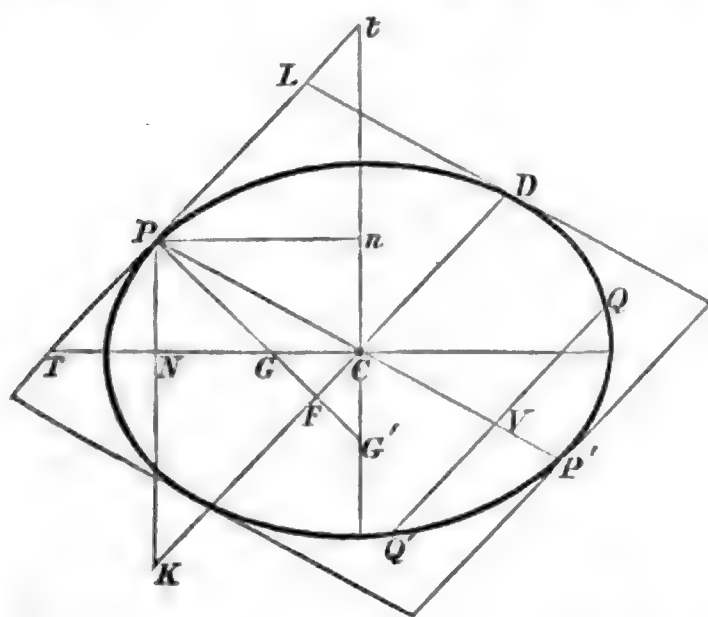
$$PF \cdot CD = AC \cdot BC.$$

PROOF.—(*Ellipse*.)—By orthogonal projection from the circle (1089).

(*Hyperbola*. Fig. 1188.)— $CL \cdot Cl = 4PH \cdot PK = CO \cdot Co$  (1192); therefore, by (VI. 15),  $\Delta LCl = OCo = AC \cdot BC$ .

If  $PF$  intersects the axes in  $G$  and  $G'$ ,

$$\mathbf{1195} \quad PF \cdot PG = BC^2 \text{ and } PF \cdot PG' = AC^2.$$



PROOF.—  $PF \cdot PG = PK \cdot PN = Cn \cdot Ct = BC^2$  (1177). Similarly for  $PF \cdot PG'$ .

$$\mathbf{1197} \text{ COR.} \quad PG \cdot PG' = CD^2 = PT \cdot Pt. \quad \text{By (1194)}$$

**1198** The diameter bisects all chords parallel to the tangent at its extremity.

PROOF.—(*Ellipse*. Fig. 1195.)—By projection from the circle (1088)  $QV = VQ'$ . (*Hyperbola*.) By (1189.)

**1199** COR. 1.—The tangents at the extremities of any chord meet on the diameter which bisects it.

PROOF.—The secants drawn through the extremities of two parallel chords meet on the diameter which bisects them (VI. 4), and the tangents are the limiting positions of the secants when the parallel chords coincide.

**1200** COR. 2.—If the tangents from a point are equal, the diameter through the point must be a principal axis. (I. 8)

**1201** COR. 3.—The chords joining any point  $Q$  on the curve with the extremities of a diameter  $PP'$ , are parallel to conjugate diameters, and are called *supplemental chords*.

For the diameter bisecting  $PQ$  is parallel to  $P'Q$  (VI. 2). Similarly the diameter bisecting  $P'Q$  is parallel to  $PQ$ .

**1202** Diameters are mutually conjugate; If  $CD$  be parallel to the tangent at  $P$ ,  $CP$  will be parallel to the tangent at  $D$ .

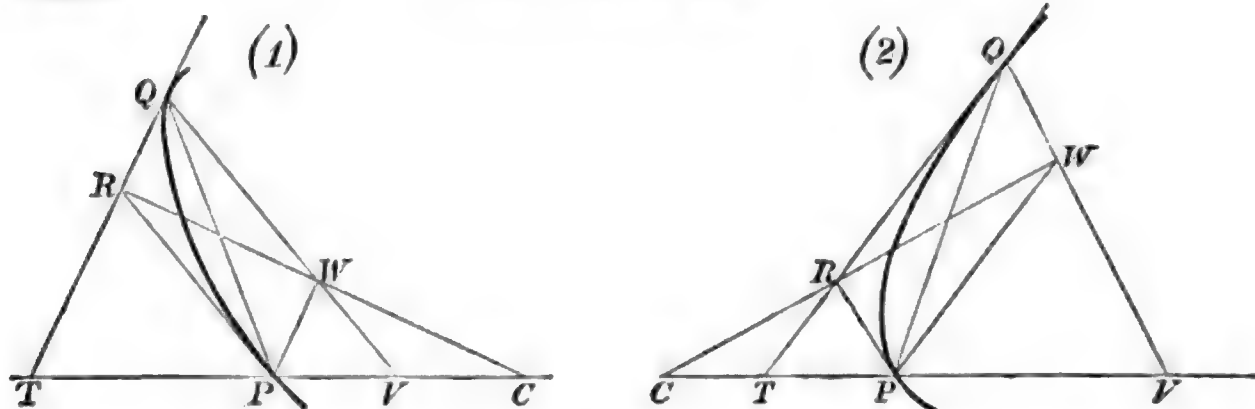
PROOF.—(*Ellipse*. Fig. 1205.)—By projection from the circle (1088).

NOTE.—Observe that, if the ellipse in the figure with its ordinates and tangents be turned about the axis  $Tt$  through the angle  $\cos^{-1}(BC \div AC)$ , it becomes the projection of the auxiliary circle with its corresponding ordinates and tangents.

(*Hyperbola*. Fig. 1188.)—By (1187, 1189) the tangents at  $P, D$  meet the asymptotes in the same point  $L$ . Therefore they are parallel to  $CD, CP$  (VI. 2.)

If  $QT$  be the tangent at  $Q$ , and  $QV$  the ordinate parallel to the tangent at any other point  $P$ ,

**1203**  $CV \cdot CT = CP^2$ .

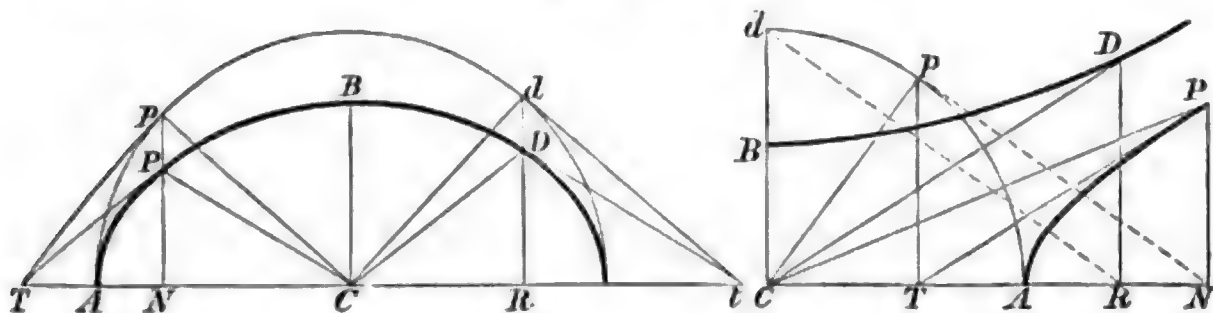


PROOF.— $CR$  bisects  $PQ$  (1199). Therefore  $PW$  is parallel to  $QR$ . Therefore, by (VI. 2),  $CV : CP = CW : CR = CP : CT$ .

**1204** COR.—Hence, to draw two tangents from a point  $T$ , we may find  $CV$  from the above equation, and draw  $QVQ'$  parallel to the tangent at  $P$  to determine the points of contact  $Q, Q'$ .

Let  $PN, DN$  be the ordinates at the extremities of conjugate diameters, and  $PT$  the tangent at  $P$ . Let the ordinates at  $N$  and  $R$  in the ellipse, but at  $T$  and  $C$  in the hyperbola, meet the auxiliary circle in  $p$  and  $d$ ; then

**1205**  $CN = dR, \quad CR = pN.$



PROOF.—(Ellipse.)  $Cp, Cd$  are parallel to the tangents at  $d$  and  $p$  (Note to 1202). Therefore  $pCd$  is a right angle. Therefore  $pNC, CRd$  are equal right-angled triangles with  $CN = dR$  and  $CR = pN$ .

(Hyperbola.)  $CN \cdot CT = AC^2$  (1170),  
and  $DR \cdot CT = 2\Delta CDT = 2CDP = AC \cdot BC$  (1194);  
 $\therefore \frac{CN}{DR} = \frac{AC}{BC} = \frac{pN}{PN}$  (1174),  $\therefore \frac{CN}{pN} = \frac{DR}{PN} = \frac{CR}{TN}$  (similar triangles).

But  $\frac{CN}{pN} = \frac{pN}{TN}$  (VI. 8);  $\therefore CR = pN$ . Also  $Cp = Cd$ ; therefore the triangles  $CpN, dCR$  are equal and similar; therefore  $CN = dR$  and  $dR$  is parallel to  $pN$ .

**1206** COR.—  $DR : dR = BC : AC.$

PROOF.—(Ellipse.) By (1174). (Hyperbola.) By the similar right-angled triangles, we have  $dR : pN = CR : TN = DR : PN$ ;  
therefore  $dR : DR = pN : PN = AC : BC$  (1174).

In the same figures,

**1207** (Ellipse.)  $CN^2 + CR^2 = AC^2; \quad DR^2 + PN^2 = BC^2.$

**1209** (Hyperbola.)  $CN^2 - CR^2 = AC^2; \quad DR^2 - PN^2 = BC^2.$

PROOF.—Firstly, from the right-angled triangle  $CNp$  in which  $pN = CR$  (1205).

Secondly, In the ellipse, by (1174),  $DR^2 + PN^2 : dR^2 + pN^2 = BC^2 : AC^2$ , and  $dR^2 + pN^2 = AC^2$ , by (1205). For the hyperbola, take difference of squares.

**1211** (*Ellipse.*)  $CP^2 + CD^2 = AC^2 + BC^2$ .

**1212** (*Hyperbola.*)  $CP^2 - CD^2 = AC^2 - BC^2$ .

PROOF.—(Figs. 1205.) By (1205—1210) and (I. 47), applied to the triangles  $CNP$ ,  $CRD$ .

The product of the focal distances is equal to the square of the semi-conjugate diameter, or

**1213**  $PS \cdot PS' = CD^2$ .

PROOF.—(*Ellipse.* Fig. 1171.)  $2PS \cdot PS' = (PS + PS')^2 - PS^2 - PS'^2 = 4AC^2 - 2CS^2 - 2CP^2$  (922, i.)  $= 2(AC^2 + BC^2 - CP^2)$  (1164)  $= 2CD^2$  (1211).

(*Hyperbola.*)—Similarly with  $2PS \cdot PS' = PS^2 + PS'^2 - (PS' - PS)^2 = \&c$ .

**1214** The products of the segments of intersecting chords  $QOq$ ,  $Q'Oq'$  are in the ratio of the squares of the diameters parallel to them, or

$$OQ \cdot Oq : OQ' \cdot Oq' = CD^2 : CD'^2.$$

PROOF.—(*Ellipse.*) By projection from the circle (1088); for the proportion is true for the circle, by (III. 35, 36).

(*Hyperbola.* Fig. 1188.) Let  $O$  be any point on  $Qq$ . Draw  $IOi$  parallel to  $Ee$ , meeting the asymptotes in  $I$  and  $i$ ; then

$$OR \cdot Or - OQ \cdot Oq = QR \cdot Qr \text{ (II. 5)} = PL^2 \text{ (1191) } \dots\dots\dots (1).$$

$$\text{Now } \frac{OR}{OI} = \frac{PL}{PE}, \text{ and } \frac{Or}{Oi} = \frac{Pl}{Pe}; \therefore \frac{OR \cdot Or}{OI \cdot Oi} = \frac{PL^2}{PE \cdot Pe} = \frac{CD^2}{BC^2} \text{ (1184).}$$

$$\text{Therefore } \frac{OR \cdot Or - PL^2}{OI \cdot Oi - BC^2} = \frac{CD^2}{EC^2}; \text{ or, by (1), } \frac{OQ \cdot Oq}{OI \cdot Oi - BC^2} = \frac{CD^2}{BC^2}.$$

Similarly for any other chord  $Q'Oq'$  drawn through  $O$ .

$$\text{Therefore } OQ \cdot Oq : OQ' \cdot Oq' = CD^2 : CD'^2.$$

**1215** COR.—The tangents from any point to the curve are in the ratio of the diameters parallel to them.

For, when  $O$  is without the curve and the chords become tangents, each product of segments becomes the square of a tangent.

**1216** If from any point  $Q$  on a tangent  $PT$  drawn to *any* conic (Fig. 1220), two perpendiculars  $QR$ ,  $QL$  be drawn to the focal distance  $PS$  and the directrix  $XM$  respectively; then

$$SR : QL = e.$$

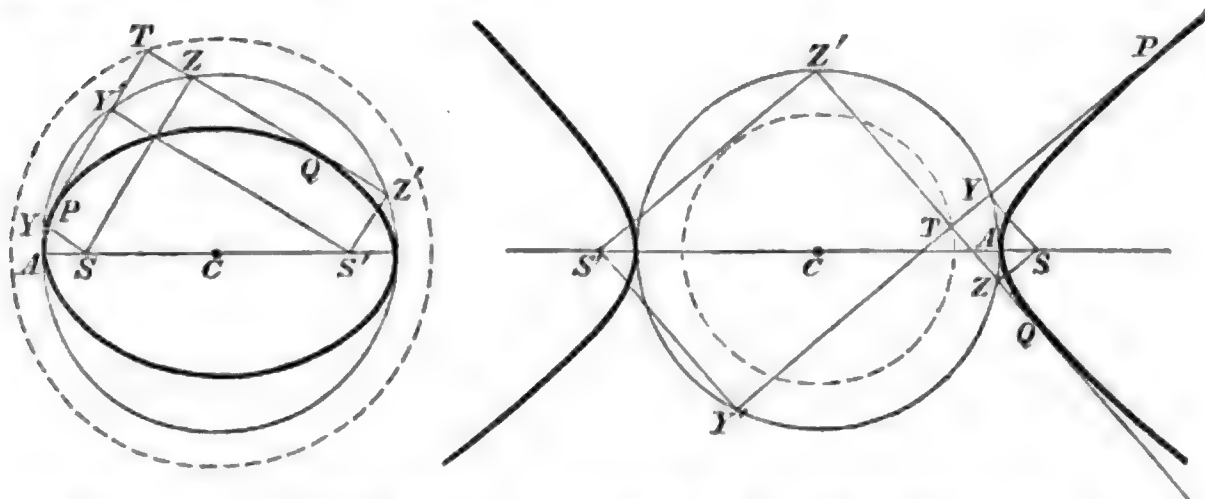
PROOF.—Since  $QR$  is parallel to  $ZS$  (1167), therefore, by (VI. 2),

$$SR : PS = QZ : PZ = QL : PM;$$

$$\text{therefore } SR : QL = PS : PM = e.$$

COR.—By applying the theorem to each of the tangents from  $Q$ , a proof of (1181) is obtained.

**1217** *The Director Circle.*—The locus of the point of intersection,  $T$ , of two tangents always at right angles is a circle called the *Director Circle*.



PROOF.—Perpendiculars from  $S, S'$  to the tangents meet them in points  $Y, Z, Y', Z'$ , which lie on the auxiliary circle. Therefore, by (II. 5, 6) and (III. 35, 36),  $TC^2 \sim AC^2 = TZ \cdot TZ' = SY \cdot SY' = BC^2$ . (1178)

Therefore  $TC^2 = AC^2 \pm BC^2$ , a constant value.

NOTE.—Theorems (1170), (1177), and (1203) may also be deduced at once for the ellipse by orthogonal projection from the circle; and, in all such cases, the analogous property of the hyperbola may be obtained by a similar projection from the rectangular hyperbola if the property has already been demonstrated for the latter curve.

**1218** If the points  $A, S$  (Fig. 1162) be fixed, while  $C$  is moved to an infinite distance, the conic becomes a parabola. Hence, any relation which has been established for parts of the curve which remain finite, when  $AC$  thus becomes infinite, will be a property of the parabola.

**1219** Theorems relating to the ellipse may generally be adapted to the parabola by eliminating the quantities which become infinite, employing the principle that *finite differences may be neglected in considering the ratios of infinite quantities*.

EXAMPLE.—In (1193), when  $P'$  is at infinity,  $VP'$  becomes  $= 2CP$ ; and in (1213)  $PS'$  becomes  $= 2CP$ . Thus the equations become

$$\frac{QV^2}{PV} = \frac{2CD^2}{CP} \quad \text{and} \quad PS = \frac{CD^2}{2CP}.$$

Therefore  $QV^2 = 4PS \cdot PV$  in the parabola.



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 THE PARABOLA.
 

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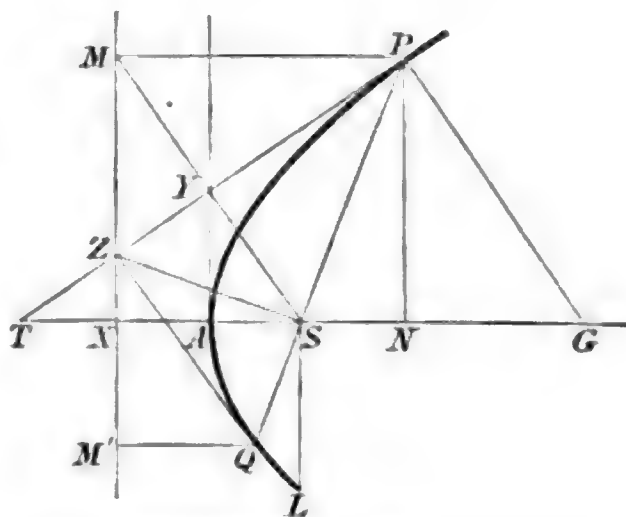
If  $S$  be the focus,  $XM$  the directrix, and  $P$  any point on the curve, the *defining property* is

**1220**  $PS = PM$

and  $e = 1.$  (1153)

**1221** Hence

$$AX = AS.$$



**1222** The Latus Rectum  $= 4AS.$

PROOF.—  $SL = SX$  (1220)  $= 2AS.$

**1223** If  $PZ$  be a tangent at  $P$ , meeting the directrix in  $Z$ , then  $PSZ$  is a right angle.

PROOF.—As in (1167); theorem (1166) applying equally to the parabola.

**1224** The tangent at  $P$  bisects the angles  $SPM, SZM.$

PROOF.— $PZ$  is common to the triangles  $PSZ, PMZ$ ;  $PS = PM$  and  $\angle PSZ = \angle PMZ$  (1223).

**1225** COR.—  $ST = SP = SG.$  (I. 29, 6)

**1226** The tangents at the extremities of a focal chord  $PQ$  intersect at right angles in the directrix.

PROOF.—(i.) They intersect in the directrix, as in (1169).

(ii.) They bisect the angles  $SZM, SZM'$  (1224), and therefore include a right angle.

**1227** The curve bisects the sub-tangent.  $AN = AT.$

PROOF.—  $ST = SP$  (1225)  $= PM = XN$ , and  $AX = AS.$

**1228** The sub-normal is half the latus rectum.  $NG = 2AS.$

PROOF.— $ST = SP = SG$  and  $TX = SN$  (1227). Subtract.

**1229**  $PN^2 = 4AS \cdot AN.$ 

PROOF.— $PN^2 = TN \cdot NG$  (VI. 8)  $= AN \cdot 2NG$  (1227)  $= 4AS \cdot AN$  (1228).  
Otherwise, by (1176) and (1165); making  $AC$  infinite. See (1219).

**1230** The tangents at  $A$  and  $P$  each bisect  $SM$ , the latter bisecting it at right angles.

PROOF.—(i.) The tangent at  $A$ , by (VI. 2), since  $AX = AS$ .

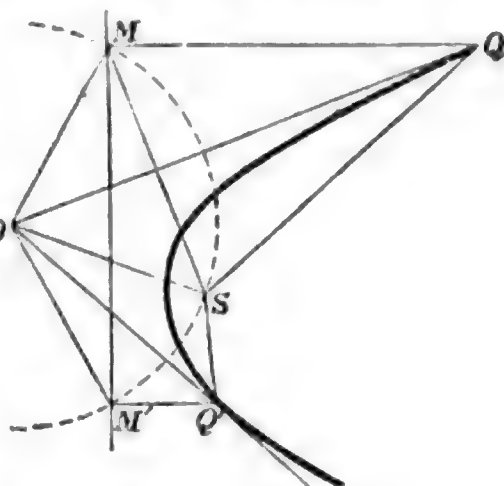
(ii.)  $PT$  bisects  $SM$  at right angles, by (I. 4), since  $PS = PM$  and  $\angle SPY = MPY$ .

**1231** COR.—  $SA : SY : SP.$  [By similar triangles.]**1232** To draw tangents from a point  $O$  to the parabola.

CONSTRUCTION.—Describe a circle, centre  $O$  and radius  $OS$ , cutting the directrix in  $M, M'$ . Draw  $MQ, M'Q'$  parallel to the axis, meeting the parabola in  $Q, Q'$ . Then  $OQ, OQ'$  will be tangents.

PROOF.— $OS, SQ = OM, MQ$  (1220); therefore, by (I. 8),  $\angle OQS = \angle OQM$ ; therefore  $OQ$  is a tangent (1224). Similarly  $OQ'$  is a tangent.

Otherwise, by (1181). When  $S'$  moves to infinity, the circle  $MM'$  becomes the directrix.

**1233** COR. 1.—The triangles  $SQO, SOQ'$  are similar, and  $SQ : SO : SQ'.$ 

PROOF.—  $\angle SQO = \angle MQO = \angle SMM' = \angle SOQ'. \quad (\text{III. 20})$   
Similarly  $\angle SQ'O = \angle SOQ.$

**1234** COR. 2.—The tangents at two points subtend equal angles at the focus; and they contain an angle equal to half the exterior angle between the focal distances of the points.

PROOF.—  $\angle OSQ = \angle OSQ',$  by (Cor. 1).

Also  $\angle QOQ' = \angle SOQ + \angle SQO = \pi - \angle OSQ = \frac{1}{2} \angle OSQ'.$

**1235** DEF.—Any line parallel to the axis of a parabola is called a *diameter*.**1236** The chord of contact  $QQ'$  of tangents from any point  $O$  is bisected by the diameter through  $O$ .

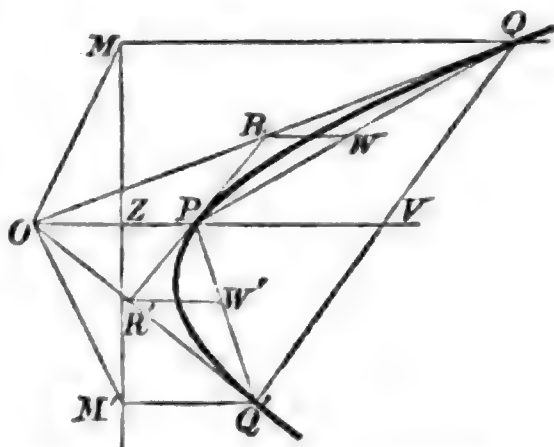
PROOF.—This proposition and the corollaries are included in (1198–1200), by the principle in (1218). An independent proof is as follows.

The construction being as in (1232), we have  $ZM = ZM'$ ; therefore  $QV = VQ'$  (VI. 2).

**1237** COR. 1.—The tangent  $RR'$  at  $P$  is parallel to  $QQ'$ ; and  $OP = PV$ .

PROOF.—Draw the diameter  $RW$ .  $QW = WP$ ; therefore  $QR = RO$  (VI. 2). Similarly  $Q'R' = R'O$ .

**1238** COR. 2.—Hence, the diameter through  $P$  bisects *all* chords parallel to the tangent at  $P$ .



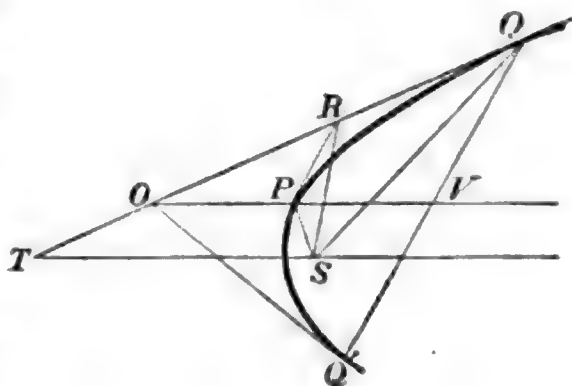
If  $QV$  be a semi-chord parallel to the tangent at  $P$ ,  
**1239**  $QV^2 = 4PS \cdot PV$ .

This is the fundamental equation of the parabola, equation (1229) being the most important form of it.

PROOF.—Let  $QO$  meet the axis in  $T$ . By similar triangles (1231),

$\angle SRP = \angle SQR = \angle STQ = \angle POR$ ;  
 and  $\angle SPR = \angle OPR$  (1224). Therefore  $PR^2 = PS \cdot PO = PS \cdot PV$  and  $QV = 2PR$ .

Otherwise: See (1219), where the equation is deduced from (1193) of the ellipse.



**1240** COR. 1.—If  $v$  be any other point, either within or without the curve, on the chord  $QQ'$ , and  $pv$  the corresponding diameter,  
 $vQ \cdot vQ' = 4pS \cdot pv$ . (II. 5)

**1241** COR. 2.—The focal chord parallel to the diameter through  $P$ , and called the *parameter* of that diameter, is equal to  $4SP$ . For  $PV$  in this case is equal to  $PS$ .

**1242** The products of the segments of intersecting chords,  $QOq$ ,  $Q'Oq'$ , are in the ratio of the parameters of the diameters which bisect the chords; or

$$OQ \cdot Oq : OQ' \cdot Oq' = PS : P'S.$$

PROOF.—By (1240), the ratio is equal to  $4PS \cdot pO : 4P'S \cdot pO$ .

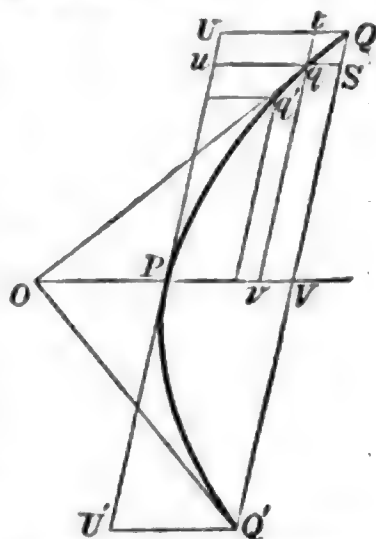
Otherwise: In the ellipse (1214), the ratio is  $= \frac{CD^2}{CD^2} = \frac{PS \cdot PS'}{P'S \cdot P'S'} = \frac{PS}{P'S}$  (1213) when  $S'$  is at infinity and the curve becomes a parabola (1219).

**1243** COR.—The squares of the tangents to a parabola from any point are as the focal distances of the points of contact.

PROOF.—As in (1215). Otherwise, by (1233) and (VI. 19).

**1244** The area of the parabola cut off by any chord  $QQ'$  is two-thirds of the circumscribed parallelogram, or of the triangle formed by the chord and the tangents at  $Q, Q'$ .

PROOF.—Through  $Q, q, q', \&c.$ , adjacent points on the curve, draw right lines parallel to the diameter and tangent at  $P$ . Let the secant  $Qq$  cut the diameter in  $O$ . Then, when  $q$  coincides with  $Q$ , so that  $Qq$  becomes a tangent, we have  $OP = PV$  (1237). Therefore the parallelogram  $Vq = 2Uq$ , by (I. 43), applied to the parallelogram of which  $OQ$  is the diagonal. Similarly  $vq' = 2uq'$ , &c. Therefore the sum of all the evanescent parallelograms on one side of  $PQ$  is equal to twice the corresponding sum on the other side; and these sums are respectively equal to the areas  $PQV, PQU$ .—(NEWTON, Sect. I., Lem. II.)



### *Practical methods of constructing the Conic.*

**1245** *To draw the Ellipse.*

Fix two pins at  $S, S'$  (Fig. 1162). Place over them a loop of thread having a perimeter  $SPS' = SS' + AA'$ . A pencil point moved so as to keep the thread stretched will describe the ellipse, by (1163).

**1246** *Otherwise.*—(Fig. 1173.) Draw  $PHK$  parallel to  $QC$ , cutting the axes in  $H, K$ .  $PK = AC$  and  $PH = BC$  (1174). Hence, if a ruler  $PHK$  moves so that the points  $H, K$  slide along the axes,  $P$  will describe the ellipse.

**1247** *To draw the Hyperbola.*

Make the pin  $S'$  (Fig. 1162) serve as a pivot for one end of a bar of any convenient length. To the free end of the bar attach one end of a thread whose length is less than that of the bar by  $AA'$ ; and fasten the other end of the thread to the pin  $S$ . A pencil point moved so as to keep the thread stretched, and touching the bar, will describe the hyperbola, by (1163).

**1248** *Otherwise:*—Lay off any scale of equal parts along both asymptotes (Fig. 1188), starting and numbering the divisions from  $C$ , in both positive and negative directions.

Join every pair of points  $L, l$ , the product of whose distances from  $C$  is the same, and a series of tangents will be formed (1192) which will define the hyperbola. See also (1289).

1249

*To draw the Parabola.*

Proceed as in (1247), with this difference: let the end of the bar, before attached to  $S'$ , terminate in a "T-square," and be made to slide along the directrix (Fig. 1220), taking the string and bar of the same length.

1250

*Otherwise:*—Make the same construction as in (1248), and join every pair of points, the algebraic sum of whose distances from the zero point of division is the same.

PROOF.—If the two equal tangents from any point  $T$  on the axis (Fig. 1239) be cut by a third tangent in the points  $R, r$ ; then  $RQ$  may be proved equal to  $rT$ , by (1233), proving the triangles  $SRQ, SrT$  equal in all respects.

1251

COR.—The triangle  $SRr$  is always similar to the isosceles triangle  $SQT$ .

1252

*To find the axes and centre of a given central conic.*

(i.) Draw a right line through the centres of two parallel chords. This line is a diameter, by (1198); and two diameters so found will intersect in the centre of the conic.

(ii.) Describe a circle having for its diameter any diameter  $PP'$  of the conic, and let the circle cut the curve in  $Q$ . Then  $PQ, P'Q$  are parallel to the axes, by (1201) and (III. 31).

1253

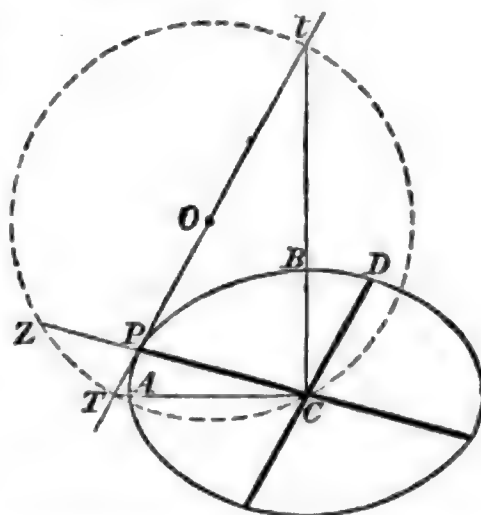
*Given two conjugate diameters,  $CP, CD$ , in position and magnitude: to construct the conic.*

On  $CP$  take  $PZ = CD^2 \div CP$ ; measuring from  $C$  in the ellipse, and towards  $C$  in the hyperbola (Fig. 1188). A circle described through the points  $C, Z$ , and having its centre  $O$  on the tangent at  $P$ , will cut the tangent in the points where it is intersected by the axes.

PROOF.—*Analysis:* Let  $AC, BC$  cut the tangent at  $P$  in  $T, t$ . The circle whose diameter is  $Tt$  will pass through  $C$  (III. 31), and will make

$$CP \cdot PZ = PT \cdot Pt \text{ (III. 35, 36) } = CD^2 \text{ (1197).}$$

Hence the construction.



### *Circle and Radius of Curvature.*

1254

DEFINITIONS.—The circle which has the same tangent with a curve at  $P$  (Fig. 1259), and which passes through another point  $Q$  on the curve, becomes the *circle of curvature* when  $Q$  ultimately coincides with  $P$ ; and its radius becomes the *radius of curvature*.

**1255** Otherwise.—The *circle of curvature* is the circle which passes through *three* coincident points on the curve at *P*.

**1256** Any chord *PH* of the circle of curvature is called a *chord of curvature* at *P*.

**1257** Through *Q* draw *RQ'* parallel to *PH*, meeting the tangent at *P* in *R*, and the circle in *Q'*, and draw *QV* parallel to *PR*. *RQ* is called a *subtense* of the arc *PQ*.

**1258** THEOREM.—Any chord of curvature *PH* is equal to the ultimate value of the square of the arc *PQ* divided by the subtense *RQ* parallel to the chord: and this is also equal to

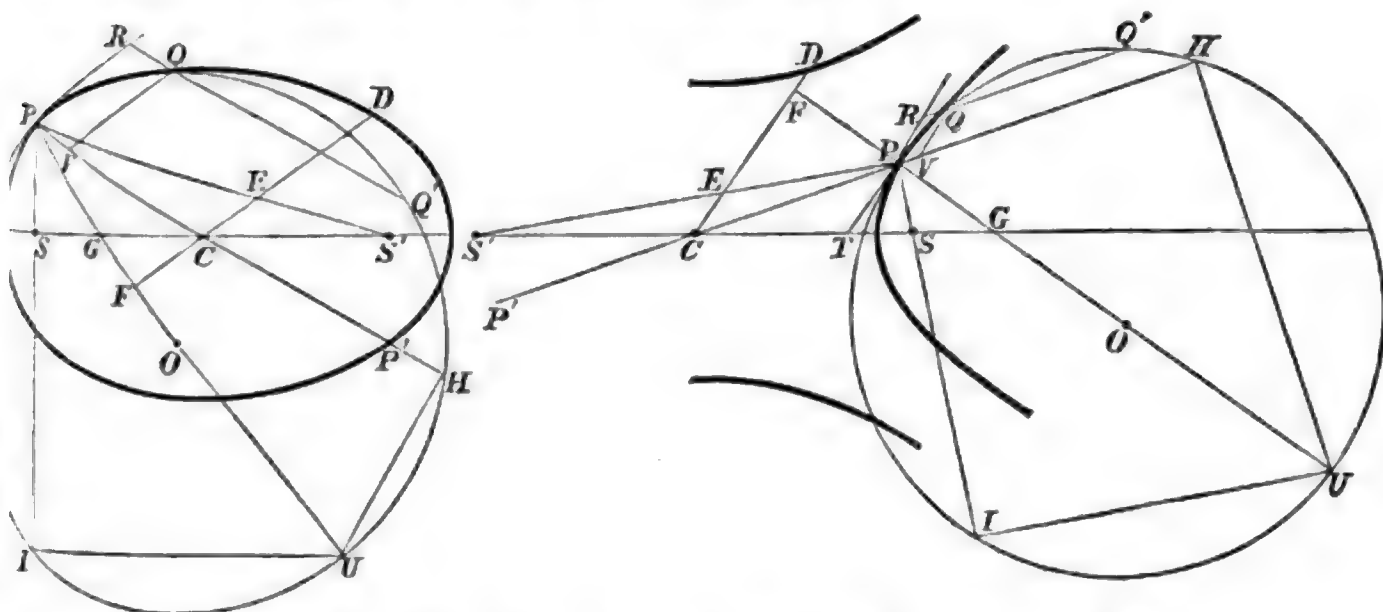
$$QV^2 \div PV.$$

PROOF.— $RQ' = RP^2 \div RQ$  (III. 36). And when *Q* moves up to *P*, *RQ'* becomes *PH*; and *RP*, *PQ*, and *QV* become equal because coincident lines.

**1259** In the ellipse or hyperbola, the semi-chords of curvature at *P*, measured along the diameter *PC*, the normal *PF*, and the focal distance *PS*, are respectively equal to

$$\frac{CD^2}{CP}, \quad \frac{CD^2}{PF}, \quad \frac{CD^2}{AC};$$

the second being the radius of curvature at *P*.



PROOF.—(i.) By (1258),  $PH = \frac{QV^2}{PV} = \frac{VP' \cdot CD^2}{CP^3}$  (1193)  $= \frac{2CD^2}{CP}$  in the limit when  $VP'$  becomes  $PF = 2CP$ .

(ii.) By the similar triangles *PHU*, *PFC* (III. 31), we have  $PU \cdot PF = CP \cdot PH = 2CD^2$ , by (i.)

(iii.) By the similar triangles *PIU*, *PFE* (1168), we have  $PI \cdot PE = PU \cdot PF = 2CD^2$ , by (ii.); and  $PE = AC$  (1179).



**1260** In the parabola, the chord of curvature at  $P$  (Fig. 1259) drawn parallel to the axis, and the one drawn through the focus, are each equal to  $4SP$ , the parameter of the diameter at  $P$  (1241).

PROOF.—By (1258). The chord parallel to the axis  $= QV^2 \div PV = 4PS$  (1239); and the two chords are equal because they make equal angles with the diameter of the circle of curvature.

**1261** COR.—The radius of curvature of the parabola at  $P$  (Fig. 1220) is equal to  $2SP^2 \div SY$ .

PROOF.—(Fig. 1259.)  $\frac{1}{2}PU = \frac{1}{2}PI \sec IPU = 2SP \sec PSY$  (Fig. 1221).

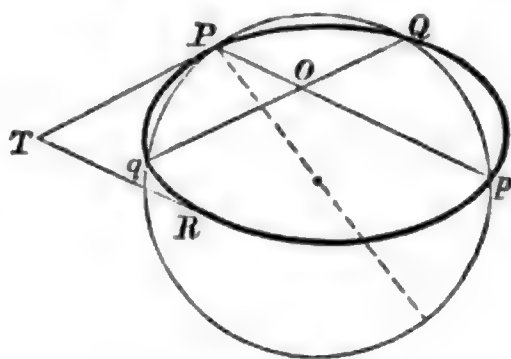
**1262** The products of the segments of intersecting chords are as the squares of the tangents parallel to them (1214–15), (1242–43).

**1263** The common chords of a circle and conic (Fig. 1264) are equally inclined to the axis; and conversely, if two chords of a conic are equally inclined to the axis, their extremities are concyclic.

PROOF.—The products of the segments of the chords being equal (III. 35, 36), the tangents parallel to them are equal (1262). Therefore, by (1200).

**1264** The common chord of any conic and of the circle of curvature at a point  $P$ , has the same inclination to the axis as the tangent at  $P$ .

PROOF.—Draw any chord  $Qq$  parallel to the tangent at  $P$ . The circle circumscribing  $PQq$  always passes through the same point  $p$  (1263), and does so, therefore, when  $Qq$  moves up to  $P$ , and the circle becomes the circle of curvature.



**1265** PROBLEM.—To find the centre of curvature at any given point of a conic.

*First Method.*—(Fig. 1264.) Draw a chord from the point making the same angle with the axis as the tangent. The perpendicular bisector of the chord will meet the normal in the centre of curvature, by (1264) and (III. 3).

**1266** *Second Method.*—Draw the normal  $PG$  and a perpendicular to it from  $G$ , meeting either of the focal distances in  $Q$ . Then a perpendicular to the focal distance drawn from  $Q$  will meet the normal in  $O$ , the centre of curvature.



Hence, the relations in (1205-6) still subsist when  $CA, CB$  are any conjugate radii. Thus universally,

$$1281 \quad PN : CR = DR : CN \text{ or } PN \cdot CN = DR \cdot CR.$$

1282 If the tangent at  $P$  meets any pair of conjugate diameters in  $T, T'$ , then  $PT \cdot PT'$  is constant and equal to  $CD^2$ .

PROOF.—Let  $CA, CB$  (Figs. 1205) be the conjugate radii, the figures being deformed through any angle. By similar triangles,

$$\left. \begin{array}{l} PT : PN = CD : DR \\ PT' : CN = CD : CB \end{array} \right\}, \text{ therefore } PT \cdot PT' : PN \cdot CN = CD^2 : DR \cdot CR.$$

Therefore  $PT \cdot PT' = CD^2$ , by (1281).

1283 If the tangent at  $P$  meets any pair of parallel tangents in  $T, T'$ , then  $PT \cdot PT' = CD^2$ , where  $CD$  is conjugate to  $CP$ .

PROOF.—Let the parallel tangents touch in the points  $Q, Q'$ . Join  $PQ, PQ', CT, CT'$ . Then  $CT, CT'$  are conjugate diameters (1199, 1201). Therefore  $PT \cdot PT' = CD^2$  (1282).

1284 COR.— $QT \cdot Q'T' = CD^2$ , where  $CD$  is the radius parallel to  $QT$ .

1285 To draw two conjugate diameters of a conic to include a given angle. Proceed as in (1252 ii.), making  $PP'$  in this case the chord of the segment of a circle containing the given angle (III. 33).

1286 The focal distance of a point  $P$  on any conic is equal to the length  $QN$  intercepted on the ordinate through  $P$  between the axis and the tangent at the extremity of the latus rectum.

PROOF.—(Fig. 1220).  $QN : NX = LS : SX = e$  and  $SP : NX = e$ .

1287 In the hyperbola (Fig. 1183).  $CO : CA = e$ . (1162, 1164).

If a right line  $PKK'$  be drawn parallel to the asymptote  $CE$ , cutting the one directrix  $XE$  in  $K$  and the other in  $K'$ ; then

$$1288 \quad SP = PK = e \cdot CN - AC; \quad S'P = PK' = e \cdot CN + AC.$$

PROOF.—From  $CR = e \cdot CN$  (1287) and  $CE = AC$  (1186).

1289 COR.—Hence the hyperbola may be drawn mechanically by the method of (1249) by merely fixing the cross-piece of the T-square at an angle with the bar equal to  $BCO$ .

1290 DEFINITION.—*Confocal conics* are conics which have the same foci.

1291 The tangents drawn to any conic from a point  $T$  on a confocal conic make equal angles with the tangent at  $T$ .

PROOF.—(Fig. 1217.) Let  $T$  be the point on the confocal conic.

$$SY : SZ = S'Z' : S'Y' \text{ (1178).}$$

Therefore  $ST$  and  $S'T$  make equal angles with the tangents  $TP, TQ$ ; and they also make equal angles with the tangent to the confocal at  $T$  (1168), therefore &c.

1292 In the construction of (1253),  $PZ$  is equal to half the chord of curvature at  $P$  drawn through the centre  $C$  (1259).

# DIFFERENTIAL CALCULUS.

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## INTRODUCTION.

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**1400 Functions.**—A quantity which depends for its value upon another quantity  $x$  is called a *function* of  $x$ . Thus,  $\sin x$ ,  $\log x$ ,  $a^x$ ,  $a^2 + ax + x^2$  are all functions of  $x$ . The notation  $y = f(x)$  expresses generally that  $y$  is a function of  $x$ .  $y = \sin x$  is a particular function.

**1401**  $f(x)$  is called a *continuous* function between assigned limits, when an indefinitely small change in the value of  $x$  always produces an indefinitely small change in the value of  $f(x)$ .

A *transcendental* function is one which is not purely algebraical, such as the exponential, logarithmic, and circular functions  $a^x$ ,  $\log x$ ,  $\sin x$ ,  $\cos x$ , &c.

If  $f(x) = f(-x)$ , the function is called an *even* function. If  $f(x) = -f(-x)$ , it is called an *odd* function.

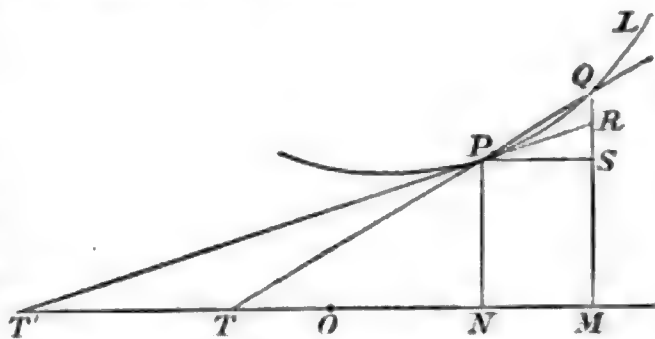
Thus,  $x^2$  and  $\cos x$  are even functions, while  $x^3$  and  $\sin x$  are odd functions of  $x$ ; the latter, but not the former, being altered in value by changing the sign of  $x$ .

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**1402 Differential Coefficient or Derivative.**—Let  $y$  be any function of  $x$  denoted by  $f(x)$ , such that any change in the value of  $x$  causes a definite change in the value of  $y$ ; then  $x$  is called the *independent variable*, and  $y$  the *dependent variable*. Let an indefinitely small change in  $x$ , denoted by  $dx$ , produce a corresponding small change  $dy$  in  $y$ ; then the ratio  $\frac{dy}{dx}$ , in the limit when both  $dy$  and  $dx$  are vanishing, is called the *differential coefficient*, or *derivative*, of  $y$  with respect to  $x$ .

**1403 THEOREM.**—The ratio  $dy : dx$  is *definite* for each value of  $x$ , and generally *different* for different values.

PROOF.—Let an abscissa  $ON$  (Def. 1160) be measured from  $O$  equal to  $x$ , and a perpendicular ordinate  $NP$  equal to  $y$ . Then, *whatever may be the form of the function*  $y = f(x)$ , as  $x$  varies, the locus of  $P$  will be *some line*  $PQL$ . Let  $OM = x'$ ,  $MQ = y'$  be values of  $x$  and  $y$  near to the former values. Let the straight line  $QP$  meet the axis in  $T$ ; and when  $Q$  coincides with  $P$ , let the final direction of  $QP$  cut the axis in  $T'$ .



Then  $\frac{QS}{SP}$  or  $\frac{y' - y}{x' - x} = \frac{PN}{NT'}$ . And, ultimately, when  $QS$  and  $SP$  vanish, they vanish in the ratio of  $PN : NT'$ . Therefore  $\frac{dy}{dx} = \frac{PN}{NT'} = \tan PTN$ , a *definite* ratio at each point of the curve, but *different* at different points.

**1404** Let  $NM$ , the increment of  $x$ , be denoted by  $h$ ; then, when  $h$  vanishes,  $\frac{dy}{dx} = \frac{f(x+h) - f(x)}{h} = f'(x)$ ,

a new function of  $x$ , called also the *first derived function*. The process of finding its value is called *differentiation*.

**1405 Successive differentiation.**—If  $\frac{dy}{dx}$  or  $f'(x)$  be differentiated with respect to  $x$ , the result is the *second differential coefficient* of  $f(x)$ , or the *second derived function*; and so on to any number of differentiations. These successive functions may be represented in any of the three following systems of notation:—

$$\begin{array}{ccccccc} \frac{dy}{dx}, & \frac{d^2y}{dx^2}, & \frac{d^3y}{dx^3}, & \frac{d^4y}{dx^4}, & \dots\dots & \frac{d^ny}{dx^n}; \\ f'(x), & f''(x), & f'''(x), & f^{iv}(x), & \dots\dots & f^n(x); \\ y_x, & y_{2x}, & y_{3x}, & y_{4x}, & \dots\dots & y_{nx}.^* \end{array}$$

The operations of differentiating a function of  $x$  once, twice, or  $n$  times, are also indicated by prefixing the symbols

$$\frac{d}{dx}, \frac{d^2}{dx^2}, \dots \frac{d^n}{dx^n}; \text{ or } \frac{d}{dx}, \left(\frac{d}{dx}\right)^2, \dots \left(\frac{d}{dx}\right)^n;$$

or, more concisely,  $d_x, d_{2x}, \dots d_{nx}$ .

\* See note to (1487).



**1406** If, *after* differentiating a function for  $x$ ,  $x$  be made zero in the result, the value may be indicated in any of the following ways:  $\frac{dy}{dx_0}$ ,  $f'(0)$ ,  $y_{x0}$ ,  $\frac{d}{dx_0}$ ,  $d_{x0}$ .

If any other constant  $a$  be substituted for  $x$  in  $y_x$ , the result may be indicated by  $y_{x,a}$ .

**1407** *Infinitesimals and Differentials.*—The evanescent quantities  $dx$ ,  $dy$  are called *infinitesimals*; and, with respect to  $x$  and  $y$ , they are called *differentials*.  $dx^2$ ,  $d^2y$  are the *second differentials* of  $x$  and  $y$ ;  $dx^3$ ,  $d^3y$  the *third*, and so on.

**1408** The successive differentials of  $y$  are expressed in terms of  $dx$  by the equations

$$dy = f'(x) dx; \quad d^2y = f''(x) dx^2; \quad \&c., \quad \text{and} \quad d^n y = f^n(x) dx^n.$$

Since  $f'(x)$  is the coefficient of  $dx$  in the value of  $dy$ , it has therefore been named *the differential coefficient of  $y$  or  $f(x)$* .<sup>\*</sup> For similar reasons  $f''(x)$  is called the *second*, and  $f^n(x)$  the  *$n^{\text{th}}$  differential coefficient of  $f(x)$* , &c.

**1409** Two infinitesimals are of the *same order* when their ratio is neither zero nor infinity.

If  $dx$ ,  $dy$  are infinitesimals of the same order,  $dx^2$ ,  $dy^2$ , and  $dx dy$  will be infinitesimals of the *second* order with respect to  $dx$ ,  $dy$ ;  $dx^3$ ,  $dx^2 dy$ , &c. will be of the *third* order, and so on.

$dx$ ,  $dx^2$ , &c. are sometimes denoted by  $\dot{x}$ ,  $\ddot{x}$ , &c.

**1410** LEMMA.—In estimating the ratio of two quantities, any increment of either which is infinitely small in comparison with the quantities may be neglected.

Hence the ratio of two infinitesimals of the same order is not affected by adding to or subtracting from either of them an infinitesimal of a higher order.

EXAMPLE.— $\frac{dy - dx^2}{dx} = \frac{dy}{dx} - dx = \frac{dy}{dx}$ , for  $dx$  is zero in comparison with the ratio  $\frac{dy}{dx}$ . Thus, in Fig. (1403), putting  $PS = dx$ ,  $QS = dy$ ; we have ultimately, by (1258),  $QR = k dx^2$ , where  $k$  is a constant. Therefore  $\frac{PN}{NT} = \frac{RS}{PS} = \frac{dy - k dx^2}{dx} = \frac{dy}{dx}$  in the limit, by the principle just enunciated; that is,  $QR$  vanishes in comparison with  $PS$  or  $QS$  even when those lines themselves are infinitely small.

\* The name is slightly misleading, as it seems to imply that  $f'(x)$  is in some sense a coefficient of  $f(x)$ .



## DIFFERENTIATION.

## DIFFERENTIATION OF A SUM, PRODUCT, AND QUOTIENT.

Let  $u, v$  be functions of  $x$ , then

$$1411 \quad \frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

$$1412 \quad \frac{d(uv)}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}.$$

$$1413 \quad \frac{d}{dx} \left( \frac{u}{v} \right) = \left( v \frac{du}{dx} - u \frac{dv}{dx} \right) \div v^2.$$

PROOF.—(i.)  $d(u+v) = (u+du+v+dv) - (u+v) = du+dv.$

(ii.)  $d(uv) = (u+du)(v+dv) - uv = vdu + u dv - dudv,$

and, by (1410),  $dudv$  disappears in the ultimate ratio to  $dx$ .

(iii.)  $d\left(\frac{u}{v}\right) = \frac{u+du}{v+dv} - \frac{u}{v} = \frac{vdu - u dv}{(v+dv)v},$

therefore &c., by (1410);  $v dv$  vanishing in comparison with  $v^2$ .

Hence, if  $u$  be a constant  $= c$ ,

$$1414 \quad \frac{d(cv)}{dx} = c \frac{dv}{dx} \quad \text{and} \quad \frac{d}{dx} \left( \frac{c}{v} \right) = -\frac{c}{v^2} \frac{dv}{dx}.$$

## DIFFERENTIATION OF A FUNCTION OF A FUNCTION.

If  $y$  be a function of  $z$ , and  $z$  a function of  $x$ ,

$$1415 \quad \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

PROOF.—Since, in all cases, the change  $dx$  causes the change  $dz$ , and the change  $dz$  causes the change  $dy$ ; therefore the change  $dx$  causes the change  $dy$  in the limit.

Differentiating the above as a product, by (1412), the successive differential coefficients of  $y$  can be formed. The first four are here subjoined for the sake of reference. Observe that  $(y_z)_z = y_{zz}$ .

$$1416 \quad y_x = y_z z_x.$$

$$1417 \quad y_{zx} = y_{zz} z_x^2 + y_z z_{zx}.$$

$$1418 \quad y_{3x} = y_{3z} z_x^3 + 3y_{2z} z_x z_{zx} + y_z z_{3x}.$$

$$1419 \quad y_{4x} = y_{4z} z_x^4 + 6y_{3z} z_x^2 z_{zx} + y_{2z} (3z_{zx}^2 + 4z_x z_{3x}) + y_z z_{4x}.$$

## DIFFERENTIATION OF A COMPOSITE FUNCTION.

If  $u$  and  $v$  be explicit functions of  $x$ , so that  $u = \phi(x)$  and  $v = \psi(x)$ ,

$$1420 \quad \frac{dF(u, v)}{dx} = \frac{dF}{du} \frac{du}{dx} + \frac{dF}{dv} \frac{dv}{dx}.$$

Here  $dF$  in the first term on the right is the change in  $F(u, v)$  produced by  $du$ , the change in  $u$ ; and  $dF$  in the second term is the change produced by  $dv$ , so that the total change  $dF(u, v)$  may be written as in (1408)

$$dF_1 + dF_2 = \frac{dF}{du} du + \frac{dF}{dv} dv.$$

## DIFFERENTIATION OF THE SIMPLE FUNCTIONS.

Since  $\frac{dy}{dx} = \frac{f(x+h) - f(x)}{h}$  when  $h$  vanishes, we have the following rule for finding its value:

**1421** RULE.—Expand  $f(x+h)$  by some known theorem in ascending powers of  $h$ ; subtract  $f(x)$ ; divide by  $h$ ; and in the result put  $h$  equal to zero.

The differential coefficients which follow are obtained by the rule and the theorems indicated.

$$1422 \quad y = x^n. \quad \frac{dy}{dx} = nx^{n-1}.$$

$$\text{PROOF.}—\text{Here } \frac{f(x+h) - f(x)}{h} = \frac{(x+h)^n - x^n}{h} = \frac{nx^{n-1}h + O(n, 2)x^{n-2}h^2 + \dots}{h}$$

$$(125) = nx^{n-1} + O(n, 2)x^{n-2}h + \dots = nx^{n-1}, \text{ when } h \text{ vanishes.}$$

**1423** COR.—

$$\frac{d^r y}{dx^r} = n(n-1) \dots (n-r+1) x^{n-r}. \quad \frac{d^n y}{dx^n} = \lfloor n \rfloor.$$

$$1424 \quad y = \log_a x; \quad \frac{dy}{dx} = \frac{1}{x \log_e a}.$$

$$\text{PROOF.}—\text{By (145), } \frac{\log_e(x+h) - \log_e x}{h} = \frac{1}{x \log_e a} \left\{ \log_e \left( 1 + \frac{h}{x} \right) \right\} \div \frac{h}{x}.$$

Expand the logarithm by (155).

$$1425 \quad \text{COR.}— \quad \frac{d^n y}{dx^n} = \frac{(-1)^{n-1} \lfloor n-1 \rfloor}{x^n \log a}. \quad \text{Put } n = -1 \text{ in (1423) and } r = n-1.$$

$$1426 \quad y = a^x; \quad \frac{dy}{dx} = a^x \log_e a.$$

PROOF.—  $\frac{a^{x+h} - a^x}{h} = \frac{a^x (a^h - 1)}{h}.$  Expand  $a^h$  by (149).

$$1427 \quad \text{COR.—} \quad \frac{d^n y}{dx^n} = a^x (\log_e a)^n.$$

	Function.	Derivative.	Method of Proof by Rule (1421) and Limits (753).
1428	$\sin x.$	$\cos x.$	Expand by (627, 629), and put $1 - \cos h = 2 \sin^2 \frac{h}{2}.$
1429	$\cos x.$	$-\sin x.$	
1430	$\tan x.$	$\sec^2 x.$	Expand by (631), observing (1410).
1431	$\cot x.$	$-\text{cosec}^2 x.$	By $\cot x = \frac{1}{\tan x}$ , and (1415).
1432	$\sec x.$	$\tan x \sec x.$	By $\sec x = \frac{1}{\cos x}$ , and (1415).
1433	$\text{cosec } x.$	$-\cot x \text{ cosec } x.$	Similarly.
1434	$\sin^{-1} x \}$	$\pm \frac{1}{\sqrt{1-x^2}}.$	If $\sin^{-1} x = y$ , $x = \sin y$ , therefore $\frac{dx}{dy} = \cos y = \sqrt{1-x^2};$
1436	$\tan^{-1} x \}$	$\pm \frac{1}{1+x^2}.$	therefore $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$
1438	$\sec^{-1} x \}$	$\pm \frac{1}{x \sqrt{x^2-1}}.$	Similarly for the rest.

## EXAMPLES.

$$1440 \quad (\sqrt{x})_s = (x^{\frac{1}{2}})_s = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}. \quad (1422)$$

$$1441 \quad \left(\frac{1}{x^n}\right)_s = (x^{-n})_s = -nx^{-n-1} = -\frac{n}{x^{n+1}}. \quad (1422)$$

$$1442 \quad \{(a+x^2)^3 (b+x^2)^3\}_s = 3(a+x^2)^2 2x(b+x^2)^3 + 2(b+x^2) 3x^2(a+x^2)^2 \\ = 6x(a+x^2)^2 (b+x^2) (b+ax+2x^2). \quad (1412, '15, '22)$$

$$1443 \quad \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)_s = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2}. \quad (1413, 1426)$$

$$1444 \quad d_s (\log \tan x)^3 = 2 \log \tan x \frac{1}{\tan x} \sec^2 x = \frac{4 \log \tan x}{\sin 2x}. \quad (1415, '24, '30).$$

Some differentiations are rendered easier by taking the logarithm of the function. For example,

$$1445 \quad y = \sqrt{\frac{1-x^2}{(1+x^2)^3}}; \text{ therefore } \log y = \frac{1}{2} \log (1-x^2) - \frac{3}{2} \log (1+x^2);$$

$$\text{therefore} \quad \frac{1}{y} \frac{dy}{dx} = \frac{1}{2} \frac{-2x}{(1-x^2)} - \frac{3}{2} \frac{2x}{(1+x^2)};$$

$$\text{therefore} \quad \frac{dy}{dx} = y \frac{-2x(2-x^2)}{1-x^4} = \frac{-2x(2-x^2)}{(1+x^2)^{\frac{3}{2}}(1-x^2)^{\frac{1}{2}}}.$$

$$1446 \quad y = (\sin x)^x; \text{ therefore } \log y = x \log \sin x;$$

$$\text{therefore} \quad \frac{1}{y} y_x = \log \sin x + \frac{x}{\sin x} \cos x; \quad (1415, '24, '28)$$

$$\text{therefore} \quad y_x = (\sin x)^x (\log \sin x + x \cot x).$$

$$\text{Otherwise, by (1420), } y_x = x (\sin x)^{x-1} \cos x + (\sin x)^x \log \sin x \quad (1426) \\ = (\sin x)^x (x \cot x + \log \sin x).$$

## SUCCESSIVE DIFFERENTIATION.

1460 *Leibnitz's Theorem.*—If  $n$  be any integer,

$$(yz)_{nx} = y_{nx} z + ny_{(n-1)x} z_x + C(n, 2) y_{(n-2)x} z_{2x} + \dots \\ \dots + C(n, r) y_{(n-r)x} z_{rx} + \dots + y z_{nx}.$$

PROOF.—By Induction (233). Differentiate the two consecutive terms

$$C(n, r) y_{(n-r)x} z_{rx} + C(n, r+1) y_{(n-r-1)x} z_{(r+1)x}$$

and four terms are obtained, the second and third of which are

$$C(n, r) y_{(n-r)x} z_{(r+1)x} + C(n, r+1) y_{(n-r)x} z_{(r+1)x} \\ = \{C(n, r) + C(n, r+1)\} y_{(n-r)x} z_{(r+1)x} = C(n+1, r+1) y_{(n+1-r-1)x} z_{(r+1)x},$$

by (102).

This is the general term of the series with  $n$  increased by unity. Similarly, by differentiating all the terms the whole series is reproduced with  $n$  increased by unity.

## DIFFERENTIAL COEFFICIENTS OF THE $n^{\text{th}}$ ORDER.

$$1461 \quad (\sin ax)_{nx} = a^n \sin \left(ax + \frac{1}{2}n\pi\right). \quad \text{By Induction}$$

$$1462 \quad (\cos ax)_{nx} = a^n \cos \left(ax + \frac{1}{2}n\pi\right). \quad \text{and (1428).}$$

$$1463 \quad (e^{ax})_{nx} = a^n e^{ax}. \quad (1426)$$

$$1464 \quad (e^{ax} y)_{nx} = e^{ax} (a + d_x)^n y,$$

where, in the expansion by the Binomial Theorem,  $d_x^r y$  is to be replaced by  $y_{rx}$ . (1460, '63)

$$1465 \quad (e^{ax} \cos bx)_{nx} = r^n e^{ax} \cos (bx + n\phi),$$

$$\text{where} \quad a = r \cos \phi \quad \text{and} \quad b = r \sin \phi.$$

PROOF.—By Induction. Differentiating once more, we obtain

$$\begin{aligned} r^n e^{ax} \{a \cos (bx + n\phi) - b \sin (bx + n\phi)\} \\ = r^{n+1} e^{ax} \{\cos \phi \cos (bx + n\phi) - \sin \phi \sin (bx + n\phi)\} \\ = r^{n+1} e^{ax} \cos (bx + n + 1\phi). \end{aligned}$$

Thus  $n$  is increased by one.

$$1466 \quad (x^{n-1} \log x)_{nx} = \underline{n-1} \div x. \quad (1460), (283)$$

$$1467 \quad \left(\frac{1-x}{1+x}\right)_{nx} = \frac{(-1)^n 2}{(1+x)^{n+1}} \underline{n}. \quad (1423)$$

$$1468 \quad (\tan^{-1} x)_{nx} = (-1)^{n-1} \underline{n-1} \sin^n \theta \sin n\theta,$$

$$\text{where} \quad \theta \doteq \cot^{-1} x.$$

PROOF.—By Induction. Differentiating again, we obtain (omitting the coefficient)

$$\begin{aligned} (n \sin^{n-1} \theta \cos \theta \sin n\theta + n \cos n\theta \sin^n \theta) \theta_x \\ = n \sin^{n-1} \theta (\sin n\theta \cos \theta + \cos n\theta \sin \theta) (-\sin^2 \theta). \end{aligned}$$

$$\text{Since, by (1437),} \quad \theta_x = -(1+x^2)^{-1} = -\sin^2 \theta.$$

$$\text{Therefore} \quad (\tan^{-1} x)_{(n+1)x} = (-1)^n \underline{n} \sin^{n+1} \theta \sin (n+1) \theta,$$

$n$  being increased by one.

$$1469 \quad \left(\frac{1}{1+x^2}\right)_{nx} = (-1)^n \underline{n} \sin^{n+1} \theta \sin (n+1) \theta. \quad (1436, 1468)$$

$$1470 \quad \left(\frac{x}{1+x^2}\right)_{nx} = (-1)^n \underline{n} \sin^{n+1} \theta \cos (n+1) \theta.$$

$$\text{PROOF.—By (1460),} \quad \left(\frac{x}{1+x^2}\right)_{nx} = x \left(\frac{1}{1+x^2}\right)_{nx} + n \left(\frac{1}{1+x^2}\right)_{(n-1)x}.$$

Then by (1469).

$$1471 \quad \text{Jacobi's Formula.}$$

$$d_{(n-1)x} (1-x^2)^{n-1} = (-1)^{n-1} 1 \cdot 3 \dots (2n-1) \sin (n \cos^{-1} x) \div n.$$

PROOF.—Let  $y = 1-x^2$ ; therefore

$$(y^{n+\frac{1}{2}})_{nx} = -(2n+1) (xy^{n-\frac{1}{2}})_{(n-1)x}. \quad \text{Also} \quad (y^{n+\frac{1}{2}})_{nx} = (yy^{n-\frac{1}{2}})_{nx}.$$

Expand each of these values by (1460) and eliminate  $(y^{n-\frac{1}{2}})_{(n-2)x}$ , the derivative of lowest order. Call the result equation (1). Now assume (1471) true for the value  $n$ . Differentiate and substitute the result, and also (1471) on the right side of equation (1) to obtain a proof by Induction.

**1472** *Theorem.*—If  $y, z$  are functions of  $x$ , and  $n$  a positive integer,

$$zy_{nx} = (yz)_{nx} - n(yz_x)_{(n-1)x} + C(n, 2)(yz_{2x})_{(n-2)x} \dots + (-1)^n yz_{nx}.$$

PROOF.—By Induction. Differentiate for  $x$ , substituting for  $z_x y_{nx}$  on the right its value by the formula itself.

## PARTIAL DIFFERENTIATION.

**1480** If  $u = f(x, y)$  be a function of two *independent* variables, any differentiation of  $u$  with respect to  $x$  *requires that  $y$  should be considered constant* in that operation, and *vice versa*.

Thus,  $\frac{d^2 u}{dx^2}$  or  $u_{2x}$  signifies that  $u$  is to be differentiated successively twice with respect to  $x$ ,  $y$  being considered constant.

**1481** The notation  $\frac{d^5 u}{dx^2 dy^3}$  or  $u_{2x3y}^*$  signifies that  $u$  is to be differentiated successively twice for  $x$ ,  $y$  being considered constant, and the result three times successively for  $y$ ,  $x$  being considered constant.

**1482** The *order* of the differentiations does not affect the final result, or  $u_{xy} = u_{yx}$ .

PROOF.—Let  $u = f(xy)$ ; then  $u_x = \frac{f(x+h, y) - f(x, y)}{h}$  in limit. (1484)

$$u_{xy} = \frac{du_x}{dy} = \frac{f(x+h, y+k) - f(x, y+k) - f(x+h, y) + f(x, y)}{hk} \text{ in limit.}$$

Now, if  $u_y$  had been first formed, and then  $u_{yx}$ , the same result would have been obtained. The proof is easily extended. Let  $u_x = v$ ;

then  $u_{xy} = v_{xy} = v_{yx} = u_{yx}$ ; and so on.

## THEORY OF OPERATIONS.

**1483** Let the symbols  $\Phi, \Psi$ , prefixed to a quantity, denote operations upon it of the *same class*, such as multiplication or differentiation. Then the law of the operation is said to be *distributive*, when

$$\Phi(x+y) = \Phi(x) + \Phi(y);$$

\* See note to (1487).



that is, the operation may be performed upon an undivided quantity, or it may be distributed by being performed upon parts of the quantity separately with the same result.

**1484** The law is said to be *commutative* when

$$\Phi\Psi x = \Psi\Phi x;$$

that is, the order of operation may be changed,  $\Phi$  operating upon  $\Psi x$  producing the same result as  $\Psi$  operating upon  $\Phi x$ .

**1485**  $\Phi^m x$  denotes the repetition of the operation  $\Phi$   $m$  times, and is equivalent to  $\Phi\Phi \dots x$  to  $m$  operations. This definition involves the index law,

$$\Phi^m \Phi^n x = \Phi^{m+n} x = \Phi^{n+m} x,$$

which merely asserts, that to perform the operation  $n$  times in succession upon  $x$ , and afterwards  $m$  times in succession upon the result, is equivalent to performing it  $m+n$  times in succession upon  $x$ .

**1486** The three laws of *Distribution*, *Commutation*, and the law of *Indices* apply to the operation of *multiplication*, and also to that of *differentiation* (1411, '12). Therefore any algebraic transformation which proceeds at every step by one or more of these laws *only*, has a valid result when for the operation of *multiplication* that of *differentiation* is substituted.

**1487** In making use of this principle, the symbol of differentiation employed is  $\frac{d}{dx}$ , or simply  $d_x$  prefixed to the quantity upon which the operation of differentiating with respect to  $x$  is to be performed. The repetition of the operation is indicated by  $\frac{d^2}{dx^2}$ ,  $\frac{d^3}{dx^3}$ ,  $\frac{d^5}{dx^2 dy^3}$ , &c., prefixed to the function. An abbreviated notation is  $d_x$ ,  $d_{2x}$ ,  $d_{3x}$ ,  $d_{2x3y}$ , &c. Since  $d_x \times d_x = d_x^2$  in the symbolic operation of multiplication, it will be requisite, in transferring the operation to differentiation, to change all such *indices* to *suffixes* when the abbreviated notation is being used.

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NOTE.—The notation  $y_x$ ,  $y_{2x}$ ,  $u_{2x3y}$ ,  $d_{2x3y}$ , &c. is an innovation. It has, however, the recommendations of definiteness, simplicity, and economy of time in writing, and of space in printing. The expression  $\frac{d^6 u}{dx^3 dy^3}$  requires at least fourteen distinct types, while its equivalent  $u_{2x3y}$  requires but seven. For

such reasons I have introduced the shorter notation experimentally in these pages.

All such abbreviated forms of differential coefficients as  $y' y'' y''' \dots$  or  $\dot{y} \ddot{y} \dddot{y} \dots$ , though convenient in practice, are incomplete expressions, because the independent variable is not specified.

The operation  $d_{2x} y$ , and the derived function  $u_{2x} y$ , would be more accurately represented by  $(d_x^2)_y$  and  $(u_x^2)_y$ , the index as usual indicating the repetition of the operation. But the former notation is simpler, and it has the advantage of separating more clearly the index of differentiation from the index of involution.

In the symbols  $y^2$  and  $y_{2x}$ , the figure 2 is an index in each case: in the first, it shows the *degree* of involution; in the second, the *order* of differentiation. The index is omitted when the degree or the order is unity, since we write  $y$  and  $y_x$ .

The suffix takes precedence of the superfix.  $y_x^2$  means the square of  $y_x$ .  $d_x(y^2)$  would be written  $(y^2)_x$  in this notation.

As a concise nomenclature for all fundamental operations is of great assistance in practice, the following is recommended:  $\frac{dy}{dx}$  or  $y_x$  may be read "*y for x,*" as an abbreviation of the phrase, "*the differential coefficient of y for, or with respect to, x.*" Similarly,  $\frac{d^2 y}{dx^2}$ ,  $\frac{d^2 u}{dx dy}$ ,  $\frac{d^3 u}{dx^2 dy^2}$ , or the shortened forms  $y_{2x}$ ,  $u_{xy}$ ,  $u_{2x} y$ , may be read "*y for two x,*" "*u for xy,*" "*u for two x, three y,*" and so forth.

The distinction in meaning between the two forms  $y_{nx}$  and  $y_{xx}$  is obvious. The first (in which  $n$  is numerical and *always an integer*) indicates  $n$  successive differentiations for  $x$ ; the second indicates two successive differentiations for the variables  $x$  and  $x$ .

The symbols  $\frac{dy}{dx_0}$  or  $y_{x0}$  and  $\frac{d^2 y}{dx_0^2}$  or  $y_{2x0}$  may be read, for shortness, "*y for x zero,*" "*y for two x zero*";  $d_{2x} y \phi(xy)$  can be read "*d for two x three y of  $\phi(xy)$ .*"

Although the notation  $v_x$  is already employed in a totally different sense in the Calculus of Finite Differences, my own experience is that the double signification of the symbol does not lead to any confusion: and this for the very reason that the two meanings are so entirely distinct. Whenever the operation of differentiation is introduced along with the subject of Finite Differences, the notation  $\frac{dy}{dx}$  must of course alone be employed.

Thus, in differentiation, we have

$$1488 \quad \text{THE DISTRIBUTIVE LAW} \quad d_x(u+v) = d_x u + d_x v. \quad (1411)$$

$$1489 \quad \text{THE COMMUTATIVE LAW} \quad d_x(d_y u) = d_y(d_x u) \\ \text{or} \quad d_{xy} u = d_{yx} u. \quad (1482)$$

$$1490 \quad \text{THE INDEX LAW} \quad d_x^m d_x^n u = d_x^{m+n} u, \\ \text{that is,} \quad d_{mx} d_{nx} u = d_{(m+n)x} u. \quad (1485)$$

**1491** EXAMPLE.—

$$(d_x - d_y)^2 = (d_x - d_y)(d_x - d_y) = d_x d_x - d_x d_y - d_y d_x + d_y d_y = d_{xx} - 2d_{xy} + d_{yy}$$

Here  $d_x d_y - d_y d_x$  or  $d_{xy} = d_{yx}$  by the commutative law. (1489)

$d_x d_x = d_{xx}$  by the index law. (1490)

Also  $(d_{xx} - 2d_{xy} + d_{yy})u = d_{xx}u - 2d_{xy}u + d_{yy}u$ , by the distributive law.

Therefore, finally,  $(d_x - d_y)^2 u = d_{xx}u - 2d_{xy}u + d_{yy}u$ .

Similarly for more complex transformations.

**1492** Thus  $d_x$  may be treated as quantitative, and operated upon as such by the laws of Algebra;  $d_x^2$  being written  $d_{xx}$ , and factors such as  $d_x d_y$ , in which the independent variables are different, being written  $d_{xy}$ , &c.

## EXPANSION OF EXPLICIT FUNCTIONS.

TAYLOR'S THEOREM.—EXPANSION OF  $f(x+h)$ .**1500**

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{1.2}f''(x) + \dots + \frac{h^n}{[n]}f^n(x+\theta h),$$

where  $\theta$  is some quantity between zero and unity, and  $n$  is any integer.

PROOF.—(i.) Assume  $f(x+h) = A + Bh + Ch^2 + \&c.$

Differentiate both sides of this equation,—first for  $x$ , and again for  $h$ ,—and equate coefficients in the two results.

**1501** (ii.) Cox's Proof.—LEMMA.—If  $f(x)$  vanishes when  $x=a$ , and also when  $x=b$ , and if  $f(x)$  and  $f'(x)$  are continuous functions between the same limits; then  $f'(x)$  vanishes for some value of  $x$  between  $a$  and  $b$ .

For  $f'(x)$  must change sign somewhere between the assigned limits (see proof of 1403), and, being continuous, it must vanish in passing from plus to minus.

**1502** Now, the expression

$$f(a+x) - f(a) - xf'(a) - \dots - \frac{x^n}{[n]}f^n(a) \\ - \frac{x^{n+1}}{[n+1]} \frac{[n+1]}{h^{n+1}} \left\{ f(a+h) - f(a) - hf'(a) - \dots - \frac{h^n}{[n]}f^n(a) \right\}$$

vanishes when  $x=0$  and when  $x=h$ . Therefore the differential coefficient with respect to  $x$  vanishes for some value of  $x$  between 0 and  $h$  by the lemma. Let  $\theta h$  be this value. Differentiate, and apply the lemma to the resulting expression, which vanishes when  $x=0$  and when  $x=\theta h$ . Perform the same process  $n+1$  times successively, writing  $\theta h$  for  $\theta\theta h$ , &c., since  $\theta$  merely stands for some quantity less than unity. The result shews that

$$f^{n+1}(a+x) - \frac{[n+1]}{h^{n+1}} \left\{ f(a+h) - f(a) - hf'(a) - \dots - \frac{h^n}{[n]}f^n(a) \right\}$$

vanishes when  $x=\theta h$ . Substituting  $\theta h$  and equating to zero, the theorem is proved.

**1503** The last term in (1500) is called the remainder after  $n$  terms. It may be obtained in either of the subjoined forms, the first being due to Lagrange,

$$\frac{h^n}{n} f^n(x+\theta h) \quad \text{or} \quad \frac{h^n}{n-1} (1-\theta)^{n-1} f^n(x+\theta h).$$

**1504** Since the coefficient  $\frac{h^n}{n}$  diminishes at last without limit as  $n$  increases (239, ii.), it follows that *Taylor's series is convergent if  $f^n(x)$  remains finite for all values of  $n$ .*

**1505** If in any expansion of  $f(x+h)$  in powers of  $h$  some index of  $h$  be *negative*, then  $f(x)$ ,  $f'(x)$ ,  $f''(x)$ , &c. all become infinite.

**1506** If the least fractional index of  $h$  lies between  $n$  and  $n+1$ ; then  $f^{n+1}(x)$  and all the following differential coefficients become infinite.

PROOF.—To obtain the value of  $f^n(x)$ , differentiate the expansion  $n$  times successively for  $h$ , and put  $h=0$  in the result.

#### MACLAURIN'S THEOREM.

Put  $x=0$  in (1500), and write  $x$  for  $h$ ; then, with the notation of (1406),

$$\mathbf{1507} \quad f(x) = f(0) + xf'(0) + \frac{x^2}{1.2} f''(0) + \dots + \frac{x^n}{n} f^n(\theta x),$$

where  $\theta$ , as before, lies between 0 and 1.

Putting  $y=f(x)$ , this may also be written

$$\mathbf{1508} \quad y = y_0 + x \frac{dy}{dx_0} + \frac{x^2}{1.2} \frac{d^2y}{dx_0^2} + \frac{x^3}{1.2.3} \frac{d^3y}{dx_0^3} + \&c.$$

**1509** NOTE.—If any function  $f(x)$  becomes infinite with a finite value of  $x$ , then  $f'(x)$ ,  $f''(x)$ , &c. all become infinite. Thus, if  $f(x) = \sec^{-1}(1+x)$ ,  $f'(x)$  is infinite when  $x=0$  (1438). Therefore  $f''(0)$ ,  $f'''(0)$ , &c. are all infinite, and  $f(x)$  cannot be expanded by this theorem.

*Bernoulli's Series.*—Put  $h=-x$  in (1500); thus,

$$\mathbf{1510} \quad f(0) = f(x) - xf'(x) + \frac{x^2}{1.2} f''(x) - \frac{x^3}{1.2.3} f'''(x) + \&c.$$

**1511** If  $\phi(y+k) = 0$  and  $\phi(y) = x$ ; then

$$k = -xy_x + \frac{x^2}{1.2} y_{2x} - \frac{x^3}{1.2.3} y_{3x} + \&c.$$

PROOF.—Let  $y = \phi^{-1}(x) = f(x)$ , and let  $y+k = f(x+h)$ ;  
therefore  $x+h = \phi(y+k) = 0$ .

Therefore  $y+k = f(0) = f(x) - xf'(x) + \frac{x^2}{1.2} f''(x) - \&c.$ , by (1510);  
which proves the theorem.

### EXPANSION OF $f(x+h, y+k)$ .

Let  $f(xy) = u$ . Then, with the notation of (1405),

**1512**

$$\begin{aligned} f(x+h, y+k) &= u + (hu_x + ku_y) + \frac{1}{1.2} (h^2 u_{2x} + 2hku_{xy} + k^2 u_{2y}) \\ &\quad + \frac{1}{1.2.3} (h^3 u_{3x} + 3h^2 ku_{2xy} + 3hk^2 u_{x2y} + k^3 u_{3y}) + \&c. \end{aligned}$$

**1513** The general term is given by  $\frac{1}{n} (hd_x + kd_y)^n u$ ,

where, in the expansion by the Binomial Theorem, each index of  $d_x$  and  $d_y$  is changed into a suffix; and the coefficients  $d_x, d_{2x}, \&c.$  are joined to  $u$  as symbols of operation (1487); thus  $u_x^3$  is to be changed into  $u_{3x}$ .

PROOF.—First expand  $f(x+h, y+k)$  as a function of  $(x+h)$  by (1500); thus,  $f(x+h, y+k) = f(x, y+k) + hf_x(x, y+k) + \frac{1}{1.2} h^2 f_{2x}(x, y+k) + \&c.$

Next, expand each term of this series as a function of  $(y+k)$ . Thus, writing  $u$  for  $f(xy)$ ,

$$\begin{aligned} f(x, y+k) &= u + ku_y + \frac{1}{2} k^2 u_{2y} + \frac{1}{3} k^3 u_{3y} + \frac{1}{4} k^4 u_{4y} + \dots \\ hf_x(x, y+k) &= hu_x + hku_{xy} + \frac{1}{2} hk^2 u_{x2y} + \frac{1}{3} hk^3 u_{x3y} + \dots \dots \dots \\ \frac{h^2}{2} f_{2x}(x, y+k) &= \frac{1}{2} h^2 u_{2x} + \frac{1}{2} h^2 ku_{2xy} + \frac{1}{2} \frac{1}{2} h^2 k^2 u_{2x2y} + \dots \dots \dots \\ \frac{h^3}{3} f_{3x}(x, y+k) &= \frac{1}{3} h^3 u_{3x} + \frac{1}{3} h^3 ku_{3xy} + \dots \dots \dots \\ \frac{h^4}{4} f_{4x}(x, y+k) &= \frac{1}{4} h^4 u_{4x} + \dots \dots \dots \end{aligned}$$

The law by which the terms of the same dimension in  $h$  and  $k$  are formed, is seen on inspection. They lie in successive diagonals; and when cleared of fractions the numerical coefficients are those of the Binomial Theorem.



The theorem may be extended inductively to a function of three or more variables. Thus, if  $u = f(x, y, z)$ , we have

$$1514 \quad f(x+h, y+k, z+l) = u + (hu_x + ku_y + lu_z) \\ + \frac{1}{2} (h^2 u_{xx} + k^2 u_{yy} + l^2 u_{zz} + 2klu_{yz} + 2lhu_{xz} + 2hku_{xy}) + \dots,$$

the general term being obtained as before from the expression

$$\frac{1}{n!} (hd_x + kd_y + ld_z)^n u.$$

1515 COR.—If  $u = f(xyz)$  be a function of several independent variables, the term  $(hu_x + ku_y + lu_z)$  proves, in conjunction with (1410), that the total change in the value of  $u$ , caused by simultaneous small changes in  $x, y, z$ , is equal to the sum of the increments of  $u$  due to the increments of  $x, y, z$  taken separately and *superposed in any order*.

This is known as the principle of the *superposition* of small quantities.

1516 To expand  $f(x, y)$  or  $f(x, y, z, \dots)$  in powers of  $x, y$ , &c., put  $x, y, z$  each equal to zero after differentiating in (1512) or (1514), and write  $x, y, \dots$  instead of  $h, k$ , &c.

1517 Observe that any term in these series may be made the last by writing  $x + \theta h$  for  $x$ ,  $y + \theta k$  for  $y$ , &c., as in (1500).

#### SYMBOLIC FORM OF TAYLOR'S THEOREM.

The expansion in (1500) is equivalent to the following

$$1520 \quad f(x+h) = e^{hd_x} f(x).$$

PROOF.—By the Exponential Theorem (150), writing the indices of  $d_x$  as suffixes (1487),

$$e^{hd_x} f(x) = (1 + hd_x + \frac{1}{2} h^2 d_{xx} + \dots) f(x) = f(x) + hf_x(x) + \frac{1}{2} h^2 f_{xx}(x) + \dots, \text{ by (1488).}$$

COR.—  $\Delta f(x) = f(x+h) - f(x) = (e^{hd_x} - 1) f(x)$ ,  
therefore  $\Delta^2 f(x) = (e^{hd_x} - 1)^2 f(x)$ ,  
and generally  $\Delta^n f(x) = (e^{hd_x} - 1)^n f(x)$ ,

the index signifying that the operation is performed  $n$  times upon  $f(x)$ .

$$1521 \quad \text{Similarly } f(x+h, y+k) = e^{hd_x + kd_y} f(x, y).$$

PROOF.—

$$e^{hd_x + kd_y} f(x, y) = \{1 + hd_x + kd_y + \frac{1}{2} (hd_x + kd_y)^2 + \frac{1}{6} (hd_x + kd_y)^3 + \dots\} f(x, y) \\ = f(x+h, y+k), \text{ by (150) and (1512).}$$



**1522** And, generally, with any number of variables,

$$f(x+h, y+k, z+l \dots) = e^{hd_x + kd_y + ld_z + \dots} f(x, y, z \dots).$$

COR.—As in (1520),

$$\Delta f(x, y, z \dots) = (e^{hd_x + kd_y + \dots} - 1) f(x, y, z \dots).$$

**1523** If  $u = f(x, y) = \phi(r, \theta)$ , where  $x = r \cos \theta$ ,  $y = r \sin \theta$ ; and if  $x' = r \cos(\theta + \omega)$ ,  $y' = r \sin(\theta + \omega)$ ; then  $f(x', y')$  is expanded in powers of  $\omega$  by the formula

$$f(x', y') = e^{\omega(xd_y - yd_x)} f(x, y).$$

PROOF.—By (1520),  $r$  being constant,

$$\phi(r, \theta + \omega) = e^{\omega d_\theta} \phi(r, \theta) = e^{\omega d_\theta} f(x, y).$$

Now  $x$  and  $y$  are functions of the single variable  $\theta$ ; therefore

$$u_\theta = u_x x_\theta + u_y y_\theta = u_x (-r \sin \theta) + u_y (r \cos \theta) = xu_y - yu_x.$$

The operation  $d_\theta$  will be transformed by the same law (1492); therefore

$$d_\theta = xd_y - yd_x; \text{ therefore}$$

$$f(x', y') = e^{\omega(xd_y - yd_x)} f(x, y) = 1 + \omega(xu_y - yu_x) + \frac{1}{2}\omega^2(x^2u_{yy} - 2xyu_{xy} + y^2u_{xx}) + \&c.$$

**1524** EXAMPLES.—The Binomial, Exponential, and Logarithmic series for  $(1+x)^n$ ,  $a^x$ , and  $\log(1+x)$ , (125, 149, 155), are obtained immediately by Maclaurin's Theorem (1507); as also the series for  $\sin x$  and  $\cos x$  (764), and  $\tan^{-1} x$  (791). The mode of proceeding, which is the same in all cases, is shewn in the following example; the test of convergency (1504) being applied when practicable.

**1525**  $\tan x = x + \frac{1}{3} x^3 + \frac{2}{15} x^5 + \frac{17}{315} x^7 + \frac{62}{2835} x^9 + \&c.$

Obtained by Maclaurin's theorem, as follows:—Let

$$\left. \begin{aligned} f(x) &= \tan x = y \\ f'(x) &= \sec^2 x = z \end{aligned} \right\} \begin{aligned} &\text{Therefore } y_x = z \text{ and } z_x = 2yz; \\ &y \text{ and } z \text{ being used for shortness.} \end{aligned}$$

$$f''(x) = 2 \sec^2 x \tan x = 2yz,$$

$$f'''(x) = 2(zy_x + yz_x) = 2(z^2 + 2y^2z),$$

$$f^{(4)}(x) = 2(4yz^2 + 4yz^2 + 4y^3z) = 8(2yz^2 + y^3z),$$

$$f^{(5)}(x) = 8(2z^3 + 8y^2z^2 + 3y^2z^2 + 2y^4z) = 8(2z^3 + 11y^2z^2 + 2y^4z),$$

$$f^{(6)}(x) = 8(12yz^3 + 22yz^3 + \dots) = 272yz^3 + \dots,$$

$$f^{(7)}(x) = 272z^4 + \dots \&c.,$$

the terms omitted involving positive powers of  $y$ , which vanish when  $x$  is zero, and which therefore need not be computed if no term of the expansion higher than that containing  $x^7$  is required.

Hence, by making  $x = 0$ , and therefore  $y = 0$  and  $z = 1$ , we obtain  
 $f(0) = 0$ ;  $f'(0) = 1$ ;  $f''(0) = 0$ ;  $f'''(0) = 2$ ;  $f^{(4)}(0) = 0$ ;  $f^{(5)}(0) = 16$ ;  
 $f^{(6)}(0) = 0$ ;  $f^{(7)}(0) = 272$ .

Thus the terms up to  $x^7$  may be written by substituting these values in (1507).

In a similar manner, may be obtained

$$1526 \quad \sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \&c. \dots\dots\dots$$

### *Methods of expansion by Indeterminate Coefficients.*

**1527** RULE I.—Assume  $f(x) = A + Bx + Cx^2 + \&c$ . Differentiate both sides of the equation. Then expand  $f'(x)$  by some known theorem, and equate coefficients in the two results to determine  $A, B, C, \&c$ .

$$1528 \quad \text{Ex. } \sin^{-1} x = x + \frac{1}{2} \frac{x^3}{3} + \frac{1.3}{2.4} \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \frac{x^7}{7} + \&c.$$

Obtained by Rule I. Assume

$$\sin^{-1} x = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \dots\dots\dots$$

Therefore, by (1434),  $(1-x^2)^{-\frac{1}{2}} = B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \dots\dots\dots$

But, by Bin. Th. (128),  $(1-x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \dots\dots\dots$

Equate coefficients; therefore  $B = 1$ ;  $C = 0$ ;  $D = \frac{1}{2.3}$ ;  $E = 0$ ;  $F = \frac{1.3}{2.4.5}$ ;

&c. By putting  $x = 0$ , we see that  $A = f(0)$  always. In this case  $A = \sin^{-1} 0 = 0$ .

In a similar manner, by Rule I.,

$$1529 \quad e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{3x^4}{4} - \frac{8x^5}{5} - \frac{3x^6}{6} + \&c. \dots\dots\dots$$

**1530** RULE II.—Assume the series, as before, with unknown coefficients. Differentiate successively until the function reappears. Then equate coefficients in the two equivalent series.

**1531** Ex.—To expand  $\sin x$  in powers of  $x$ .

$$\text{Assume } \sin x = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \dots\dots\dots$$

$$\text{Differentiate twice, } \cos x = B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \dots\dots\dots$$

$$-\sin x = 2C + 3.2Dx + 4.3Ex^2 + 5.4Fx^3 + \dots\dots\dots$$

Put  $x = 0$  in the first two equations; therefore  $A = 0$ ,  $B = 1$ .

Equate coefficients in the first and third series.

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Thus  $-2C = A, \therefore C = 0; -3.2D = B, \therefore D = -\frac{1}{2.3};$

$-4.3E = C, \therefore E = 0; -5.4F = D, \therefore F = \frac{1}{1.2.3.4}, \&c.$

Therefore  $\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \&c.,$  as in (764).

**1532** RULE III.—Differentiate the equation  $y = f(x)$  twice with respect to  $x$ , and combine the results so as to form an equation in  $y, y_x$ , and  $y_{xx}$ . Next assume  $y = A + Bx + Cx^2 + \&c.$  Differentiate twice, and substitute the three values of  $y, y_x, y_{xx}$  so obtained in the former equation. Lastly, equate coefficients in the result to determine in succession  $A, B, C, \&c.$

**1533** Ex.—To expand  $\sin m\theta$  and  $\cos m\theta$  in ascending powers of  $\sin \theta$  or  $\cos \theta$ .

These series are given in (775–779). They may be obtained by Rule III. as follows:—

Put  $x = \sin \theta$  and  $y = \sin m\theta = \sin (m \sin^{-1} x).$

Therefore  $y_x = \cos (m \sin^{-1} x) \frac{m}{\sqrt{1-x^2}} \quad (1434) \dots\dots\dots (i.)$

$$y_{xx} = -\sin (m \sin^{-1} x) \frac{m^3}{1-x^2} + \cos (m \sin^{-1} x) \frac{mx}{(1-x^2)^{\frac{3}{2}}}.$$

Therefore, eliminating  $\cos (m \sin^{-1} x), \quad (1-x^2) y_{xx} - xy_x + m^2 y = 0 \dots\dots\dots (ii.)$

Let  $y = A + A_1 x + A_2 x^2 + \dots\dots + A_n x^n + \dots\dots\dots (iii.)$

Differentiate twice, and put the values of  $y, y_x$ , and  $y_{xx}$  in equation (ii.);

thus  $0 = m^2 (A + A_1 x + A_2 x^2 + A_3 x^3 + \dots + A_n x^n + \dots\dots\dots)$   
 $-x (A_1 + 2A_2 x + 3A_3 x^2 + \dots + nA_n x^{n-1} + \dots\dots\dots)$   
 $+ (1-x^2) \{2A_2 + 2.3A_3 x + \dots\dots\dots\}$   
 $+ (n-1) nA_n x^{n-2} + n(n+1) A_{n+1} x^{n-1} + (n+1)(n+2) A_{n+2} x^n + \dots\}.$

Equating the collected coefficients of  $x^n$  to zero, we get the relation

$$A_{n+2} = \frac{n^2 - m^2}{(n+1)(n+2)} A_n \dots\dots\dots (iv.)$$

Now, when  $x=0, y=0$ ; therefore  $A=0$ , by (iii.). And when  $x=0, y_x=m$ , by (i.); and therefore  $A_1=m$ , by differentiating (iii.). The relation (iv.) furnishes the remaining coefficients by making  $n$  equal to 0, 1, 2, 3, &c. in succession.

$\cos m\theta$  is obtained in a similar way.

**1534** RULE IV.—Form the equation in  $y, y_x$ , and  $y_{xx}$ , as in Rule III. Take the  $n^{\text{th}}$  derivative of this equation by applying Leibnitz's formula (1460) to the terms, and an equation in

$y_{(n+2)x}$ ,  $y_{(n+1)x}$ , and  $y_{nx}$  is obtained. Put  $x=0$  in this; and employ the resulting formula to calculate in succession  $y_{3x0}$ ,  $y_{4x0}$ , &c. in Maclaurin's expansion (1507).

$$1535 \quad \text{Ex.} \quad e^{a \sin^{-1} x} = 1 + ax + \frac{a^2}{1.2} x^2 + \frac{a(a^2+1)}{1.2.3} x^3 \\ + \frac{a^2(a^2+2^2)}{1.2.3.4} x^4 + \frac{a(a^2+1)(a^2+3^2)}{1.2.3.4.5} x^5 + \&c.$$

Obtained by Rule IV. Writing  $y$  for the function, the relation found is  
 $(1-x^2) y_{2x} - xy_x - a^2 y = 0.$

Differentiating  $n$  times, by (1460), we get

$$(1-x^2) y_{(n+2)x} - (2n+1) xy_{(n+1)x} - (a^2+n^2) y_{nx} = 0.$$

Therefore  $y_{(n+2)x0} = (a^2+n^2) y_{nx0}$ , a formula which produces the coefficients in Maclaurin's expansion in succession when  $y_{x0}$  and  $y_{2x0}$  have been calculated.

### 1536 ARBOGAST'S METHOD OF EXPANDING $\phi(z)$ ,

where  $z = a + a_1 x + \frac{a_2}{1.2} x^2 + \frac{a_3}{1.2.3} x^3 + \&c. \dots\dots\dots (i.)$

Let  $y = \phi(z)$ . When  $x=0$ ,  $y = \phi(a)$ ; therefore, by Maclaurin's theorem (1508),

$$y = \phi(a) + xy_{x0} + \frac{x^2}{1.2} y_{2x0} + \frac{x^3}{1.2.3} y_{3x0} + \&c. \dots\dots (ii.)$$

Hence, in the values of  $y_x$ ,  $y_{2x}$ , &c., at (1416),  $x$  has to be put  $= 0$ .

Now, when  $x=0$ ,  $z=a$ ; therefore  $y_x$ ,  $y_{2x}$ , &c. become  $\phi'(a)$ ,  $\phi''(a)$ , &c.; and  $z_{x0}$ ,  $z_{2x0}$ ,  $z_{3x0}$ , &c. become  $a_1$ ,  $a_2$ ,  $a_3$ , &c. Hence

$$y_{x0} = \phi'(a) a_1,$$

$$y_{2x0} = \phi''(a) a_1^2 + \phi'(a) a_2,$$

$$y_{3x0} = \phi'''(a) a_1^3 + 3\phi''(a) a_1 a_2 + \phi'(a) a_3, \&c.$$

### 1537 EXAMPLE.—To expand $\log(a+bx+cx^2+dx^3+\&c.)$ .

Here  $a_1 = b$ ,  $a_2 = 2c$ ,  $a_3 = 6d$ ,  $\phi'(a) = \frac{1}{a}$ ,  $\phi''(a) = -\frac{1}{a^2}$ ,  $\phi'''(a) = \frac{2}{a^3}$ .

Therefore  $y_{x0} = \frac{b}{a}$ ,  $y_{2x0} = -\frac{b^2}{a^2} + \frac{2c}{a}$ ,  $y_{3x0} = \frac{2b^3}{a^3} - \frac{6bc}{a^2} + \frac{6d}{a}$ .

Therefore, substituting in (ii.), we obtain as far as four terms,

$$\log(a+bx+cx^2+\dots) = \log a + \frac{b}{a} x + \left(\frac{c}{a} - \frac{b^2}{2a^2}\right) x^2 + \left(\frac{b^3}{3a^3} - \frac{bc}{a^2} + \frac{d}{a}\right) x^3 +$$

**1538** Ex. 2.—To expand  $(a + a_1x + a_2x^2 + \dots + a_nx^n)^r$  in powers of  $x$ .

Arbogast's method may be employed; otherwise, we may proceed as follows. Assume  $(a + a_1x + a_2x^2 + \dots + a_nx^n)^r = A_0 + A_1x + A_2x^2 + \dots$

Differentiate for  $x$ ; divide the equation by the result; clear of fractions, and equate coefficients of like powers of  $x$ .

### BERNOULLI'S NUMBERS.

$$\mathbf{1539} \quad \frac{x}{e^x - 1} = 1 - \frac{x}{2} + B_2 \frac{x^2}{2} - B_4 \frac{x^4}{4} + B_6 \frac{x^6}{6} - \&c.,$$

where  $B_2, B_4, \&c.$  are known as Bernoulli's numbers. Their values, as far as  $B_{18}$ , are

$$B_2 = \frac{1}{6}, \quad B_4 = \frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = \frac{1}{30}, \quad B_{10} = \frac{5}{66},$$

$$B_{12} = \frac{691}{2730}, \quad B_{14} = \frac{7}{6}, \quad B_{16} = \frac{3617}{510}, \quad B_{18} = \frac{43867}{798}.$$

They are found in succession from the formula

$$\mathbf{1540} \quad nB_{n-1} + C(n, 2)B_{n-2} + C(n, 3)B_{n-3} + \dots$$

$$\dots + C(n, 2)B_2 - \frac{1}{2}n + 1 = 0,$$

the odd numbers  $B_3, B_5, \&c.$  being all zero.

PROOF.—Let  $y = \frac{x}{e^x - 1}$ . Then, by (1508),

$$y = y_0 + y_{x0}x + y_{2x0} \frac{x^2}{2} + y_{3x0} \frac{x^3}{3} + y_{4x0} \frac{x^4}{4} + \&c.$$

Here  $y_{rx0} = (-1)^{r+1}B_r$ . Now  $y_0 = 1$  and  $y_{x0} = -\frac{1}{2}$ , by (1587). Also  $ye^x = y + x$ . Therefore, by (1460), differentiating  $n$  times,

$$e^x \{y_{nx} + ny_{(n-1)x} + C(n, 2)y_{(n-2)x} + \dots + ny_x + y\} = y_{nx}.$$

Therefore  $ny_{(n-1)x0} + C(n, 2)y_{(n-2)x0} + \dots + ny_{x0} + y_0 = 0$ .

Substitute  $B_{n-1}, B_{n-2}, \&c.$ , and we get the formula required.

$B_3, B_5, B_7, \&c.$  will all be found to vanish. It may be proved, *a priori*, that this will be the case: for

$$\mathbf{1541} \quad \frac{x}{e^x - 1} + \frac{x}{2} = \frac{x}{2} \frac{e^x + 1}{e^x - 1}.$$

Therefore the series (1539) wanting its second term is the expansion of the expression on the right. But that expression is an *even* function of  $x$  (1401); changing the sign of  $x$  does not alter its value. Therefore the series in question contains *no odd powers of  $x$*  after the first.

**1542** The connexion between Bernoulli's numbers and the sums of the powers of the natural numbers in (276) is seen by expanding  $(1 - e^x)^{-1}$  in powers of  $e^x$ , and each term afterwards by the Exponential Theorem (150).

1543

$$\frac{x}{e^x+1} = \frac{x}{2} - B_2(2^2-1)\frac{x^2}{\underline{2}} + B_4(2^4-1)\frac{x^4}{\underline{4}} - B_6(2^6-1)\frac{x^6}{\underline{6}} + \&c.$$

1544

$$\frac{e^x-1}{e^x+1} = 2 \left\{ B_2(2^2-1)\frac{x}{\underline{2}} - B_4(2^4-1)\frac{x^3}{\underline{4}} + B_6(2^6-1)\frac{x^5}{\underline{6}} - \&c. \right\}.$$

PROOF.  $\frac{x}{e^x+1} = \frac{x}{e^x-1} - \frac{2x}{e^{2x}-1}$  and  $\frac{e^x-1}{e^x+1} = 1 - \frac{2}{x} \frac{x}{e^x+1}$ , and by (1539).

$$1545 \quad 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \dots = \frac{2^{2n-1}\pi^{2n}}{1.2 \dots 2n} B_{2n}.$$

PROOF.—In the expansion of  $\frac{x}{2} \frac{e^x+1}{e^x-1}$  (1540) substitute  $2i\theta$  for  $x$ , and it becomes the expansion of  $\theta \cot \theta$  (770). Obtain a second expansion by differentiating the logarithm of equation (815, *sin  $\theta$  in factors*). Expand each term of the result by the Binomial Theorem, and equate coefficients of like powers of  $\theta$  in the two expansions.

## STIRLING'S THEOREM.

$$1546 \quad \phi(x+h) - \phi(x) = h\phi'(x) + A_1h \{ \phi'(x+h) - \phi'(x) \} \\ + A_2h^2 \{ \phi''(x+h) - \phi''(x) \} + \&c.,$$

where  $A_{2n} = (-1)^n B_{2n} \div \underline{2n}$  and  $A_{2n+1} = 0$ .

PROOF.— $A_1, A_2, A_3$  &c. are determined by expanding each function of  $x+h$  by (1500), and then equating coefficients of like powers of  $x$ . Thus

$$\frac{1}{\underline{2}} - A_1 = 0; \quad \frac{1}{\underline{3}} - \frac{A_1}{\underline{2}} - A_2 = 0; \quad \frac{1}{\underline{4}} - \frac{A_1}{\underline{3}} - \frac{A_2}{\underline{2}} - A_3 = 0; \quad \&c.$$

To obtain the general relation between the coefficients: put  $\phi(x) = e^x$ , since  $A_1, A_2$  &c. are independent of the form of  $\phi$ . Equation (1546) then

produces  $\frac{h}{e^h-1} = 1 - A_1h - A_2h^2 - A_3h^3 - \&c.;$

and, by (1539), we see that, for values of  $n$  greater than zero,

$$A_{2n+1} = 0 \quad \text{and} \quad A_{2n} = (-1)^n B_{2n} \div \underline{2n}.$$

## BOOLE'S THEOREM.

$$1547 \quad \phi(x+h) - \phi(x) = A_1h \{ \phi'(x+h) + \phi'(x) \} \\ + A_2h^2 \{ \phi''(x+h) + \phi''(x) \} + \&c.$$

PROOF.— $A_1, A_2, A_3$  &c. are found by the same method as that employed in Stirling's Theorem.



For the general relation between the coefficients, as before, make  $\phi(x) = e^x$ , and equation (1547) then produces

$$\frac{e^h - 1}{e^h + 1} = A_1 h + A_2 h^2 + A_3 h^3 + \&c.;$$

and, by comparing this with (1544), we see that

$$A_{2n} = 0 \quad \text{and} \quad A_{2n-1} = (-1)^{n-1} B_{2n} \frac{2^{2n} - 1}{2n}.$$

## EXPANSION OF IMPLICIT FUNCTIONS.

**1550** DEFINITION.—An equation  $f(x, y) = 0$  constitutes  $y$  an *implicit* function of  $x$ . If  $y$  be obtained in terms of  $x$  by solving the equation,  $y$  becomes an *explicit* function of  $x$ .

**1551** LEMMA.—If  $y$  be a function of two independent variables  $x$  and  $z$ ,

$$d_x \{F(y) y_z\} = d_z \{F(y) y_x\}.$$

PROOF.—By performing the differentiations, we obtain

$$F'(y) y_z y_x + F(y) y_{zx} \quad \text{and} \quad F'(y) y_x y_z + F(y) y_{xz},$$

which are evidently equal, by (1482).

## LAGRANGE'S THEOREM.

**1552** Given  $y = z + x\phi(y)$ , the expansion of  $u = f(y)$  in powers of  $x$  is

$$f(y) = f(z) + x\phi(z)f'(z) + \dots + \frac{x^n}{n} \frac{d^{n-1}}{dz^{n-1}} [\{\phi(z)\}^n f'(z)] +$$

PROOF.—Expand  $u$  as a function of  $x$ , by (1507); thus, with the notation of (1406),

$$u = u_0 + x u_{x0} + \frac{x^2}{2} u_{2x0} + \dots + \frac{x^n}{n} u_{nx0} + \&c.$$

Here  $u_0$  is evidently  $f(z)$ .

Differentiating the equation  $y = z + x\phi(y)$  for  $x$  and  $z$  in turn, we have

$$y_x = \phi(y) + x\phi'(y) y_x \quad \text{and} \quad y_z = 1 + x\phi'(y) y_z.$$

Therefore  $y_x = \phi(y) y_z$ ; and, since  $u_x = f'(y) y_x$  and  $u_z = f'(y) y_z$ ,

therefore also  $u_x = \phi(y) u_z$  ..... (i.)

The following equation may now be proved by induction, equation (i.) being its form when  $n = 1$ .

Assume that  $u_{nx} = d_{(n-1)x} [\{\phi(y)\}^n u_z]$  ..... (ii.)

Therefore  $u_{(n+1)x} = d_{(n-1)x} d_x [\{\phi(y)\}^n u_z]$  (1482)

$$= d_{(n-1)x} d_x [\{\phi(y)\}^n u_z] \quad (1551) = d_{nx} [\{\phi(y)\}^{n+1} u_z], \text{ by (i.)}$$

Thus,  $n$  becomes  $n+1$ . But equation (ii.) is true when  $n=1$ ; for then it is equation (i.); therefore it is universally true.

Now, since in equations (i.) and (ii.) the differentiations on the right are all effected with respect to  $z$ ,  $x$  may be made zero *before* differentiating instead of *after*. But, when  $x=0$ ,  $u_x = f'(z)$  and  $\phi(y) = \phi(z)$ , therefore equations (i.) and (ii.) give

$$u_{x0} = \phi(z) f'(z); \quad u_{nx0} = d_{(n-1)z} [\{\phi(z)\}^n f'(z)].$$

**1553** Ex. 1.—Given  $y^3 - ay + b = 0$ : to expand  $\log y$  in powers of  $\frac{1}{a}$ .

Here  $y = \frac{b}{a} + \frac{y^3}{a}$ ; therefore, in Lagrange's formula,

$$x = \frac{1}{a}; \quad z = \frac{b}{a}; \quad f(y) = \log y; \quad \phi(y) = y^3; \quad \text{and} \quad y = z + xy^3.$$

Therefore  $u_0 = \log z; \quad u_{x0} = z^3 \frac{1}{z} = z^2;$

$$u_{nx0} = d_{(n-1)z} \left( z^{2n} \frac{1}{z} \right) = (3n-1)(3n-2) \dots (2n+1) z^{2n}.$$

Therefore, substituting the values of  $x$  and  $z$ , (1552) becomes

$$\log y = \log \frac{b}{a} + \frac{b^2}{a^2} \frac{1}{a} + \dots + \frac{(3n-1)(3n-2) \dots (2n+1)}{1.2 \dots n} \frac{b^{2n}}{a^{2n}} \frac{1}{a^n} + \dots$$

**1554** Ex. 2.—Given the same equation: to expand  $y^n$  in powers of  $\frac{1}{a}$ .

$f(y)$  is now  $y^n$ , and, proceeding as in the last example, we find

$$y^n = \frac{b^n}{a^n} \left\{ 1 + n \frac{b^2}{a^2} \frac{1}{a} + \frac{n(n+5)}{1.2} \frac{b^4}{a^4} \frac{1}{a^2} + \frac{n(n+7)(n+8)}{1.2.3} \frac{b^6}{a^6} \frac{1}{a^3} \right. \\ \left. + \frac{n(n+9)(n+10)(n+11)}{1.2.3.4} \frac{b^8}{a^8} \frac{1}{a^4} + \&c. \right\}$$

If  $n=1$ ,  $y = \frac{b}{a} \left( 1 + \frac{b^2}{a^2} \frac{1}{a} + 3 \frac{b^4}{a^4} \frac{1}{a^2} + 12 \frac{b^6}{a^6} \frac{1}{a^3} + 55 \frac{b^8}{a^8} \frac{1}{a^4} + \&c. \dots \right)$

### CAYLEY'S SERIES FOR $\frac{1}{\phi(z)}$ .

**1555**

$$\frac{1}{\phi(z)} = A - \frac{A^2 z}{1.2} [z\phi(z)]_{zz} + \dots + \frac{A^n (-z)^{n-1}}{1.2 \dots n} [z\{\phi(z)\}^{n-1}]_{nz} + \dots,$$

where  $A = \frac{1}{\phi(0)}.$

PROOF.—Differentiate Lagrange's expansion (1552) for  $z$ , noting that  $\frac{dy}{dz} = \frac{1}{1-x\phi'(y)}$ . Replace  $x$  by  $\frac{y-z}{\phi(y)}$ . Put  $f(y) = \frac{y}{\phi(y)}$ ; and therefore  $f'(z) = \frac{z}{\phi(z)}$ , since  $f$  is an arbitrary function. Then make  $y=0$ .

## LAPLACE'S THEOREM.

**1556** To expand  $f(y)$  in powers of  $x$  when

$$y = F\{z + x\phi(y)\}.$$

**RULE.**—Proceed as in Lagrange's Theorem, merely substituting  $F(z)$  for  $z$  in the formula.

**1557** Ex. 3.—To expand  $e^y$  in powers of  $x$  when  $y = \log(z + x \sin y)$ .

Here  $f(y) = e^y$ ;  $F(z) = \log z$ ;  $\phi(y) = \sin y$ ;

In the value of  $u_{n+1}$  (1552),  $\phi(z)$  becomes  $\phi\{F(z)\} = \sin \log z$ ;  
 $f(z)$  becomes  $f\{F(z)\} = e^{\log z} = z$ ; therefore  $f'(z) = 1$ .

Thus the expansion becomes

$$e^y = z + x \sin \log z + \dots + \frac{x^n}{n} d_{(n-1),1} (\sin \log z)^n.$$

**1558** Ex. 4.—Given  $\sin y = x \sin(y + a)$ : to expand  $y$  in powers of  $x$ .

Here  $y = \sin^{-1}(x \sin(y + a))$ , with  $z = 0$ .

$$f(y) = y; \quad F(z) = \sin^{-1} z; \quad \phi(y) = \sin(y + a).$$

$\phi(z)$  in (1552) becomes  $\phi\{F(z)\} = \sin(\sin^{-1} z + a)$ .

$f(z)$  becomes  $f\{F(z)\} = F(z) = \sin^{-1} z$ ; therefore  $F'(z) = (1 - z^2)^{-\frac{1}{2}}$ .

Thus  $y = x \sin(\sin^{-1} z + a) (1 - z^2)^{-\frac{1}{2}}$

$$+ \frac{1}{2} x^2 d_z \{\sin^2(\sin^{-1} z + a) (1 - z^2)^{-\frac{1}{2}}\} + \frac{1}{6} x^3 d_z^2 \{\sin^3(\sin^{-1} z + a) (1 - z^2)^{-\frac{1}{2}}\} +$$

with  $z$  put = 0 after differentiating. The result is, as in (796),

$$y = x \sin a + \frac{1}{2} x^2 \sin 2a + \frac{1}{3} x^3 \sin 3a + \&c.$$

## BURMANN'S THEOREM.

**1559** To expand one function  $f(y)$  in powers of another function  $\psi(y)$ .

**RULE.**—Put  $x = \psi(y)$  in Lagrange's expansion, and therefore  $\phi(y) = (y - z) \div \psi(y)$ ; therefore

$$\begin{aligned} \mathbf{1560} \quad f(y) &= f(z) + \psi(y) \left\{ \frac{y - z}{\psi(y)} f'(y) \right\}_{y=z} + \dots \\ &\dots + \frac{\{\psi(y)\}^n}{[n]} \frac{d^{n-1}}{dy^{n-1}} \left\{ \left( \frac{y - z}{\psi(y)} \right)^n f'(y) \right\}_{y=z} + \&c. \end{aligned}$$

Here  $y = z$  signifies that after differentiating  $z$  is to be substituted for  $y$ .

**1561** COR. 1.—Since  $x = \psi(y)$ ,  $y = \psi^{-1}(x)$ ; therefore (1560) becomes, by writing  $x$  for  $\psi(y)$ ,

$$f\{\psi^{-1}(x)\} = f(z) + \dots + \frac{x^n}{n} \frac{d^{n-1}}{dy^{n-1}} \left\{ \left( \frac{y-z}{\psi(y)} \right)^n f'(y) \right\}_{y=z} + \dots$$

But since the variable  $y$  is changed into  $z$  after differentiating, it is immaterial what letter is written for  $y$  in the second factor of the general term.

**1562** COR. 2.—If  $f(y)$  be simply  $y$ , the equation becomes

$$\psi^{-1}(x) = z + x \left( \frac{y-z}{\psi(y)} \right)_{y=z} + \dots + \frac{x^n}{n} \frac{d^{n-1}}{dy^{n-1}} \left\{ \left( \frac{y-z}{\psi(y)} \right)^n \right\}_{y=z} + \dots$$

**1563** COR. 3.—If  $z=0$ , so that  $y = x\phi(y)$ , we obtain the expansion of an inverse function,

$$\psi^{-1}(x) = x \left( \frac{y}{\psi(y)} \right)_{y=0} + \dots + \frac{x^n}{n} \frac{d^{n-1}}{dy^{n-1}} \left\{ \left( \frac{y}{\psi(y)} \right)^n \right\}_{y=0} + \dots$$

**1564** Ex. 5.—The series (1528) for  $\sin^{-1}x$  may be obtained by this formula; thus,

Let  $\sin^{-1}x = y$ , therefore  $x = \sin y = \psi(y)$ , in (1563); therefore

$$\sin^{-1}x = x \left( \frac{y}{\sin y} \right)_{y=0} + \frac{x^3}{1.2} \left( \frac{y^3}{\sin^3 y} \right)_{y=0} + \frac{x^5}{1.2.3} \left( \frac{y^5}{\sin^5 y} \right)_{y=0} + \&c.$$

**1565** Ex. 6.—If  $y = \frac{x}{1 + \sqrt{1-x^2}} = \frac{1 - \sqrt{1-x^2}}{x}$ , then, by Lagrange's theorem (1552), since  $y = \frac{x}{2} + \frac{y^3}{2}$ , we find

$$y^n = \left( \frac{x}{2} \right)^n + n \left( \frac{x}{2} \right)^{n+2} + \dots + \frac{n | n+2r-1 }{r | n+r} \left( \frac{x}{2} \right)^{n+2r} + \dots$$

Put  $x = 2\sqrt{t}$ , thus

$$\mathbf{1566} \quad \left( \frac{1 - \sqrt{1-4t}}{2} \right)^n = t^n + n t^{n+1} + \dots + \frac{n | n+2r-1 }{r | n+r} t^{n+r} + \dots$$

Change the sign of  $n$ , thus

$$\mathbf{1567} \quad \left( \frac{1 + \sqrt{1-4t}}{2} \right)^n = 1 - nt + \dots (-1)^r \frac{n | n-r-1 }{r | n-2r} t^r \pm \dots$$

This last series, continued to  $\frac{n}{2} + 1$  or  $\frac{n+1}{2}$  terms, according as  $n$  is even or odd, is equal to the sum of the two series, as appears by the Binomial theorem.

Also, by Lagrange's Theorem,

$$1568 \quad \log y = \log \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \dots + \frac{|2r-1|}{|r| |r|} \left(\frac{x}{2}\right)^{2r} + \dots$$

or, by putting  $x = 2\sqrt{t}$ ,

$$1569 \quad \log \frac{1 - \sqrt{(1-4t)}}{2t} = t + \dots + \frac{|2r-1|}{|r| |r|} t^r + \dots$$

1570 Ex. 7.—Given  $xy = \log y$ ; to expand  $y$  in powers of  $x$ .

The equation can be adapted as follows:

$$y = e^{xy}, \quad \text{therefore} \quad xy = xe^{xy}.$$

Put  $xy = y'$ , therefore  $y' = xe^{y'}$ , from which, by putting  $z = 0$  in (1552),  $y'$  may be expanded, and therefore  $y$ .

Ex. 8.—To expand  $e^{xy}$  in powers of  $ye^{by}$ .

Here  $x = ye^{by}$ ,  $\phi(y) = \frac{y^{-z}}{ye^{by}} = e^{-by}$ , if we take  $z = 0$ . Therefore

$$1571 \quad e^{xy} = 1 + aye^{by} + a(a-2b) \frac{y^2 e^{2by}}{1.2} + a(a-3b)^2 \frac{y^3 e^{3by}}{1.2.3} + \dots$$

#### ABEL'S THEOREM.

1572 If  $\phi(x)$  be a function developable in powers of  $e^x$ ; then

$$\begin{aligned} \phi(x+a) = \phi(x) + a\phi'(x+b) + \frac{a(a-2b)}{1.2} \phi''(x+2b) + \dots \\ \dots + \frac{a(a-rb)^{r-1}}{1.2 \dots r} \phi^r(x+rb) + \dots \end{aligned}$$

PROOF.—Let  $\phi(y) = A_0 + A_1 e^y + A_2 e^{2y} + A_3 e^{3y} + \dots$  (i.)  
Put  $y = 0, 1, 2, 3$ , &c. in (1571), and multiply the results respectively by  $A_0, A_1 e^x, A_2 e^{2x}$ , &c. Then the theorem is proved by equation (i.).

1573 COR.—If  $\phi(x) = x^n$ , Abel's formula gives

$$\begin{aligned} (x+a)^n = x^n + na(x+b)^{n-1} + C(n,2)a(a-2b)(x+2b)^{n-2} + \dots \\ \dots + C(n,r)a(a-rb)^{r-1}(x+rb)^{n-r} + \&c. \end{aligned}$$

#### INDETERMINATE FORMS.

1580 Forms  $\frac{0}{0}, \frac{\infty}{\infty}$ . RULE.—If  $\frac{\phi(x)}{\psi(x)}$  be a fraction which takes either of these forms when  $x = a$ ; then  $\frac{\phi(a)}{\psi(a)} = \frac{\phi'(a)}{\psi'(a)}$  or  $\frac{\phi''(a)}{\psi''(a)}$ , the first determinate fraction obtained by differentiating

the numerator and denominator simultaneously and substituting  $a$  for  $x$  in the result.

**1581** But at any stage of the process the fraction may be reduced to its simplest form before the next differentiation. See example (1589).

PROOF.—(i.) By Taylor's theorem (1500), since  $\phi(a) = 0 = \psi(a)$ ,

$$\frac{\phi(a+h)}{\psi(a+h)} = \frac{\phi(a) + h\phi'(a+\theta h)}{\psi(a) + h\psi'(a+\theta h)} = \frac{\phi'(a+\theta h)}{\psi'(a+\theta h)} = \frac{\phi'(a)}{\psi'(a)},$$

when  $h$  vanishes.

(ii.) If  $\phi(a) = \psi(a) = \infty$ , 
$$\frac{\phi(a)}{\psi(a)} = \frac{1}{\frac{1}{\psi(a)}} \div \frac{1}{\phi(a)},$$

which is of the first form, and therefore

$$= \frac{\psi'(a)}{\{\psi(a)\}^2} \div \frac{\phi'(a)}{\{\phi(a)\}^2} \quad (1414) = \frac{\{\phi(a)\}^2}{\{\psi(a)\}^2} \frac{\psi'(a)}{\phi'(a)}. \quad \text{Therefore } \frac{\phi(a)}{\psi(a)} = \frac{\phi'(a)}{\psi'(a)}.$$

**1582** Vanishing fractions in Algebra are of the indeterminate form just considered, and may be evaluated by the rule, or by rejecting the vanishing factor common to the numerator and denominator.

Ex.—When  $x = a$ ; 
$$\frac{x^3 - a^3}{x^2 - a^2} = \frac{0}{0} = \frac{(x-a)(x^2 + ax + a^2)}{(x-a)(x+a)} = \frac{3a^2}{2a} = \frac{3}{2}a.$$

**1583** Form  $0 \times \infty$ . RULE.—If  $\phi(x) \times \psi(x)$  takes this form when  $x = a$ , put  $\phi(a) \times \psi(a) = \phi(a) \div \frac{1}{\psi(a)}$ , which is of the form  $\frac{0}{0}$ .

**1584** Forms  $0^0, \infty^0, 1^\infty$ . RULE.—If  $\{\phi(x)\}^{\psi(x)}$  takes any of these forms when  $x = a$ , find the limit of the logarithm of the expression. For the logarithm  $= \psi(a) \log \phi(a)$ , which, in each case, is of the form  $0 \times \infty$ .

**1585** Form  $\infty - \infty$ . RULE.—If  $\phi(x) - \psi(x)$  takes this form when  $x = a$ , we have  $e^{\phi(a) - \psi(a)} = \frac{e^{-\psi(a)}}{e^{-\phi(a)}} = \frac{0}{0}$ ; and if the value of this expression be found to be  $c$ , by (1580), the required value will be  $\log c$ .

**1586** Otherwise:  $\phi(a) - \psi(a) = \phi(a) \left\{ 1 - \frac{\psi(a)}{\phi(a)} \right\}$ , which is of the form  $\infty \times 0$  (1583).



**1587** Ex. 1.—With  $x = 0$ ,  $y = \frac{x}{e^x - 1} = \frac{0}{0} = \frac{1}{e^x}$  (1580)  $= 1$ .

Also, with  $x = 0$ ,

$$y_{x0} = \frac{e^x - 1 - xe^x}{(e^x - 1)^2} = \frac{0}{0} = \frac{e^x - e^x - xe^x}{2(e^x - 1)e^x} = \frac{-x}{2(e^x - 1)} = \frac{0}{0} = -\frac{1}{2e^x} = -\frac{1}{2}.$$

**1588** Ex. 2.—With  $x = 1$ ;  
 $\cot(\pi x) \log x = \frac{\log x}{\tan(\pi x)} = \frac{0}{0} = \frac{\cos^2(\pi x)}{\pi x}$  (1580)  $= \frac{1}{\pi}$ .

**1589** Ex. 3.—With  $x = 0$ ;  
 $x^m (\log x)^n = \frac{(\log x)^n}{x^{-m}} = \frac{\infty}{\infty} = \frac{n(\log x)^{n-1}}{-mx^{-m}} = \frac{|n|}{(-m)^n x^{-m}} = 0$ ,

by (1581), differentiating  $n$  times and reducing the fraction to its simplest form after each differentiation.

**1590** Ex. 4.—With  $x = 0$ ;  $y = (1 + ax)^{\frac{1}{x}} = 1^{\infty}$ .  
 By (1584),  $\log y = \frac{\log(1 + ax)}{x} = \frac{0}{0} = \frac{a}{1 + ax}$  (1580)  $= a$ ;  $\therefore y = e^a$ .

**1591** Ex. 5.—With  $x = \pi$ ;  $y = (\pi - x)^{\sin x} = 0^0$ .  
 By (1584),  $\log y = \sin x \log(\pi - x) = \frac{\log(\pi - x)}{\operatorname{cosec} x} = \frac{-\infty}{\infty} = \frac{\sin^2 x}{(\pi - x) \cos x}$   
 (1580)  $= \frac{0}{0} = \frac{2 \sin x \cos x}{-\cos x - (\pi - x) \sin x} = 0$ ;  $\therefore y = 1$ .

**1592** If  $f(x)$  and  $x$  become infinite together, then

$$\frac{f(x)}{x} = f'(x) = f(x+1) - f(x).$$

$$\text{PROOF.—} \frac{f(x)}{x} = \frac{\infty}{\infty} = \frac{f'(x)}{1} \text{ (1580) } = \frac{f(x+1) - f(x)}{1} \text{ (1404),}$$

since, when  $x = \infty$ ,  $h$  may be taken  $= 1$ .

### *Indeterminate forms involving two variables.*

**1592** RULE.—First: If the values  $x=a$ ,  $y=b$  make the fraction  $\frac{\phi(x, y)}{\psi(x, y)} = \frac{0}{0}$ ; the true value is  $= \frac{\phi_x}{\psi_x}$ , if  $\phi_y$  and  $\psi_y$  both vanish.

**1593** Secondly: If  $\phi_x : \psi_x = \phi_y : \psi_y = k$ , the true value of the fraction is  $k$ .

PROOF.—(i.) By (1703)  $\frac{\phi(x, y)}{\psi(x, y)} = \frac{\phi_x + \phi_y y_x}{\psi_x + \psi_y y_x}$ , and  $y$  being an arbitrary function of  $x$ ,—that is, independent of  $x$ ,—the value of the fraction is indeterminate unless  $\phi_y$  and  $\psi_y$  both vanish.

(ii.) If we substitute  $\phi_y = k\psi_y$  and  $\phi_x = k\psi_x$ , the fraction becomes  $= k$ .

## JACOBIANS.

**1600** Let  $u, v, w$  be  $n$  functions of  $n$  variables  $x, y, z$  ( $n=3$ ). The following determinant notation is adopted:—

$$\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \equiv \frac{d(uvw)}{d(xyz)}, \quad \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} \equiv \frac{d(xyz)}{d(uvw)}.$$

The first determinant is called the *Jacobian* of  $u, v, w$  with respect to  $x, y, z$ , and is also denoted by  $J(uvw)$ , or simply by  $J$ .

**1601** THEOREM.—  $\frac{d(uvw)}{d(xyz)} \times \frac{d(xyz)}{d(uvw)} = 1.$

PROOF.—If the product of the two determinants be formed by the rule in (570), first changing the columns into rows in the second determinant (559), the first column of the resulting determinant will be

$$\left. \begin{aligned} u_x x_u + u_y y_u + u_z z_u &= u_u \\ u_x x_v + u_y y_v + u_z z_v &= u_v \\ u_x x_w + u_y y_w + u_z z_w &= u_w \end{aligned} \right\}, \text{ and the whole determinant will be } \begin{vmatrix} u_u & u_v & u_w \\ u_v & v_v & v_w \\ u_w & v_w & w_w \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

**1602** If  $u, v, w$  are  $n$  functions of  $n$  variables  $\alpha, \beta, \gamma$  ( $n=3$ ), and  $\alpha, \beta, \gamma$ , functions of  $x, y, z$ ;

$$\frac{d(uvw)}{d(\alpha\beta\gamma)} \times \frac{d(\alpha\beta\gamma)}{d(xyz)} = \frac{d(uvw)}{d(xyz)}.$$

PROOF.—Form the product of the two determinants, changing columns into rows in the second as in (1601). The first column of the resulting determinant will be

$$\left. \begin{aligned} u_x \alpha_x + u_\beta \beta_x + u_\gamma \gamma_x &= u_x \\ u_x \alpha_y + u_\beta \beta_y + u_\gamma \gamma_y &= u_y \\ u_x \alpha_z + u_\beta \beta_z + u_\gamma \gamma_z &= u_z \end{aligned} \right\}, \text{ and the whole determinant will be } \begin{vmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{vmatrix} = \frac{d(uvw)}{d(xyz)},$$

since rows and columns may be transposed (559).

**1603** COR.—If  $\alpha, \beta, \gamma$  are only given as implicit functions of  $x, y, z$ , by the equations  $\phi=0, \chi=0, \psi=0$ , involving the six variables; then

$$\frac{d(\phi\chi\psi)}{d(\alpha\beta\gamma)} \times \frac{d(\alpha\beta\gamma)}{d(xyz)} = (-1)^3 \frac{d(\phi\chi\psi)}{d(xyz)}.$$

PROOF.—By (1737),  $\phi_x \alpha_x + \phi_\beta \beta_x + \phi_\gamma \gamma_x = -\phi_x$ , where  $\phi_x$  is the partial derivative of  $\phi$ . Thus  $u_x, v_x, w_x$  in the determinant, are now replaced by  $-\phi_x, -\chi_x, -\psi_x$ ; so with  $y$  and  $z$ ; and by changing the sign of each element, the factor  $(-1)^3$  is introduced (562).

**1604** If  $u, v, w, n$  functions of  $n$  variables  $x, y, z$  ( $n=3$ ), be transformed into functions of  $\xi, \eta, \zeta$  by the linear substitutions

$$\left. \begin{aligned} x &= a_1\xi + a_2\eta + a_3\zeta \\ y &= b_1\xi + b_2\eta + b_3\zeta \\ z &= c_1\xi + c_2\eta + c_3\zeta \end{aligned} \right\}; \quad \text{then} \quad \frac{d(urw)}{d(\xi\eta\zeta)} = M \frac{d(urw)}{d(xyz)},$$

$$\text{or} \quad J' = MJ,$$

where  $M$  is the determinant  $(a_1b_2c_3)$  called the *modulus* of transformation (573).

$$\text{PROOF.}— J = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \quad M = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad J' = \begin{vmatrix} u_\xi & u_\eta & u_\zeta \\ v_\xi & v_\eta & v_\zeta \\ w_\xi & w_\eta & w_\zeta \end{vmatrix}$$

Form the product  $MJ$  by the rule in (570). The first element of the resulting determinant is  $u_xa_1 + u_yb_1 + u_zc_1 = u_x\xi + u_y\eta + u_z\zeta = u_\xi$ . Similarly for each element. Then transpose rows and columns, and the determinant  $J'$  is obtained.

**1605** When the modulus is unity, the transformation is said to be *unimodular*.

**1606** If, in (1600),  $\phi(uvw) = 0$ , where  $\phi$  is some function; then  $J(uvw) = 0$ ; and conversely.

PROOF.—Differentiate  $\phi$  for  $x, y$ , and  $z$  separately, thus

$$\phi_x u_x + \phi_y v_x + \phi_z w_x = 0;$$

similarly  $y$  and  $z$ ; and the eliminant of the three equations is  $J(uvw) = 0$ .

**1607** If  $u=0, v=0, w=0$  be a number of homogeneous equations of dimensions  $m, n, p$  in the same number of variables  $x, y, z$ ; then  $J(uvw)$  vanishes, and if the dimensions are equal  $J_x, J_y, J_z$  also vanish.

$$\text{PROOF.}— \text{By (1624), } \left. \begin{aligned} xu_x + yu_y + zu_z &= mu \\ xv_x + yv_y + zv_z &= nv \\ xw_x + yw_y + zw_z &= pw \end{aligned} \right\}; \quad \therefore Jx = A_1mu + B_1nv + C_1pw.$$

By (582),  $A_1, B_1, C_1$  being the minors of the first column of  $J$ . Therefore, if  $u, v, w$  vanish,  $J$  also vanishes.

Again, differentiating the last equation,  $J + xJ_x = A_1mu_x + B_1nv_x + C_1pw_x$ . Therefore, if  $m = n = p$ ,  $J + xJ_x = m(A_1u_x + B_1v_x + C_1w_x) = mJ$ . Therefore  $J_x$  vanishes when  $J$  does.

**1608** If  $u=0, v=0, w=0$  are three homogeneous equations of the second degree in  $x, y, z$ , their eliminant will be the determinant of the sixth order formed by taking the eliminant of the six equations  $u, v, w, J_x, J_y, J_z$ .

PROOF.— $J$  is of the third degree, and therefore  $J_x, J_y, J_z$  are of the second degree, and they vanish because  $u, v, w$  vanish, by (1607). Hence  $u, v, w, J_x, J_y, J_z$  form six equations of the form  $(x, y, z)^2 = 0$ .

**1609** If  $n$  variables  $x, y, z$  ( $n=3$ ) are connected with  $n$  other variables  $\xi, \eta, \zeta$ , by as many equations  $u=0, v=0, w=0$ ;

then 
$$\frac{dx}{d\xi} \frac{dy}{d\eta} \frac{dz}{d\zeta} = \frac{d(uvw)}{d(\xi\eta\zeta)} \div \frac{d(uvw)}{d(xyz)}.$$

PROOF.—By (562) we have

$$\frac{dx}{d\xi} \frac{dy}{d\eta} \frac{dz}{d\zeta} \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} u_x x_\xi & u_y y_\eta & u_z z_\zeta \\ v_x x_\xi & v_y y_\eta & v_z z_\zeta \\ w_x x_\xi & w_y y_\eta & w_z z_\zeta \end{vmatrix} = \begin{vmatrix} u_\xi & u_\eta & u_\zeta \\ v_\xi & v_\eta & v_\zeta \\ w_\xi & w_\eta & w_\zeta \end{vmatrix}$$

## QUANTICS.

**1620** DEFINITION.—A *Quantic* is a homogeneous function of any number of variables: if of *two, three* variables, &c., it is called a *binary, ternary* quantic, &c. The following will illustrate the notation in use. The binary quantic

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3$$

is denoted by  $(a, b, c, d \chi x, y)^3$  when the numerical coefficients are those in the expansion of  $(x+y)^3$ . When the numerical coefficients are all unity, the same quantic is written  $(a, b, c, d \chi x, y)^3$ . When the coefficients are not mentioned, the notation  $(x, y)^3$  is employed.

### EULER'S THEOREM OF QUANTICS.

If  $u = f(x, y)$  be a binary quantic of the  $n^{\text{th}}$  degree, then

**1621** 
$$xu_x + yu_y = nu.$$

**1622** 
$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = n(n-1)u.$$

...      ...      ...      ...

**1623** 
$$(xd_x + yd_y)^r u = n(n-1) \dots (n-r+1)u. \quad (1492)$$

PROOF.—In (1512) put  $h = ax, k = ay$ ; then, because the function is homogeneous, the equation becomes

$$(1+a)^n u = u + a(xu_x + yu_y) + \frac{1}{2}a^2(x^2u_{xx} + 2xyu_{xy} + y^2u_{yy}) + \dots$$

Expand  $(1+a)^n$ , and equate coefficients of powers of  $a$ .

The theorem may be extended to *any* quantic, the quantities on the right remaining unaltered. Thus, in a ternary quantic  $u$  of the  $n^{\text{th}}$  degree,

$$1624 \quad xu_x + yu_y + zu_z = nu; \quad \text{and generally}$$

$$1625 \quad (xd_x + yd_y + zd_z)^r u = n(n-1) \dots (n-r+1)u.$$

#### DEFINITIONS.

**1626** The *Eliminant* of  $n$  quantics in  $n$  variables is the function of the coefficients obtained by putting all the quantics equal to zero and eliminating the variables (583, 586).

**1627** The *Discriminant* of a quantic is the eliminant of its first derivatives with respect to each of the variables (Ex. 1631).

**1628** An *Invariant* is a function of the coefficients of an equation whose value is not altered by linear transformation of the equation, excepting that the function is multiplied by the modulus of transformation (Ex. 1632).

**1629** A *Covariant* is a quantic derived from another quantic, and such that, when both are subjected to the same linear transformation, the resulting quantics are connected by the same process of derivation (Ex. 1634).

**1630** A *Hessian* is the Jacobian of the first derivatives of a function.

Thus, the Hessian of a ternary quantic  $u$ , whose first derivatives are  $u_x, u_y, u_z$ , is

$$\frac{d(u_x u_y u_z)}{d(xyz)} = \begin{vmatrix} u_{xx} & u_{xy} & u_{xz} \\ u_{yx} & u_{yy} & u_{yz} \\ u_{zx} & u_{zy} & u_{zz} \end{vmatrix}.$$

**1631** Ex.—Take the *binary cubic*  $u = ax^3 + 3bx^2y + 3cxy^2 + dy^3$ . Its first derivatives are

$$\begin{aligned} u_x &= 3ax^2 + 6bxy + 3cy^2, & \begin{vmatrix} 3a & 6b & 3c & 0 \\ 0 & 3a & 6b & 3c \\ 3b & 6c & 3d & 0 \\ 0 & 3b & 6c & 3d \end{vmatrix} \\ u_y &= 3bx^2 + 6cxy + 3dy^2. \end{aligned}$$

Therefore (1627) the *discriminant* of  $u$  is the annexed determinant, by (587).

**1632** The determinant is also an invariant of  $u$ , by (1638); that is, if  $u$  be transformed into  $v$  by putting  $x = \alpha\xi + \beta\eta$  and  $y = \alpha'\xi + \beta'\eta$ ; and, if a corresponding determinant be formed with the coefficients of  $v$ , the new determinant will be equal to the original one multiplied by  $(\alpha\beta' - \alpha'\beta)^3$ .

**1633** Again,  $u_{2x} = 6ax + 6by$ ,  $u_{2y} = 6cx + 6dy$ ,  $u_{xy} = 6bx + 6cy$ .

Therefore, by (1630), the Hessian of  $u$  is

$$\begin{aligned} u_{2x}u_{2y} - u_{xy}^2 &= (ax + by)(cx + dy) - (bx + cy)^2 \\ &= (ac - b^2)x^2 + (ad - bc)xy + (bd - c^2)y^2. \end{aligned}$$

**1634** And this is also a *covariant*; for, if  $u$  be transformed into  $v$ , as before, then the result of transforming the Hessian by the same equations will be found to be equal to  $v_{2x}v_{2y} - v_{xy}^2$ . See (1652).

**1635** If a quantic,  $u = f(x, y, z \dots)$ , involving  $n$  variables can be expressed as a function of the second degree in  $X_1, X_2 \dots X_{n-1}$ , where the latter are linear functions of the variables, the discriminant vanishes.

PROOF.—Let  $u = \phi X_1^2 + \psi X_1 X_2 + \chi X_1 X_3 + \&c.$ ,  
where  $X_1 = a_1x + b_1y + c_1z + \&c.$

The derivatives  $u_x, u_y, \&c.$  must contain one of the factors  $X_1 X_2 \dots X_{n-1}$  in every term, and therefore must have, for common roots, the roots of the simultaneous equations  $X_1 = 0, X_2 = 0, \dots X_{n-1} = 0$ ;  $n - 1$  equations being required to determine the ratios of the  $n$  variables. Therefore the discriminant of  $u$ , which (1627) is the eliminant of the equations  $u_x = 0, u_y = 0, \&c.$ , vanishes, by (588).

**1636** COR. 1.—If a binary quantic contains a square factor, the discriminant vanishes; and conversely.

Thus, in Example (1631), if  $u$  has a factor of the form  $(Ax + By)^2$ , the determinant there written vanishes.

**1637** COR. 2.—If any quadric is resolvable into two factors, the discriminant vanishes.

An independent proof is as follows:—

Let  $u = XY$  be the quadric, where

$$X = (ax + by + cz + \dots), \quad Y = (a'x + b'y + c'z + \dots).$$

The derivatives  $u_x, u_y, u_z$  are each of the form  $pX + qY$ , and therefore have for common roots the roots of the simultaneous equations  $X = 0, Y = 0$ . Therefore the eliminant of  $u_x = 0, u_y = 0, \&c.$  vanishes (1627).

**1638** The discriminant of a binary quantic is an invariant.

PROOF.—A square factor remains a square factor after linear transformation. Hence, by (1636), if the discriminant vanishes, the discriminant of the transformed equation vanishes, and must therefore contain the former discriminant as a factor (see 1628). Thus the determinant in (1631) is an invariant of the quantic  $u$ .

The discriminant of the ternary quadric

**1639** 
$$u = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$
  
2 P



is the eliminant of the equations

$$1640 \quad \left. \begin{aligned} \frac{1}{2}u_x &= ax + hy + gz = 0 \\ \frac{1}{2}u_y &= hx + by + fz = 0 \\ \frac{1}{2}u_z &= gx + fy + cz = 0 \end{aligned} \right\}; \quad \text{that is, the determinant } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$1641 \quad = abc + 2fgh - af^2 - bg^2 - ch^2 \equiv \Delta.$$

1642 The following notation will frequently be employed.

The determinant will be denoted by  $\Delta$ , and its minors by  $A, B, C, F, G, H$ . Their values are readily found by differentiating  $\Delta$  with respect to  $a, b, c, f, g, h$ ; thus,

$$A = bc - f^2 = \Delta_a, \quad B = ca - g^2 = \Delta_b, \quad C = ab - h^2 = \Delta_c, \\ F = gh - af = \frac{1}{2}\Delta_f, \quad G = hf - bg = \frac{1}{2}\Delta_g, \quad H = fg - ch = \frac{1}{2}\Delta_h.$$

1643 The reciprocal determinant is equal to  $\Delta^2$  or

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}^2 \quad \text{By (575).}$$

1644 The discriminant of the quaternary quadric

$$u = ax^2 + by^2 + cz^2 + dw^2 + 2fyz + 2gzx + 2hxy \\ + 2pxw + 2qyw + 2rzw$$

is the eliminant of the equations

$$1645 \quad \left. \begin{aligned} \frac{1}{2}u_x &= ax + hy + gz + pw = 0 \\ \frac{1}{2}u_y &= hx + by + fz + qw = 0 \\ \frac{1}{2}u_z &= gx + fy + cz + rw = 0 \\ \frac{1}{2}u_w &= px + qy + rz + dw = 0 \end{aligned} \right\}; \quad \text{that is, the determinant } \begin{vmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & d \end{vmatrix}$$

The determinant will be denoted by  $\Delta'$ , and, by decomposing it by (568), we have

$$1646 \quad \Delta' = d \cdot \Delta - Ap^2 - Bq^2 - Cr^2 - 2Fqr - 2Grp - 2Hpq.$$

$$1647 \quad \Delta' = \frac{1}{2} (p\Delta'_p + q\Delta'_q + r\Delta'_r) + d \cdot \Delta.$$

1648 THEOREM.—If  $\phi(xy)$  be a quantic of an even degree,

$$\phi(d_y, -d_x) \phi(x, y)$$

is an invariant of the quantic.

PROOF.—Let the linear substitutions (1628) be

$$x = a\xi + b\eta, \quad y = a'\xi + b'\eta \dots\dots\dots(i.)$$

Solve for  $\xi$  and  $\eta$ . Find  $\xi_x, \xi_y, \eta_x, \eta_y$ , and substitute in the two equations

$$d_y = d_\xi \xi_y + d_\eta \eta_y; \quad d_x = d_\xi \xi_x + d_\eta \eta_x.$$

The result is

$$d_y = \{ad_\eta + b(-d_\xi)\} \div M; \quad -d_x = \{a'd_\eta + b'(-d_\xi)\} \div M \dots\dots(ii.),$$

where  $M = ab' - a'b$ , the modulus of transformation. Equations (i.) and (ii.) are parallel, and show that the operations  $d_y$  and  $-d_x$  can be transformed in the same way as the quantities  $x$  and  $y$ ; that is, if  $\phi(x, y)$  becomes  $\psi(\xi, \eta)$ , then  $\phi(d_y, -d_x)$  becomes  $\psi(d_\eta, -d_\xi) \div M^n$ , where  $n$  is the degree of the quantic  $\phi$ . But  $\phi(d_y, -d_x) \phi(x, y)$  is a function of the coefficients *only* of the quantic  $\phi$ , since the order of differentiation of each term is the same as the degree of the term; therefore the function is an invariant, by definition (1628).

**1649** EXAMPLE.—Let  $\phi(x, y) = ax^4 + bx^3y + cxy^3 + fy^4$ . The quantic must first be completed; thus,  $\phi(xy) = ax^4 + bx^3y + cx^2y^2 + cxy^3 + fy^4$ , ( $c = 0$ ); then  $\phi(d_y, -d_x) \phi(x, y) = (ad_{xy} - bd_{yxx} + cd_{xyx} - ed_{yxx} + fd_{xx}) \phi(x, y)$

$$= a.24f - b.6e + c.4c - c.6b + f.24a = 4(12af - 3be + c^2).$$

Therefore  $12af - 3be + c^2$  is an invariant of  $\phi$ , and  $= (12AF - 3BE + C^2) \div M^4$ , where  $A, B, C, E, F$  are the coefficients of any equation obtained from  $\phi$  by a linear transformation.

But if the degree of the quantic be odd, these results vanish identically.

**1650** Similarly, if  $\phi(x, y), \psi(x, y)$  are two quantics of the same degree, the functions

$$\phi(d_y, -d_x) \psi(x, y) \quad \text{and} \quad \psi(d_y, -d_x) \phi(x, y)$$

are both invariants.

**1651** Ex.—If  $\phi = ax^2 + 2bxy + cy^2$  and  $\psi = a'x^2 + 2b'xy + c'y^2$ ; then  $(ad_{xy} - 2bd_{xy} + cd_{xx})(a'x^2 + 2b'xy + c'y^2) = ac' + ca' - 2bb'$ , an invariant.

**1652** A Hessian is a covariant of the original quantic.

PROOF.—Let a ternary quantic  $u$  be transformed by the linear substitutions in (1604); so that  $u = \phi(x, y, z) = \psi(\xi, \eta, \zeta)$ . The Hessians of the two functions are  $\frac{d(u_x u_y u_z)}{d(xyz)}$  and  $\frac{d(u_\xi u_\eta u_\zeta)}{d(\xi\eta\zeta)}$  (1630). Now

$$\frac{d(u_\xi u_\eta u_\zeta)}{d(\xi\eta\zeta)} = M \frac{d(u_x u_y u_z)}{d(xyz)} = M \frac{d(u_x u_y u_z)}{d(\xi\eta\zeta)} = M^3 \frac{d(u_x u_y u_z)}{d(xyz)}.$$

The second transformation is seen at once from the form of the determinant by merely transposing rows and columns; the first and third are by theorem (1604). Therefore, by definition (1629), the Hessian of  $u$  is a covariant.

**1653** *Cogredients*.—Variables are *cogredient* when they are subjected to the same linear transformation; thus,  $x, y$  are

cogredient with  $x', y'$  when

$$\left. \begin{array}{l} x = a\xi + b\eta \\ y = c\xi + d\eta \end{array} \right\} \quad \text{and} \quad \left. \begin{array}{l} x' = a\xi' + b\eta' \\ y' = c\xi' + d\eta' \end{array} \right\}.$$

**1654** *Emanents*.—If in any quantic  $u = \phi(x, y)$ , we change  $x$  into  $x + \rho x'$ , and  $y$  into  $y + \rho y'$ , where  $x', y'$  are cogredient with  $x, y$ ; then, by (1512),

$$\begin{aligned} & \phi(x + \rho x', y + \rho y') \\ &= u + \rho(x'd_x + y'd_y)u + \frac{1}{2}\rho^2(x'd_x + y'd_y)^2u + \&c., \end{aligned}$$

and the coefficients of  $\rho, \rho^2, \rho^3, \dots$  are called the first, second, third, ... *emanents* of  $u$ .

**1655** The emanents, of the typical form  $(x'd_x + y'd_y)^n u$ , are all covariants of the quantic  $u$ .

PROOF.—If, in  $\phi(x, y)$ , we first make the substitutions which lead to the emanent, and afterwards make the cogredient substitutions, we change

$$x \text{ into } x + \rho x', \text{ and this into } a\xi + b\eta + \rho(a\xi' + b\eta').$$

And if the order of these operations be reversed, we change

$$x \text{ into } a\xi + b\eta, \text{ and this into } a(\xi + \rho\xi') + b(\eta + \rho\eta').$$

The two results are identical, and it follows that, if  $\phi(x, y)$  be transformed by the same operations in reversed order, the coefficients of the powers of  $\rho$  in the two expansions will be equal, since  $\rho$  is indeterminate. Therefore, by the definition (1629), each emanent is a covariant.

**1656** For definitions of *contragredients* and *contravariants*, see (1813–4).

For other theorems on invariants, see (1794), and the Article on Invariants in Section XII.

## IMPLICIT FUNCTIONS.

### IMPLICIT FUNCTIONS OF ONE INDEPENDENT VARIABLE.

If  $y$  and  $z$  be functions of  $x$ , the successive application of formula (1420) gives, for the first, second, and third derivatives of the function  $\phi(y, z)$  with the notation of (1405),

**1700**  $\phi_x(yz) = \phi_y y_x + \phi_z z_x.$

**1701**  $\phi_{xx}(yz) = \phi_{yy} y_x^2 + 2\phi_{yz} y_x z_x + \phi_{zz} z_x^2 + \phi_y y_{xx} + \phi_z z_{xx}.$

$$1702 \quad \phi_{3x}(yz) = \phi_{3y}y_x^3 + 3\phi_{2yz}y_x^2z_x + 3\phi_{y2z}y_xz_x^2 + \phi_{3z}z_x^3 \\ + 3(\phi_{2y}y_x + \phi_{yz}z_x)y_{2x} + 3(\phi_{2z}z_x + \phi_{zy}y_x)z_{2x} \\ + \phi_yy_{3x} + \phi_zz_{3x}.$$

By making  $z = x$  in the last three formulæ, and consequently  $z_x = 1$ ,  $z_{2x} = 0$ , or else by differentiating independently, we obtain

$$1703 \quad \phi_x(xy) = \phi_yy_x + \phi_x.$$

$$1704 \quad \phi_{2x}(xy) = \phi_{2y}y_x^2 + 2\phi_{xy}y_x + \phi_{2x} + \phi_yy_{2x}.$$

$$1705 \quad \phi_{3x}(xy) = \phi_{3y}y_x^3 + 3\phi_{2yx}y_x^2 + 3\phi_{y2x}y_x + \phi_{3x} \\ + 3(\phi_{2y}y_x + \phi_{xy})y_{2x} + \phi_yy_{3x}.$$

1706 In these formulæ the notation  $\phi_x$  is used where the differentiation is partial, while  $\phi_x(x, y)$  is used to denote the complete derivative of  $\phi(x, y)$  with respect to  $x$ . Each successive partial derivative of the function  $\phi(y, z)$  (1700) is itself treated as a function of  $y$  and  $z$ , and differentiated as such by formula (1420).

Thus, the differentiation of the product  $\phi_yy_x$  in (1700) produces

$$(\phi_y)_xy_x + \phi_yy_{2x} = (\phi_{2y}y_x + \phi_{yz}z_x)y_x + \phi_yy_{2x}.$$

The function  $\phi_y$  involves  $y$  and  $z$  by implication. If it should not in fact contain  $z$ , for instance, then the partial derivative  $\phi_{yz}$  vanishes. On the other hand,  $y_x$ ,  $y_{2x}$ , &c. are independent of  $z$ ; and  $z_x$ ,  $z_{2x}$ , &c. are independent of  $y$ .

#### DERIVED EQUATIONS.

1707 If  $\phi(xy) = 0$ , its successive derivatives are also zero, and the expansions (1703–5) are then called the *first, second, and third derived equations* of the primitive equation  $\phi(xy) = 0$ .

In this case, those equations give, by eliminating  $y_x$ ,

$$1708 \quad \frac{dy}{dx} = -\frac{\phi_x}{\phi_y}; \quad \frac{d^2y}{dx^2} = \frac{2\phi_{xy}\phi_x\phi_y - \phi_{2x}\phi_y^2 - \phi_{2y}\phi_x^3}{\phi_y^3}.$$

1710 Similarly, by eliminating  $y_x$  and  $y_{2x}$ , equation (1705) would give  $y_{3x}$  in terms of the partial derivatives of  $\phi(xy)$ . See the note following (1732).

$$1711 \quad \text{If } \phi(xy) = 0 \text{ and } \frac{dy}{dx} = 0; \quad \frac{d^2y}{dx^2} = -\frac{\phi_{2x}}{\phi_y};$$

$$1712 \quad \text{and} \quad \frac{d^3y}{dx^3} = \frac{3\phi_{2x}\phi_{xy} - \phi_{3x}\phi_y}{\phi_y^2}.$$

PROOF.—By (1708),  $\phi_x = 0$ . Therefore (1704) and (1705) give these values of  $y_{2x}$  and  $y_{3x}$ .

**1713** If  $\phi_x$  and  $\phi_y$  both vanish,  $y_x$  in (1708) is indeterminate. In this case it has two values given by the second derived equation (1704), which becomes a quadratic in  $y_x$ .

**1714** If  $\phi_{2x}$ ,  $\phi_{xy}$ , and  $\phi_{2y}$  also vanish, proceed to the third derived equation (1705), which now becomes a cubic in  $y_x$ , giving three values, and so on.

**1715** Generally, when all the partial derivatives of  $\phi(x, y)$  of orders less than  $n$  vanish for certain values  $x=a$ ,  $y=b$ , we have, by (1512),  $\phi(a, b)$  being zero,

$$\phi(a+h, b+k) = \frac{1}{n!} (hd_x + kd_y)^n \phi(xy)_{a,b} + \text{terms of}$$

higher orders which may be neglected in the limit. ( $x, y$  are here put  $= a, b$  after differentiation.) Now, with the notation of 1406),

$$h\phi_{x,a} + k\phi_{y,b} = 0;$$

therefore

$$\frac{k}{h} = -\frac{\phi_{x,a}}{\phi_{y,b}} = \frac{dy}{dx};$$

the values of which are therefore given by the equation

**1716**  $(hd_x + kd_y)^n \phi(x, y)_{a,b} = 0.$

**1717** If  $y_x$  becomes indeterminate through  $x$  and  $y$  vanishing, observe that  $\frac{dy}{dx} = \frac{y}{x}$  in this case, and that the value of the latter fraction may often be more readily determined by algebraic methods.

If  $x$  and  $y$  in the function  $\phi(x, y)$  are connected by the equation  $\psi(x, y) = 0$ ,  $y$  is thereby made an implicit function of  $x$ , and we have

**1718**  $\phi_x(x, y) = \frac{\phi_x \psi_y - \psi_x \phi_y}{\psi_y}.$

**1719**  $\phi_{2x}(x, y) = \{(\phi_{2y} \psi_y - \psi_{2y} \phi_y) \psi_x^2 + (\phi_{2x} \psi_y - \psi_{2x} \phi_y) \psi_y^2 - 2(\phi_{yx} \psi_y - \psi_{yx} \phi_y) \psi_x \psi_y\} \div \psi_y^3.$

PROOF.—(i.) Differentiate both  $\phi$  and  $\psi$  for  $x$ , by (1703), and eliminate  $y_x$ .  
(ii.) Differentiate also, by (1704), and eliminate  $y_x$  and  $y_{2x}$ .

If  $u, y, z$  are functions of  $x$ , then, as in (1700),

**1720**  $\phi_x(uyz) = \phi_u u_x + \phi_y y_x + \phi_z z_x.$

$$1721 \quad \phi_{2x}(uyz) = \phi_{2u}u_x^2 + \phi_{2y}y_x^2 + \phi_{2z}z_x^2 \\ + 2\phi_{yz}y_xz_x + 2\phi_{zu}z_xu_x + 2\phi_{uy}u_xy_x \\ + \phi_{uu}u_{2x} + \phi_{yy}y_{2x} + \phi_{zz}z_{2x}.$$

1722 To obtain  $\phi_x(xyz)$  and  $\phi_{2x}(xyz)$ , make  $u=x$  in the above equations.

Let  $U = \phi(x, y, z, \xi)$  be a function of four variables connected by three equations  $u=0, v=0, w=0$ , so that one of the variables,  $\xi$ , may be considered independent.

1723 We have, by differentiating for  $\xi$ ,

$$\left. \begin{aligned} \phi_x x_\xi + \phi_y y_\xi + \phi_z z_\xi + \phi_\xi - U_\xi &= 0 \\ u_x x_\xi + u_y y_\xi + u_z z_\xi + u_\xi &= 0 \\ v_x x_\xi + v_y y_\xi + v_z z_\xi + v_\xi &= 0 \\ w_x x_\xi + w_y y_\xi + w_z z_\xi + w_\xi &= 0 \end{aligned} \right\}; \quad \frac{dU}{d\xi} = \frac{d(\phi uvw)}{d(xyz\xi)} \frac{1}{J};$$

where  $J = \frac{d(uvw)}{d(xyz)}$ .

$$1724 \quad \frac{dx}{d\xi} = -\frac{d(uvw)}{d(\xi yz)} \frac{1}{J}; \quad \frac{dy}{d\xi} = -\frac{d(uvw)}{d(x\xi z)} \frac{1}{J}; \\ \frac{dz}{d\xi} = -\frac{d(uvw)}{d(xy\xi)} \frac{1}{J}.$$

Observe that  $U_\xi$  stands for the *complete* and  $\phi_\xi$  for the *partial* derivative of the function  $U$ .

PROOF.—(i.)  $U_\xi$  is found by taking the eliminant of the four equations, separating the determinant into two terms by means of the element  $\phi_\xi - U_\xi$ , and employing the notation in (1600).

(ii.)  $x_\xi, y_\xi$ , and  $z_\xi$  are found by solving the last three of the same equations, by (582).

#### IMPLICIT FUNCTIONS OF TWO INDEPENDENT VARIABLES.

1725 If the equation  $\phi(x, y, z) = 0$  alone be given,  $y$  may be considered an implicit function of  $x$  and  $z$ . Since  $x$  and  $z$  are independent, we may make  $z$  constant and differentiate for  $x$ ; thus, for a variation in  $x$  *only*, the equations (1703–5) are produced again with  $\phi(x, y, z)$  in the place of  $\phi(x, y)$ .

1726 If  $x$  be made constant,  $z$  must replace  $x$  in those equations as the independent variable.

Again, by differentiating the equation  $\phi(xyz) = 0$  first for  $x$ , making  $z$  constant, and the result for  $z$ , making  $x$  constant,



we obtain

$$1727 \quad \phi_{xz} + \phi_{yx}y_z + \phi_{yz}y_x + \phi_{zy}y_xy_z + \phi_yy_{xz} = 0.$$

From this and the values of  $y_x$  and  $y_z$ , by (1708),

$$1728 \quad y_{xz} = \frac{\phi_{yx}\phi_y\phi_z + \phi_{yz}\phi_y\phi_x - \phi_{xz}\phi_y^2 - \phi_x\phi_z\phi_{zy}}{\phi_y^3}.$$

**1729** If  $x, y, z$  in the function  $\phi(x, y, z)$  be connected by the relation  $\psi(x, y, z) = 0$ ,  $y$  may be taken as a function of two independent variables  $x$  and  $z$ . We may therefore make  $z$  constant, and the values of  $\phi_x(x, y, z)$  and  $\phi_{zx}(x, y, z)$  are identical with those in (1718, '19) if  $x, y, z$  be substituted for  $x, y$  in each function.

**1730** By changing  $x$  into  $z$  the same formulæ give the values of  $\phi_z(x, y, z)$  and  $\phi_{zx}(x, y, z)$ .

**1731** On the same hypothesis, if the value of  $\phi_x(x, y, z)$ , in forming which  $z$  has been made constant, be now differentiated for  $z$  while  $x$  is made constant, each partial derivative  $\phi_x, \psi_y$ , &c. in (1718) must be differentiated as containing  $x, y$ , and  $z$ , of which three variables  $x$  is now constant and  $y$  is a function of  $z$ .

The result is

$$1732 \quad \phi_{xz}(x, y, z) = \{(\phi_{xz}\psi_y - \psi_{xz}\phi_y)\psi_y^3 - (\phi_{yx}\psi_y - \psi_{yx}\phi_y)\psi_z\psi_y^2 - (\phi_{yz}\psi_y - \psi_{yz}\phi_y)\psi_x\psi_y^2 + (\phi_{zy}\psi_y - \psi_{zy}\phi_y)\psi_x\psi_z\psi_y\} \div \psi_y^3.$$

In a particular instance it is generally easier to apply such rules for differentiating directly to the example proposed, than to deduce the result in a functional form for the purpose of substituting in it the values of the partial derivatives.

**1734** EXAMPLE.—Let  $\phi(x, y, z) = lx + my + nz$  and  $\psi(x, y, z) = x^2 + y^2 + z^2 = 1$ ,  $x$  and  $z$  being the independent variables;  $\phi_x(x, y, z)$  and  $\phi_{zx}(x, y, z)$  are required. Differentiating  $\phi$ , considering  $z$  constant,

$$\phi_x(x, y, z) = l + m \frac{dy}{dx} = l - m \frac{x}{y}; \quad \text{since } \frac{dy}{dx} = -\frac{\psi_x}{\psi_y} = -\frac{x}{y};$$

$$\phi_{zx}(x, y, z) = -m \frac{y - x\eta_z}{y^2} = -m \frac{y^2 + x^2}{y^3};$$

a result which is otherwise obtained from formula (1719) by substituting the values

$$\phi_x = l, \quad \phi_y = m, \quad \phi_z = n; \quad \phi_{zx} = \phi_{zy} = \phi_{xz} = 0;$$

$$\psi_x = 2x, \quad \psi_y = 2y, \quad \psi_z = 2z; \quad \psi_{zx} = \psi_{zy} = \psi_{xz} = 2.$$

Again, to find  $\phi_{zx}(x, y, z)$ , differentiate for  $z$ , considering  $x$  constant in the function

$$\frac{d\phi}{dx} = l - m \frac{x}{y}; \quad \text{thus } \frac{d^2\phi}{dx dz} = +m \frac{x\eta_z}{y^2} = -m \frac{xz}{y^3}, \quad \text{since } \frac{dz}{dx} = -\frac{\psi_z}{\psi_x} = -\frac{z}{x}.$$

**1735** Let  $U = \phi(x, y, z, \xi, \eta)$  be a function of five variables connected by three equations  $u=0, v=0, w=0$ ; so that *two* of the variables  $\xi, \eta$  may be considered independent. Making  $\eta$  constant, the equations in (1723), and the values obtained for  $U_\xi, x_\xi, y_\xi, z_\xi$ , hold good in the present case for the variations due to a variation in  $\xi$ , observing that  $\phi, u, v, w$  now stand for functions of  $\eta$  as well as of  $\xi$ .

**1736** The corresponding values of  $U_\eta, x_\eta, y_\eta, z_\eta$  are obtained by changing  $\xi$  into  $\eta$ .

IMPLICIT FUNCTIONS OF  $n$  INDEPENDENT VARIABLES.

**1737** The same method is applicable to the general case of a function of  $n$  variables connected by  $r$  equations  $u=0, v=0, w=0 \dots$  &c.

The equations constitute *any*  $n-r$  of the variables we please, *independent*: let these be  $\xi, \eta, \zeta \dots$ . The remaining  $r$  variables will be *dependent*: let these be  $x, y, z \dots$ ; and let the function be  $U \equiv \phi(x, y, z \dots \xi, \eta, \zeta \dots)$ .

For a variation in  $\xi$  *only*, there will be the derivative of the function  $U$ , and  $r$  derived equations as under.

$$\begin{aligned} \mathbf{1738} \quad & \phi_x x_\xi + \phi_y y_\xi + \phi_z z_\xi + \dots + \phi_\xi = U_\xi, \\ & u_x x_\xi + u_y y_\xi + u_z z_\xi + \dots + u_\xi = 0, \\ & v_x x_\xi + v_y y_\xi + v_z z_\xi + \dots + v_\xi = 0, \\ & \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \&c., \end{aligned}$$

involving the  $r$  implicit functions  $x_\xi, y_\xi, z_\xi, \&c.$  The solution of the  $r$  equations, as in (1724), gives

$$\mathbf{1739} \quad \frac{dx}{d\xi} = - \frac{d(uvw \dots)}{d(\xi yz \dots)} \frac{1}{J}; \quad \frac{dy}{d\xi} = - \frac{d(uvw \dots)}{d(x \xi z \dots)} \frac{1}{J}; \quad \&c.,$$

$$\mathbf{1740} \quad \text{where } J = \frac{d(uvw \dots)}{d(xyz \dots)}. \quad \text{Also } \frac{dU}{d\xi} = \frac{d(\phi uvw)}{d(xy z \xi)} \frac{1}{J}.$$

The last value being found exactly as in (1723).

**1741** With  $\xi$  replaced by  $\eta$  we have in like manner the values of  $x_\eta, y_\eta, z_\eta, U_\eta$ ; and similarly with each of the independent variables in turn.

**1742** If there be  $n$  variables and but one equation  $\phi(x, y, z \dots) = 0$ , there will be  $n-1$  independent and one depen-

dent variable. Let  $y$  be dependent. Then for a variation in  $x$  only of the remaining variables, the equations (1703-5) apply to the present case,  $\phi$  standing for  $\phi(x, y, z \dots)$ . If  $x$  be replaced by each of the remaining independents in turn, there will be, in all,  $n-1$  sets of derived equations.

### CHANGE OF THE INDEPENDENT VARIABLE.

If  $y$  be any function of  $x$ , and if the independent variable  $x$  be changed to  $t$ , and if  $t$  be afterwards put equal to  $y$ , the following formulæ of substitution are obtained, in which  $p \equiv y_x$ :

$$1760 \quad \frac{dy}{dx} = \frac{y_t}{x_t} = \frac{1}{x_y}.$$

Differentiating these fractions, we get

$$1762 \quad \frac{d^2y}{dx^2} = \frac{y_{2t}x_t - x_{2t}y_t}{x_t^3} = -\frac{x_{2y}}{x_y^3} = pp_y.$$

$$1765 \quad \frac{d^3y}{dx^3} = \frac{x_t \{y_{3t}x_t - x_{3t}y_t\} - 3x_{2t} \{y_{2t}x_t - x_{2t}y_t\}}{x_t^5}.$$

$$1766 \quad = \frac{3x_{2y}^2 - x_y x_{3y}}{x_y^5} = pp_y^2 + p^2 p_{2y}.$$

EX.—If  $x = r \cos \theta$  and  $y = r \sin \theta$ ; then

$$1768 \quad \frac{dy}{dx} = \frac{r_\theta \sin \theta + r \cos \theta}{r_\theta \cos \theta - r \sin \theta}; \quad \frac{d^2y}{dx^2} = \frac{r^2 + 2r_\theta^2 - rr_{2\theta}}{(r_\theta \cos \theta - r \sin \theta)^3}.$$

PROOF.—Writing  $\theta$  for  $t$  in (1760) and (1762), we have to find  $x_\theta, y_\theta, x_{2\theta}, y_{2\theta}$ ; thus,

$$\begin{aligned} x_\theta &= r_\theta \cos \theta - r \sin \theta; & x_{2\theta} &= r_{2\theta} \cos \theta - 2r_\theta \sin \theta - r \cos \theta; \\ y_\theta &= r_\theta \sin \theta + r \cos \theta; & y_{2\theta} &= r_{2\theta} \sin \theta + 2r_\theta \cos \theta - r \sin \theta. \end{aligned}$$

Substituting these values, the above results are obtained.

To change the variable from  $x$  to  $t$  in  $(a+bx)^n y_{nx}$ , where  $(a+bx) = e^t$ , employ the formula

$$1770 \quad (a+bx)^n y_{nx} = b^n (d_t - \overline{n-1}) (d_t - \overline{n-2}) \dots (d_t - 1) y_t.$$

in which, multiplication by  $d_t$  by the index law signifies the repetition of the operation  $d_t$  (1492).

PROOF.—  $d_t \{ (a+bx)^n y_{nx} \} = \{ n(a+bx)^{n-1} b y_{nx} + (a+bx)^n y_{(n+1)x} \} x_t$ .

Now  $bx_t = e' = a+bx$ . Substitute this, and denote  $(a+bx)^n y_{nx}$  by  $U_n$ ;

therefore  $d_t(U_n) = nU_n + \frac{1}{b} U_{n+1}$ , or  $U_{n+1} = b(d_t - n)U_n$ .

Therefore  $U_n = b(d_t - \overline{n-1})U_{n-1}$  and  $U_{n-1} = b(d_t - \overline{n-2})U_{n-2}$ , &c.,

and finally  $U_1 = b(d_t - 1)U_1$ .

But  $U_1 = (a+bx)y_x = bx_t y'_x = by_t$ .

Therefore  $U_n = b^n (d_t - \overline{n-1})(d_t - \overline{n-2}) \dots (d_t - 1)y_t$ .

**1771** COR.— $(a+x)^n y_{nx}$  and  $x^n y_{nx}$  are transformed by the same formula by putting  $b=1$ .

**1772** Let  $V \equiv F(x, y)$ , where  $x, y$  are connected with  $\xi, \eta$  by the equations  $u=0, v=0$ . It is required to change the independent variables  $x, y$  to  $\xi, \eta$  in the functions  $V_x$  and  $V_y$ .

**1773** RULE.—To find the value of  $V_x$ —Differentiate  $V, u, v$ , each with respect to  $x$ , considering  $\xi, \eta$  functions of the independent variables  $x, y$ ; and form the eliminant of the resulting equations; thus,

$$\left. \begin{aligned} V_\xi \xi_x + V_\eta \eta_x - V_x &= 0 \\ u_\xi \xi_x + u_\eta \eta_x + u_x &= 0 \\ v_\xi \xi_x + v_\eta \eta_x + v_x &= 0 \end{aligned} \right\}; \quad \therefore \begin{vmatrix} V_\xi & V_\eta & -V_x \\ u_\xi & u_\eta & u_x \\ v_\xi & v_\eta & v_x \end{vmatrix} = 0.$$

Similarly, to find  $V_y$ .

**1775** Ex.—Let  $x = r \cos \theta$  and  $y = r \sin \theta$ ; then

$$V_x = V_r \cos \theta - V_\theta \frac{\sin \theta}{r}, \quad V_y = V_r \sin \theta + V_\theta \frac{\cos \theta}{r}.$$

PROOF.—  $u \equiv r \cos \theta - x, \quad v \equiv r \sin \theta - y$ , and the determinant in (1774) takes the form annexed by writing  $r$  and  $\theta$  instead of  $\xi$  and  $\eta$ . A similar determinant gives  $V_y$ .

To find  $V_{xx}$ , substitute  $V_r \cos \theta - V_\theta \frac{\sin \theta}{r}$  in the place of  $V$  in the value of  $V_x$ ; and in differentiating for  $r$  and  $\theta$ , consider  $V_r$  and  $V_\theta$  as functions of both  $r$  and  $\theta$ . Similarly, to find  $V_{xy}$  and  $V_{yy}$ . Thus,

$$\mathbf{1777} \quad V_{xx} = V_{xx} \cos^2 \theta + \left( \frac{1}{r} V_\theta - V_{r\theta} \right) \frac{2 \sin \theta \cos \theta}{r} + V_r \frac{\sin^2 \theta}{r} + V_{\theta\theta} \frac{\sin^2 \theta}{r^2}.$$

$$\mathbf{1778} \quad V_{yy} = V_{xx} \sin^2 \theta - \left( \frac{1}{r} V_\theta - V_{r\theta} \right) \frac{2 \sin \theta \cos \theta}{r} + V_r \frac{\cos^2 \theta}{r} + V_{\theta\theta} \frac{\cos^2 \theta}{r^2}.$$

By addition these equations give

$$\mathbf{1779} \quad V_{xx} + V_{yy} = V_{xx} + \frac{1}{r} V_r + \frac{1}{r^2} V_{\theta\theta}.$$

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**1780** Given  $V \equiv f(x, y, z)$  and  $\xi, \eta, \zeta$  known functions of  $x, y, z$ ;  $V_\xi, V_\eta, V_\zeta$  are expressed in terms of  $V_x, V_y, V_z$  by the formulæ

$$\frac{dV}{d\xi} = \frac{d(V\eta\zeta)}{d(xyz)} \div J, \quad \frac{dV}{d\eta} = \frac{d(\xi V\zeta)}{d(xyz)} \div J, \quad \frac{dV}{d\zeta} = \frac{d(\xi\eta V)}{d(xyz)} \div J.$$

PROOF.—Differentiate  $V$  as a function of  $\xi, \eta, \zeta$  with respect to independent variables  $x, y, z$ . The annexed equations are the result. Solve these by (582) with the notation of (1600).

$$\begin{cases} V_\xi \xi + V_\eta \eta_x + V_\zeta \zeta_x = V_x \\ V_\xi \xi_y + V_\eta \eta_y + V_\zeta \zeta_y = V_y \\ V_\xi \xi_z + V_\eta \eta_z + V_\zeta \zeta_z = V_z \end{cases}$$

**1781** Given  $V \equiv f(x, y, z)$ , where  $x, y, z$  are involved with  $\xi, \eta, \zeta$  in three equations  $u=0, v=0, w=0$ , it is required to change the variables to  $\xi, \eta, \zeta$  in  $V_x, V_y$ , and  $V_z$ .

Applying Rule (1773) to the case of three variables, we have

$$\left. \begin{aligned} V_\xi \xi_x + V_\eta \eta_x + V_\zeta \zeta_x - V_x &= 0 \\ u_\xi \xi_x + u_\eta \eta_x + u_\zeta \zeta_x + u_x &= 0 \\ v_\xi \xi_x + v_\eta \eta_x + v_\zeta \zeta_x + v_x &= 0 \\ w_\xi \xi_x + w_\eta \eta_x + w_\zeta \zeta_x + w_x &= 0 \end{aligned} \right\}, \quad \therefore \begin{vmatrix} V_\xi & V_\eta & V_\zeta & -V_x \\ u_\xi & u_\eta & u_\zeta & u_x \\ v_\xi & v_\eta & v_\zeta & v_x \\ w_\xi & w_\eta & w_\zeta & w_x \end{vmatrix} = 0.$$

The determinant gives  $V_x$  in terms of  $V_\xi, V_\eta, V_\zeta$  and the derivatives of  $u, v, w$ .  $V_y$  and  $V_z$  are found in an analogous manner.

**1782** Similarly with  $n$  equations between  $2n$  variables.

**1783** Ex.—Given

$$x = r \sin \theta \cos \phi; \quad y = r \sin \theta \sin \phi; \quad z = r \cos \theta.$$

The equations  $u, v, w$  become \*

$$r \sin \theta \cos \phi - x = 0; \quad r \sin \theta \sin \phi - y = 0; \quad r \cos \theta - z = 0.$$

Writing  $r, \theta, \phi$  instead of  $\xi, \eta, \zeta$ , the determinant becomes

$$\begin{vmatrix} V_r & V_\theta & V_\phi & -V_z \\ \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi & -1 \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi & 0 \\ \cos \theta & -r \sin \theta & 0 & 0 \end{vmatrix} = 0. *$$

From which  $V_z$  is obtained. Similarly,  $V_y$  and  $V_x$ ; and, by an exactly similar process, the converse forms for  $V_r, V_\theta$ , and  $V_\phi$ . The results are

\* In writing out a determinant like the above, it will be found expeditious in practice to have the columns written on separate slips of paper in order to be able to transpose them readily. Thus, to find the coefficient of  $V_r$ , bring the second column to the left side, and, since this changes the sign of the determinant, transpose any two other columns, so that the coefficient of  $V_r$  may be read off in the standard form as the minor of the first element of the determinant.

$$\begin{aligned}
 1784 \quad V_z &= V_r \sin \theta \cos \phi + V_\theta \frac{\cos \theta \cos \phi}{r} - V_\phi \frac{\sin \phi}{r \sin \theta}. \\
 1785 \quad V_\theta &= V_r \sin \theta \sin \phi + V_\phi \frac{\cos \theta \sin \phi}{r} + V_z \frac{\cos \phi}{r \sin \theta}. \\
 1786 \quad V_\phi &= V_r \cos \theta - V_z \frac{\sin \theta}{r}. \\
 1787 \quad V_r &= V_z \sin \theta \cos \phi + V_\theta \sin \theta \sin \phi + V_\phi \cos \theta. \\
 1788 \quad V_z &= V_r r \cos \theta \cos \phi + V_\theta r \cos \theta \sin \phi - V_\phi r \sin \theta. \\
 1789 \quad V_\phi &= -V_z r \sin \theta \sin \phi + V_\theta r \sin \theta \cos \phi.
 \end{aligned}$$

1790 To find  $V_z$  directly; solve the equations  $u, v, w$ , in (1783), for  $r, \theta$ , and  $\phi$ ; the solution in this case being practicable; thus,

$$r = \sqrt{(x^2 + y^2 + z^2)}, \quad \theta = \tan^{-1} \frac{\sqrt{(x^2 + y^2)}}{z}, \quad \phi = \tan^{-1} \frac{y}{x}.$$

Find  $r_x, \theta_x, \phi_x$  from these, and substitute in  $V_z = V_r r_x + V_\theta \theta_x + V_\phi \phi_x$ . Similarly,  $V_\theta$  and  $V_\phi$ . Also  $V_r = V_z x_r + V_\theta y_r + V_\phi z_r$ . Similarly,  $V_\theta$  and  $V_\phi$ .

1791 To obtain  $V_{xy}$ , substitute the value of  $V_z$  in the place of  $V$ , in the value of  $V_\theta$ , in (1785), and, in differentiating  $V_r, V_\theta, V_\phi$ , consider each of these quantities a function of  $r, \theta$ , and  $\phi$ .

To change the variables to  $r, \theta$ , and  $\phi$ , in  $V_{2x} + V_{2y} + V_{2z}$ , the equations (1783) still subsisting. Result—

$$\begin{aligned}
 1792 \quad & V_{2x} + V_{2y} + V_{2z} \\
 &= V_{2r} + \frac{2}{r} V_r + \frac{1}{r^2} (V_\theta \cot \theta + V_{2\theta} + V_{2\phi} \operatorname{cosec}^2 \theta).
 \end{aligned}$$

PROOF.—Put  $r \sin \theta = \rho$ , so that  $x = \rho \cos \phi$  and  $y = \rho \sin \phi$ , therefore, by (1779),  $V_{2x} + V_{2y} = V_{2\rho} + \frac{1}{\rho} V_\rho + \frac{1}{\rho^2} V_{2\phi} \dots \dots \dots$  (i.).

Also, since  $z = r \cos \theta$  and  $\rho = r \sin \theta$ , we have, by the same formula,

$$V_{2z} + V_{2\rho} = V_{2r} + \frac{1}{r} V_r + \frac{1}{r^2} V_{2\theta} \dots \dots \dots \text{(ii.)}.$$

Add together (i.) and (ii.), and eliminate  $V_\rho$ , by (1776), which gives

$$V_\rho = V_r \sin \theta + V_\theta \frac{\cos \theta}{r}.$$

If  $V$  be a function of  $n$  variables  $x, y, z \dots$  connected by the single relation,  $x^2 + y^2 + z^2 + \dots = r^2 \dots \dots \dots$  (i.).

$$1793 \quad V_{2x} + V_{2y} + V_{2z} + \&c. = V_{2r} + \frac{n-1}{r} V_r.$$



PROOF.—  $V_s = V_r r_s = V_r \frac{x}{r}$ , since  $r_s = \frac{x}{r}$ , by differentiating (i.),

therefore  $V_{2s} = V_{rs} \frac{x}{r} + V_r \frac{r - x r_s}{r^2} = V_{2r} \frac{x^2}{r^2} + V_r \left( \frac{1}{r} - \frac{x^2}{r^3} \right)$ .

Similarly  $V_{2y} = V_{2r} \frac{y^2}{r^2} + V_r \left( \frac{1}{r} - \frac{y^2}{r^3} \right)$ , &c.

Thus, by addition,

$$V_{2s} + V_{2y} + \&c. = V_{2r} \frac{x^2 + y^2 + \dots}{r^2} + V_r \left( \frac{n}{r} - \frac{x^2 + y^2 + \dots}{r^3} \right) = V_{2r} + \frac{n-1}{r} V_r.$$

### LINEAR TRANSFORMATION.

**1794** If  $V \equiv f(x, y, z)$ , and if the equations  $u, v, w$  in (1781) take the forms

$$\left. \begin{aligned} x &= a_1 \xi + b_1 \eta + c_1 \zeta \\ y &= a_2 \xi + b_2 \eta + c_2 \zeta \\ z &= a_3 \xi + b_3 \eta + c_3 \zeta \end{aligned} \right\}, \quad \text{then} \quad \begin{cases} \Delta \xi = A_1 x + A_2 y + A_3 z, \\ \Delta \eta = B_1 x + B_2 y + B_3 z, \\ \Delta \zeta = C_1 x + C_2 y + C_3 z, \end{cases}$$

by (582),  $\Delta$  being the determinant  $(a_1 b_2 c_3)$ , and  $A_1$  the minor of  $a_1$ , &c.

**1795** The operations  $d_x, d_y, d_z$  will now be transformed by the first set of equations below; and  $d_\xi, d_\eta, d_\zeta$  by the second set.

$$\left. \begin{aligned} d_x &= (A_1 d_\xi + B_1 d_\eta + C_1 d_\zeta) \div \Delta \\ d_y &= (A_2 d_\xi + B_2 d_\eta + C_2 d_\zeta) \div \Delta \\ d_z &= (A_3 d_\xi + B_3 d_\eta + C_3 d_\zeta) \div \Delta \end{aligned} \right\}, \quad \left. \begin{aligned} d_\xi &= a_1 d_x + a_2 d_y + a_3 d_z \\ d_\eta &= b_1 d_x + b_2 d_y + b_3 d_z \\ d_\zeta &= c_1 d_x + c_2 d_y + c_3 d_z \end{aligned} \right\}.$$

PROOF.—By  $d_x = d_\xi x_\xi + d_\eta x_\eta + d_\zeta x_\zeta$  and  $d_\xi = d_x x_\xi + d_y y_\xi + d_z z_\xi$ ; and the values of  $x_\xi, x_\eta$ , &c., from the preceding equations.

**1797** From (1795),  $V_\xi = a_1 V_x + a_2 V_y + a_3 V_z$ . Operating again upon  $V_\xi$ , we have

$$V_{2\xi} = (a_1 d_x + a_2 d_y + a_3 d_z) V_\xi = a_1 (V_\xi)_x + a_2 (V_\xi)_y + a_3 (V_\xi)_z,$$

and by substituting the value of  $V_\xi$ , and similarly with  $V_{2\eta}$ ,  $V_{2\zeta}$ , we obtain the formulæ,

**1798**

$$\left. \begin{aligned} V_{2\xi} &= a_1^2 V_{2x} + a_2^2 V_{2y} + a_3^2 V_{2z} + 2a_2 a_3 V_{yz} + 2a_3 a_1 V_{zx} + 2a_1 a_2 V_{xy} \\ V_{2\eta} &= b_1^2 V_{2x} + b_2^2 V_{2y} + b_3^2 V_{2z} + 2b_2 b_3 V_{yz} + 2b_3 b_1 V_{zx} + 2b_1 b_2 V_{xy} \\ V_{2\zeta} &= c_1^2 V_{2x} + c_2^2 V_{2y} + c_3^2 V_{2z} + 2c_2 c_3 V_{yz} + 2c_3 c_1 V_{zx} + 2c_1 c_2 V_{xy} \end{aligned} \right\}.$$

## ORTHOGONAL TRANSFORMATION.

**1799** If the transformation is orthogonal (584), we have

$$x^2 + y^2 + z^2 = \xi^2 + \eta^2 + \zeta^2;$$

and since, by (582, 584),  $\Delta = 1$ ,  $A_1 = a_1$ , &c.; equations (1794) now become

$$\begin{aligned} \mathbf{1800} \quad & \left. \begin{aligned} x &= a_1\xi + b_1\eta + c_1\zeta \\ y &= a_2\xi + b_2\eta + c_2\zeta \\ z &= a_3\xi + b_3\eta + c_3\zeta \end{aligned} \right\}, & \left. \begin{aligned} \xi &= a_1x + a_2y + a_3z \\ \eta &= b_1x + b_2y + b_3z \\ \zeta &= c_1x + c_2y + c_3z \end{aligned} \right\}. \end{aligned}$$

And equations (1795) become

$$\mathbf{1802} \quad \left. \begin{aligned} d_x &= a_1d_\xi + b_1d_\eta + c_1d_\zeta \\ d_y &= a_2d_\xi + b_2d_\eta + c_2d_\zeta \\ d_z &= a_3d_\xi + b_3d_\eta + c_3d_\zeta \end{aligned} \right\}, & \left. \begin{aligned} d_\xi &= a_1d_x + a_2d_y + a_3d_z \\ d_\eta &= b_1d_x + b_2d_y + b_3d_z \\ d_\zeta &= c_1d_x + c_2d_y + c_3d_z \end{aligned} \right\}.$$

The double relations between  $x, y, z$  and  $\xi, \eta, \zeta$ , in the six equations of (1800–1), and the similar relations in (1802–3) between  $d_x, d_y, d_z$  and  $d_\xi, d_\eta, d_\zeta$ , are indicated by a single diagram in each case; thus,

$$\mathbf{1804} \quad \begin{array}{c} \xi \quad \eta \quad \zeta \\ \begin{array}{c} x \\ y \\ z \end{array} \left[ \begin{array}{ccc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array} \right] \end{array} \quad \begin{array}{c} d_\xi \quad d_\eta \quad d_\zeta \\ \begin{array}{c} d_x \\ d_y \\ d_z \end{array} \left[ \begin{array}{ccc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array} \right] \end{array}$$

**1806** Hence, when the transformation is orthogonal, the quantities  $x, y, z$  are *cogredient* with  $d_x, d_y, d_z$ , by the definition (1653).

**1807** Extending the definition in (1629), it follows that any function  $u \equiv \phi(x, y, z)$ , when orthogonally transformed, has, for a covariant, the function  $\phi(d_x, d_y, d_z)u$ . That is, if by the transformation,

$$u = \phi(x, y, z) = \psi(\xi, \eta, \zeta),$$

then also  $\phi(d_x, d_y, d_z)u = \psi(d_\xi, d_\eta, d_\zeta)u$ .

**1808** But if  $u$  be a quantic, then, as shown in (1648),  $\phi(d_x, d_y, d_z)u$  is always a function of the coefficients only of  $u$ , and the covariant is, in this case, an invariant.

**1809** Ex.—Let  $u$  or  $\phi(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ ,  
 $\therefore \phi(d_x, d_y, d_z)u = au_{xx} + bu_{yy} + cu_{zz} + 2fu_{yz} + 2gu_{zx} + 2hu_{xy}$   
 $= 2\{a^2 + b^2 + c^2 + 2f^2 + 2g^2 + 2h^2\}$ , and this is an invariant of  $u$ .

**1810** When  $V \equiv f(x, y, z)$  is orthogonally transformed,

$$V_{2x} + V_{2y} + V_{2z} = V_{2\xi} + V_{2\eta} + V_{2\zeta}.$$

PROOF.—By adding together equations (1798), and by the relations

$$a_1^2 + b_1^2 + c_1^2 = 1, \text{ \&c., and } a_1 a_2 + b_1 b_2 + c_1 c_2 = 0, \text{ \&c.,}$$

established in (584).

**1811** If two functions  $u, v$  be subjected to the same orthogonal transformation, so that

$$u \equiv \phi(x, y, z) = \Phi(\xi, \eta, \zeta) \text{ and } v \equiv \psi(x, y, z) = \Psi(\xi, \eta, \zeta);$$

then

$$\phi(d_x, d_y, d_z) v = \Phi(d_\xi, d_\eta, d_\zeta) v.$$

**1812** Ex.—Let  $u = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \equiv \phi$   
 $= a'\xi^2 + b'\eta^2 + c'\zeta^2 + 2f'\eta\zeta + 2g'\zeta\xi + 2h'\xi\eta \equiv \Phi,$

and let  $v = x^2 + y^2 + z^2 = \xi^2 + \eta^2 + \zeta^2 \equiv \psi$  and  $\Psi$ .

Then  $\phi(d_x, d_y, d_z) v = av_{2x} + bv_{2y} + cv_{2z} + 2fv_{yz} + 2gv_{zx} + 2hv_{xy},$

and  $\Phi(d_\xi, d_\eta, d_\zeta) v = a'v_{2\xi} + b'v_{2\eta} + c'v_{2\zeta} + 2f'v_{\eta\zeta} + 2g'v_{\zeta\xi} + 2h'v_{\xi\eta}.$

But  $v_{2x} = 2$ , and  $v_{yz} = 0$ , &c. Hence the theorem gives  $a + b + c = a' + b' + c'$ ; in other words,  $a + b + c$  is an invariant.

**1813** *Contragredient*.—When the transformation is not orthogonal, (1795) shows that  $d_x$  is not transformed by the same, but by a *reciprocal* substitution, in which  $a_1, b_1, c_1$  are replaced by the corresponding minors  $A_1, B_1, C_1$ . In this case  $d_x, d_y, d_z$  are said to be *contragredient* to  $x, y, z$ .

**1814** *Contravariant*.—If, in (1629), the quantics are subjected to a *reciprocal* transformation instead of *the same*, we obtain the definition of a *contravariant*.

**1815** When  $z$  is a function of two independent variables  $x$  and  $y$ , the following notation is often used:

$$\begin{aligned} \frac{dz}{dx} &\equiv p, & \frac{dz}{dy} &\equiv q, & \frac{dp}{dx} = \frac{d^2z}{dx^2} &\equiv r, \\ \frac{dp}{dy} = \frac{dq}{dx} = \frac{dz}{dx dy} &\equiv s, & \frac{dq}{dy} = \frac{d^2z}{dy^2} &\equiv t. \end{aligned}$$

Let  $\phi(x, y, z) = 0$ . It is required to change the independent variables from  $x, y$  to  $z, y$ . The formulæ of trans-

formation are

$$1816 \quad \frac{dx}{dz} = \frac{1}{p}; \quad \frac{dx}{dy} = -\frac{q}{p}; \quad \frac{d^2x}{dz^2} = -\frac{r}{p^3};$$

$$1819 \quad \frac{d^2x}{dy^2} = \frac{2spq - tp^2 - rq^2}{p^3}; \quad \frac{d^2x}{dydz} = \frac{qr - ps}{p^3}.$$

PROOF.—Formulæ (1761, 1763) give  $x_z$  and  $x_{zz}$ , because, since  $y$  remains constant,  $\phi$  may be considered a function of only two variables,  $x$  and  $z$ .

Formulæ (1708–9) give  $x_y$  and  $x_{yy}$  in terms of partial derivatives of  $\phi$ , since  $z$  is now constant, and  $\phi$  may be taken as a function of the two variables  $x$  and  $y$ .

But  $\phi(x, y, z) = 0$  is equivalent to  $\psi(x, y) - z = 0$ ; and the partial derivatives of  $\phi$  with respect to  $x$  and  $y$  are the same as those of  $\psi$ ; and therefore the same as those of  $z$  when  $x$  and  $y$  are the independent variables. Hence  $z$  may be written for  $\phi$  in the formulæ.

$$\text{Lastly, } \frac{d^2x}{dydz} = \left(-\frac{q}{p}\right)_z = \left(-\frac{q}{p}\right)_x \frac{dx}{dz} = \frac{qp_x - pq_x}{p^3} \frac{1}{p} = \frac{qr - ps}{p^3}.$$

The independent variable is here changed from  $z$  to  $x$ , without reference to the equation  $\phi = 0$ ; and this is allowable, because  $y$  is constant for the time being in either case.

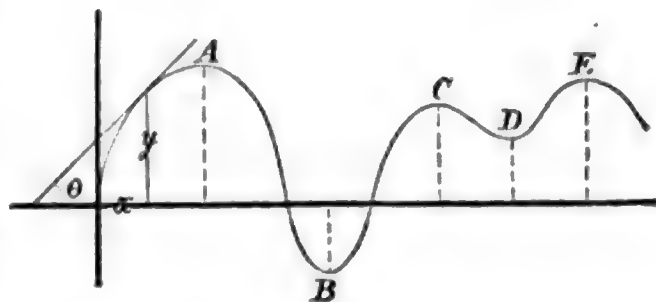
## MAXIMA AND MINIMA.

### *Maxima and Minima values of a function of one independent variable.*

**1830** DEFINITION.—A function  $\phi(x)$  has a *maximum* value when some value  $x=a$  makes  $\phi(x)$  *greater* than it is made by any value of  $x$  indefinitely near to  $a$ . Similarly for a *minimum* value, reading *less* for *greater*.

**1831** ILLUSTRATION.—If the ordinate  $y$  in the figure be always drawn  $= f(x)$ , it has maximum values at  $A, C, E$ , and minimum at  $B$  and  $D$  (1403).

NOTE.—For the algebraic determination of maxima and minima values by a quadratic equation, see (58).



**1832** RULE I.—A function  $\phi(x)$  is a maximum or minimum when  $\phi'(x)$  vanishes, and changes its sign as  $x$  increases from plus to minus or from minus to plus respectively.

**1833** RULE II.—Otherwise  $\phi(x)$  is a maximum or minimum when an odd number of consecutive derivatives of  $\phi(x)$  vanish, and the next is minus or plus respectively.

PROOF.—(i.) The tangent to the curve in the last figure becomes parallel to the  $x$  axis at the points  $A, B, C, D, E$  as  $x$  increases; therefore, by (1493),  $\tan \theta$ , which is equal to  $f'(x)$ , vanishes at those points, while its sign changes in the manner described.

(ii.) Let  $f^n(x)$  be the first derivative of  $f(x)$  which does not vanish when  $x = a$ ,  $n$  being even; therefore, by (1500),  $f(a \pm h) = f(a) + \frac{h^n}{n} f^n(a + \theta h)$ .

The last term retains the sign of  $f^n(a)$ , when  $h$  is small enough, whether  $h$  be positive or negative, since  $n$  is even. Therefore  $f(x)$  diminishes for any small variation of  $x$  from the value  $a$  if  $f^n(a)$  be negative, but increases if  $f^n(a)$  be positive. Hence the rule.

**1834** NOTE.—Before applying the rule for discovering a maximum or minimum, we may evidently—

- (i.) reject any constant factor of the function;
- (ii.) raise it to any constant power, paying attention to sign;
- (iii.) take its reciprocal; maximum becoming minimum, and vice versa;
- (iv.) take the logarithm of a positive function.

**1835** Ex. 1.—Let  $\phi(x) = x^7 - 7x^4 - 35x + 1$ ,

therefore  $\phi'(x) = 7x^6 - 28x^3 - 35 = 7(x^3 - 5)(x^3 + 1)$ .

Also  $\phi''(x) = 7(6x^5 - 12x^2)$ . Therefore  $x = \sqrt[3]{5}$  makes  $\phi'(x)$  vanish, and  $\phi''(x)$  positive; and therefore makes  $\phi(x)$  a minimum.

**1836** Ex. 2.—Let  $\phi(x) = (x-3)^{13}(x-2)^{11}$ . Here

$\phi'(x) = 14(x-3)^{13}(x-2)^{11} + 11(x-3)^{14}(x-2)^{10} = (x-3)^{13}(x-2)^{10}(25x-61)$ , and we know, by (444) or by (1460), that, when  $x = 3$ , the first thirteen derivatives of  $\phi(x)$  vanish; and 13 is an odd number. Therefore  $\phi(x)$  is either a maximum or minimum when  $x = 3$ .

To determine which, examine the change of sign in  $\phi'(x)$ . Now  $(x-3)^{13}$  changes from negative to positive as  $x$  increases from a value a little less than 3 to a value a little greater, while the other factors of  $\phi'(x)$  remain positive. Therefore, by the rule,  $\phi(x)$  is a minimum when  $x = 3$ .

Again, as  $x$  passes through the value 2,  $\phi'(x)$  does not change sign, 10 being even. Therefore  $x = 2$  gives no maximum or minimum value of  $\phi(x)$ .

Lastly, as  $x$  passes through the value  $\frac{61}{25}$  the signs of the three factors in  $\phi'(x)$  change from  $(-)(+)(-)$  to  $(-)(+)(+)$ ; that is,  $\phi'(x)$  changes from  $+$  to  $-$ ; and, consequently,  $\phi(x)$  is a maximum.

**1837** Ex. 3.—Let  $\phi(x, y) = x^4 + 2x^3y - y^3 = 0$ . To find limiting values of  $y$ .

Here  $y$  is given only as an implicit function of  $x$ . Differentiating, in order to employ formulæ (1708, 1711),

$$\phi_x = 4x^3 + 4xy, \quad \phi_{2x} = 12x^2 + 4y, \quad \phi_y = 2x^3 - 3y^2;$$

$y_x = 0$  makes  $\phi_x = 0$ . Solving this equation with  $\phi(x, y) = 0$ , we get  $x = \pm 1$ ,  $y = -1$  when  $y_x$  vanishes.

And then  $y_{2x} = -\frac{\phi_{2x}}{\phi_y} = -\frac{12-4}{2-3} = 8$ , positive; therefore, when  $x = \pm 1$ ,  $y$  has  $-1$  for a minimum value.

Similarly, by making  $y$  the independent variable, it may be shewn that, when  $y = -\frac{8}{9}$ ,  $x$  has both the maximum and minimum values  $\pm \frac{4}{9}\sqrt{6}$ .

**1838** A limiting value of  $\phi(x, y)$ ,  
subject to the condition  $\psi(x, y) = 0$ ..... (i.),  
is obtained from the equation  $\phi_x \psi_y = \phi_y \psi_x$  ..... (ii.)

Simultaneous values of  $x$  and  $y$ , found by solving equations (i.) and (ii.), correspond to a maximum or minimum value of  $\phi$ .

PROOF.—By (1718),  $\phi$  being virtually a function of  $x$  only; and, by (1832),  $\phi_x(xy) = 0$ .

**1839** Ex.—Let  $\phi(x, y) = xy$  and  $\psi(x, y) = 2x^3 - xy + y^3 = 0$ .....(i.)  
Equation (ii.) becomes  $y(3y^2 - x) = x(6x^2 - y)$ .  
Solving this with (i.), we find  $y^3 = 2x^3$  and  $x^2(4x - \sqrt[3]{2})$ .

Therefore  $x = \frac{1}{4}\sqrt[3]{2}$ ,  $y = \frac{1}{4}\sqrt[3]{4}$  are values corresponding to a *maximum* value of  $\phi$ . That it is a maximum, and not a minimum, is seen by inspecting equation (i.)

**1840** Most geometrical problems can be treated in this way, and the alternative of maximum or minimum decided by the nature of the case. Otherwise the sign of  $\phi_{2x}(xy)$  may be examined by formula (1719) for the criterion, according to the rule.

### *Maxima and Minima values of a function of two independent variables.*

**1841** RULE I.—A function  $\phi(x, y)$  is a maximum or minimum when  $\phi_x$  and  $\phi_y$  both vanish and change their signs from plus to minus or from minus to plus respectively, as  $x$  and  $y$  increase.

**1842** RULE II.—Otherwise,  $\phi_x$  and  $\phi_y$  must vanish;  $\phi_{2x}\phi_{2y} - \phi_{xy}^2$  must be positive, and  $\phi_{2x}$  or  $\phi_{2y}$  must be negative for a maximum and positive for a minimum value of  $\phi$ .

PROOF.—By (1512), writing  $A, B, C$  for  $\phi_{2x}, \phi_{xy}, \phi_{2y}$ , we have, for small changes  $h, k$  in the values of  $x$  and  $y$ ,

$$\phi(x+h, y+k) - \phi(x, y) = h\phi_x + k\phi_y + \frac{1}{2}(Ah^2 + 2Bhk + Ck^2) + \text{terms which may be neglected, by (1410).}$$



Hence, as in the proof of (1833), in order that changing the sign of  $h$  or  $k$  shall not have the effect of changing the sign of the right side of the equation, the first powers must disappear, therefore  $\phi_x$  and  $\phi_y$  must vanish. The next term may be written, by completing the square, in the form  $\frac{k^2}{2A} \left\{ \left( A \frac{h}{k} + B \right)^2 + AC - B^2 \right\}$ ; and, to ensure this quantity retaining its sign for all values of the ratio  $h : k$ ,  $AC - B^2$  must be positive.  $\phi$  will then be a maximum or minimum according as  $A$  in the denominator is negative or positive.

It is clear that  $A$  and  $B$  might have been transposed in the proof. Hence  $B$  must have the same sign as  $A$ .

**1843** A limiting value of  $\phi(x, y, z)$ ,  
subject to the condition  $\psi(x, y, z) = 0 \dots\dots\dots (i.)$ ,  
is obtained from the two equations

**1844**  $\phi_x \psi_y = \phi_y \psi_x \dots\dots (ii.)$ ,  $\phi_z \psi_y = \phi_y \psi_z \dots\dots (iii.)$ ;

**1846** or, as they may be written,  $\frac{\phi_x}{\psi_x} = \frac{\phi_y}{\psi_y} = \frac{\phi_z}{\psi_z} \dots\dots (iv.)$

Simultaneous values of  $x, y, z$ , found by solving equations (i., ii., iii.), correspond to a maximum or minimum value of  $\phi$ .

PROOF.—By (1841),  $\phi$  being considered a function of two independent variables  $x$  and  $z$ , and, by (1729, 1730),

$\phi_x(x, y, z) = 0$  gives (ii.), and  $\phi_z(x, y, z) = 0$  gives (iii.)

The criterion of maximum or minimum in (1842) may also be applied without eliminating  $y$  by employing the values of  $\phi_{xx}$  and  $\phi_{zz}$  in (1719, '30).

**1847** Ex.—Let  $\phi(x, y, z) = x^2 + y^2 + z^2$   
and  $\psi(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy - 1 = 0 \dots (i.)$   
Equations (ii.) and (iii.) here become

$$\frac{x}{ax + hy + gz} = \frac{y}{hx + by + fz} = \frac{z}{gx + fy + cz} = \frac{1}{R}, \text{ say, (iv.)}$$

Therefore, by proportion (70) and by (i.),  $x^2 + y^2 + z^2 = R^{-1} = \phi$ .

From equations (iv.) we have

$$\mathbf{1848} \quad \left. \begin{array}{l} ax + hy + gz = Rx \\ hx + by + fz = Ry \\ gx + fy + cz = Rz \end{array} \right\}; \quad \therefore \quad \begin{vmatrix} a-R & h & g \\ h & b-R & f \\ g & f & c-R \end{vmatrix} = 0,$$

**1849** or  $(R-a)(R-b)(R-c) + 2fgh$   
 $-(R-a)f^2 - (R-b)g^2 - (R-c)h^2 = 0$ , or (see 1641)

$$R^3 - R^2(a+b+c) + R(bc+ca+ab-f^2-g^2-h^2) - \Delta = 0,$$

This cubic in  $R$  is the eliminant of the three equations in  $x, y, z$ . It is called the *discriminating cubic* of the quadric (i.), and its roots are the reciprocals of the maxima and minima values of  $x^2 + y^2 + z^2$ .

**1850** To show that the roots of the discriminating cubic are all real.

Let  $R_1, R_2$  be the roots of the quadratic equation

**1851** 
$$\begin{vmatrix} R-b & f \\ f & R-c \end{vmatrix} \equiv (R-b)(R-c) - f^2 = 0 \dots\dots\dots(v.)$$

$R_1 > b$  and  $c$ , and  $b$  and  $c > R_2$ .

Make  $R = R_1$  in the cubic, and the result is negative, being minus a square quantity, by (v.). Make  $R = R_2$ , and the result is positive. Therefore the cubic has real roots between each pair of the consecutive values  $+\infty, R_1, R_2, -\infty$ ; that is, three real roots. But since the roots are in order of magnitude, the first must be a maximum value of  $R$ , the third a minimum, and the intermediate root neither a maximum nor a minimum.

*Maxima and Minima values of a function of three or more variables.*

**1852** Let  $\phi(xyz)$  be a function of three variables. Let  $\phi_{1x}, \phi_{2y}, \phi_{3z}, \phi_{yz}, \phi_{zx}, \phi_{xy}$  be denoted by  $a, b, c, f, g, h$ ; and let  $A, B, C, F, G, H$  be the corresponding minors of the determinant  $\Delta$ , as in (1642).

**1853** RULE I.— $\phi(x, y, z)$  is a maximum or minimum when  $\phi_x, \phi_y, \phi_z$  all vanish and change their signs from plus to minus or from minus to plus respectively, as  $x, y$ , and  $z$  increase. Otherwise—

**1854** RULE II.—The first derivatives of  $\phi$  must vanish;  $A$  and its coefficient in the reciprocal determinant of  $\Delta$  must be positive; and  $\phi$  will be a maximum or minimum according as  $a$  is negative or positive. Or, in the place of  $A$  and  $a$ , read  $B$  and  $b$  or  $C$  and  $c$ .

PROOF.—Pursuing the method of (1842), let  $\xi, \eta, \zeta$  be small changes in the values of  $x, y, z$ . By (1514),

$$\begin{aligned} \phi(x+\xi, y+\eta, z+\zeta) - \phi(x, y, z) \\ = \xi\phi_x + \eta\phi_y + \zeta\phi_z + \frac{1}{2}(a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta) + \&c. \end{aligned}$$

For constancy of sign on the right,  $\phi_x, \phi_y, \phi_z$  must vanish. The quadric may then be re-arranged as under by first completing the square of the terms in  $\xi$ , and then collecting the terms in  $\zeta, \eta$ , and completing the square. It

thus becomes 
$$\frac{1}{2a} \left\{ (a\xi + h\eta + g\zeta)^2 + \frac{(C\eta - F\zeta)^2}{C} + \frac{BC - F^2}{C} \zeta^2 \right\}.$$

Hence, for constancy of sign for all values of  $\xi, \eta, \zeta$ , it is necessary that  $O$  and  $BC - F^2$  should be positive. This makes  $B$  also positive. By symmetry, it is evident that  $A, B, C, BC - F^2, CA - G^2, AB - H^2$  will all be positive. The sign of  $a$  in the first factor then determines, as in (1842), whether  $\phi$  is a maximum or a minimum.

**1855** The condition may be put otherwise. Since  $BC - F^2 = a\Delta$  by (577), the condition that  $BC^2 - F^2$  must be positive is equivalent to the condition that  $a$  and  $\Delta$  must have the same sign. Hence we have also the following rule:—

**1856** RULE III.— $\phi_x, \phi_y, \phi_z$  must vanish; the second of the four determinants below must be positive, and the first and third must have the same sign: that sign being negative for a maximum and positive for a minimum value of  $\phi(x, y, z)$ .

$$\mathbf{1857} \quad \phi_{2x}, \quad \begin{vmatrix} \phi_{2x} & \phi_{xy} \\ \phi_{yx} & \phi_{2y} \end{vmatrix}, \quad \begin{vmatrix} \phi_{2x} & \phi_{xy} & \phi_{xz} \\ \phi_{yx} & \phi_{2y} & \phi_{yz} \\ \phi_{zx} & \phi_{zy} & \phi_{2z} \end{vmatrix}, \quad \begin{vmatrix} \phi_{2x} & \phi_{xy} & \phi_{xz} & \phi_{xw} \\ \phi_{yx} & \phi_{2y} & \phi_{yz} & \phi_{yw} \\ \phi_{zx} & \phi_{zy} & \phi_{2z} & \phi_{zw} \\ \phi_{wx} & \phi_{wy} & \phi_{wz} & \phi_{2w} \end{vmatrix}.$$

**1858** The theorem can be extended in a similar manner to  $\phi(x, y, z, w, \dots)$ , a function of any number of variables. Form the successive Hessians of  $\phi$  (1630) for one, two, three, &c. variables in order as shown above; then—

**1859** RULE.—In order that  $\phi(x, y, z, w, \dots)$  may be a maximum or minimum,  $\phi_x, \phi_y, \phi_z, \phi_w$ , &c. must vanish; the Hessians of an even order must be positive; and those of an odd order must have the same sign, that sign being negative for a maximum and positive for a minimum value of the function  $\phi$ .

For a demonstration in full, see Williamson's *Diff. Calc.*, 4th Edit., p. 433.

**1860** Ex.—Required a limiting value of the function

$$u = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2px + 2qy + 2rz + d.$$

The condition in the rule produces equations (1), (2), (3). Equation (4) results from Euler's theorem (1624), thus; introducing a fourth variable  $w$ , as in (1645), we have  $xu_x + yu_y + zu_z + wu_w = 2u$ , which reduces to (4) by means of (1), (2), (3), and the value of  $u$ , putting  $w = 1$ .

$$\mathbf{1861} \quad \left. \begin{aligned} \frac{1}{2}u_x &= ax + hy + gz + p = 0 \dots\dots(1) \\ \frac{1}{2}u_y &= hx + by + fz + q = 0 \dots\dots(2) \\ \frac{1}{2}u_z &= gx + fy + cz + r = 0 \dots\dots(3) \\ px + qy + rz + d &= u \dots\dots(4) \end{aligned} \right\}; \quad \therefore \begin{vmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & d-u \end{vmatrix} = 0.$$

The determinant is the eliminant of the four equations, by (583), and is equivalent, by the method of (1724, Proof. i.), to  $\Delta' - \Delta u = 0$ , or  $u = \Delta' \div \Delta$  (Notation of 1646).

To determine whether this value of  $u$  is a maximum or minimum, either of the conditions in (1854, '6) may be applied; and since, in this example,  $u_x = 2a$ ,  $u_y = 2b$ , &c., the letters  $a, b, c, f, g, h$  may be considered identical with those in the rule.

**1862** To determine a limiting value of  $\phi(x, y, z, \dots)$ , a function of  $m$  variables connected by  $n$  equations  $u_1 = 0, u_2 = 0, \dots u_n = 0$ .

**RULE.** — Assume  $n$  undetermined multipliers  $\lambda_1, \lambda_2, \dots \lambda_n$  with the following  $m$  equations:—

$$\begin{aligned} \phi_x + \lambda_1 (u_1)_x + \lambda_2 (u_2)_x + \dots + \lambda_n (u_n)_x &= 0, \\ \dots \quad \phi_y + \lambda_1 (u_1)_y + \lambda_2 (u_2)_y + \dots + \lambda_n (u_n)_y &= 0, \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \end{aligned}$$

making in all  $m+n$  equations in  $m+n$  quantities,  $x, y, z, \dots$  and  $\lambda_1, \lambda_2, \dots \lambda_n$ . The values of  $x, y, z, \dots$ , found from these equations, correspond to a maximum or minimum value of  $\phi$ .

**PROOF.**—Differentiate  $\phi$  and  $u_1, u_2, \dots u_n$  on the hypothesis that  $x, y, z, \dots$  are arbitrary functions of an independent variable  $t$ . Multiply the resulting equations, excepting the first, by  $\lambda_1, \lambda_2, \dots \lambda_n$  in order, and add them to the value of  $\phi_t$ . The coefficients of  $x_t, y_t, z_t, \dots$  may now be equated to zero, since the functions of  $t$  are arbitrary, producing the equations in the rule.

**1863** Ex. 1.—To find the limiting values of  $r^2 = x^2 + y^2 + z^2$  with the conditions  $Ax^2 + By^2 + Cz^2 = 1$  and  $lx + my + nz = 0$ .

Here  $m = 3, n = 2$ ; and, choosing  $\lambda$  and  $\mu$  for the multipliers, the equations in the rule become

$$\left. \begin{aligned} 2x + 2A\lambda x + \mu l &= 0 \dots (1) \\ 2y + 2B\lambda y + \mu m &= 0 \dots (2) \\ 2z + 2C\lambda z + \mu n &= 0 \dots (3) \end{aligned} \right\} \begin{aligned} &\text{Multiply (1), (2), (3) respectively by} \\ &x, y, z, \text{ and add; thus } \mu \text{ disappears,} \\ &\text{and we obtain} \end{aligned}$$

$$x^2 + y^2 + z^2 + (Ax^2 + By^2 + Cz^2) \lambda = 0, \quad \text{therefore } \lambda = -r^2.$$

Substitute this in (1), (2), (3); solve for  $x, y, z$ , and substitute their values in  $lx + my + nz = 0$ .

**1864** The result is  $\frac{l^2}{Ar^2-1} + \frac{m^2}{Br^2-1} + \frac{n^2}{Cr^2-1} = 0$ , a quadratic in  $r^2$ .

The roots are the maximum and minimum values of the square of the radius vector of a central section of the quadric  $Ax^2 + By^2 + Cz^2 = 1$  made by the plane  $lx + my + nz = 0$ .

**1865** Ex. 2.—To find the maximum value of  $u = (x+1)(y+1)(z+1)$ , subject to the condition  $N = a^x b^y c^z$ .

This is equivalent to finding a maximum value of  
 $\log(x+1) + \log(y+1) + \log(z+1),$   
 subject to the condition  $\log N = x \log a + y \log b + z \log c.$

The equations in the rule become

$$\frac{1}{x+1} + \lambda \log a = 0; \quad \frac{1}{y+1} + \lambda \log b = 0; \quad \frac{1}{z+1} + \lambda \log c = 0.$$

By eliminating  $\lambda$ , these are seen to be equivalent to equations (1846).

Multiplying up and adding the equations, we find  $\lambda$ , and thence  $x+1$ ,  $y+1$ ,  $z+1$ ; the values of which, substituted in  $u$ , give, for its maximum value,

$$u = \{\log(Nabc)\}^3 \div 3 \log a^3 \log b^3 \log c^3.$$

Compare (374), where  $a, b, c$  and  $x, y, z$  are integers.

### Continuous Maxima and Minima.

**1866** If  $\phi_x$  and  $\phi_y$ , in (1842), have a common factor, so that

$$\phi_x = P\psi(x, y), \quad \phi_y = Q\psi(x, y) \dots\dots\dots (i.),$$

where  $P$  and  $Q$  may also be functions of  $x$  and  $y$ ; then the equation  $\psi(x, y) = 0$  determines a continuous series of values of  $x$  and  $y$ . For all these values  $\phi$  is constant, but, at the same time, it may be a maximum or a minimum *with respect to any other contiguous values of  $\phi$* , obtained by taking  $x$  and  $y$  so that  $\psi(xy)$  shall not vanish.

**1867** In this case,  $\phi_{2x}\phi_{2y} - \phi_{xy}^2$  vanishes with  $\psi$ , so that the criterion in Rule II. is not applicable.

PROOF.—Differentiating equation (i.), we have

$$\left. \begin{aligned} \phi_{2x} &= P_x\psi + P\psi_x \\ \phi_{2y} &= Q_y\psi + Q\psi_y \end{aligned} \right\}, \quad \left. \begin{aligned} \phi_{xy} &= P_y\psi + P\psi_y \\ \phi_{yx} &= Q_x\psi + Q\psi_x \end{aligned} \right\}.$$

If from these values we form  $\phi_{2x}\phi_{2y} - \phi_{xy}^2$ ,  $\psi$  will appear as a factor of the expression.

**1868** Ex.—Take  $z$  as  $\phi(xy)$  in the equation

$$z^2 = a^2 - b^2 + 2b\sqrt{(x^2 + y^2)} - x^2 - y^2 \dots\dots\dots (i.),$$

$$\therefore \quad zz_x = x \left( \frac{b}{\sqrt{(x^2 + y^2)}} - 1 \right) \quad \text{and} \quad zz_y = y \left( \frac{b}{\sqrt{(x^2 + y^2)}} - 1 \right).$$

The common factor equated to zero gives  $x^2 + y^2 = b^2$ , and therefore  $z = \pm a \dots (ii.)$  Here  $a$  is a continuous maximum value of  $z$ , and  $-a$  a continuous minimum.

Equation (i.) represents, in Coordinate Geometry, the surface of an anchor ring, the generating circle of radius  $a$  having its centre at a distance  $b$  from the axis of revolution  $Z$ . Equations (ii.) give the loci of the highest and lowest points of the surface.

For the application of the Differential Calculus to the Theory of Curves, see the Sections on *Coordinate Geometry*.



# INTEGRAL CALCULUS.

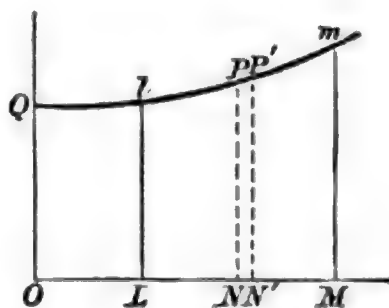
## INTRODUCTION.

**1900** The operations of *differentiation* and *integration* are the converse of each other. Let  $f(x)$  be the *derivative* of  $\phi(x)$ ; then  $\phi(x)$  is called the *integral* of  $f(x)$  with respect to  $x$ . These converse relations are expressed in the notations of the Differential and Integral Calculus, by

$$\frac{d\phi(x)}{dx} = f(x) \quad \text{and by} \quad \int f(x) dx = \phi(x).$$

**1901** THEOREM.—Let the curve  $y = f(x)$  be drawn as in (1403), and any ordinates  $Ll$ ,  $Mm$ , and let  $OL = a$ ,  $OM = b$ ; then the area  $LMml = \phi(b) - \phi(a)$ .

PROOF.—Let  $ON$  be any value of  $x$ , and  $PN$  the corresponding value of  $y$ , and let the area  $ONPQ = A$ ; then  $A$  is some function of  $x$ . Also, if  $NN' = dx$ , the elemental area  $NN'P'P = dA = ydx$  in the limit; therefore  $\frac{dA}{dx} = y$ . Thus  $A$  is that function of  $x$  whose derivative for each value of  $x$  is  $y$  or  $f(x)$ ; therefore  $A = \phi(x) + C$ , where  $C$  is any constant. Consequently the area  $LMml = \phi(b) - \phi(a)$ , whatever  $C$  may be.



The demonstration assumes that there is only one function  $\phi(x)$  corresponding to a given derivative  $f(x)$ . This may be formally proved.

If possible, let  $\psi(x)$  have the same derivative as  $\phi(x)$ ; then, with the same coordinate axes, two curves may be drawn so that the areas defined as above, like  $LMml$ , shall be  $\phi(x)$  and  $\psi(x)$  respectively, each area vanishing with  $x$ . If these curves do not coincide, then, for a given value of  $x$ , they have different ordinates, that is,  $\phi'(x)$  and  $\psi'(x)$  are different, contrary to the hypothesis. The curves must therefore coincide, that is,  $\phi(x)$  and  $\psi(x)$  are identical.



**1902** Since  $\phi(b) - \phi(a)$  is the sum of all the elemental areas like  $NN'P'P$  included between  $Ll$  and  $Mm$ , that is, the sum of the elements  $ydx$  or  $f(x)dx$  taken for all values of  $x$  between  $a$  and  $b$ , this result is written

$$\int_a^b f(x) dx = \phi(b) - \phi(a).$$

**1903** The expression on the left is termed a *definite integral* because the limits  $a, b$  of the integration are assigned.\* When the limits are not assigned, the integral is called *indefinite*.

**1904** By taking the constant  $C=0$  in (1901), we have the area

$$ONPQ = \phi(x) = \int f(x) dx.$$

NOTE.—In practice, the constant should always be added to the result of an integration when no limits are assigned.

#### MULTIPLE INTEGRALS.

**1905** Let  $f(x, y, z)$  be a function of three variables; then the notation

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x, y, z) dx dy dz$$

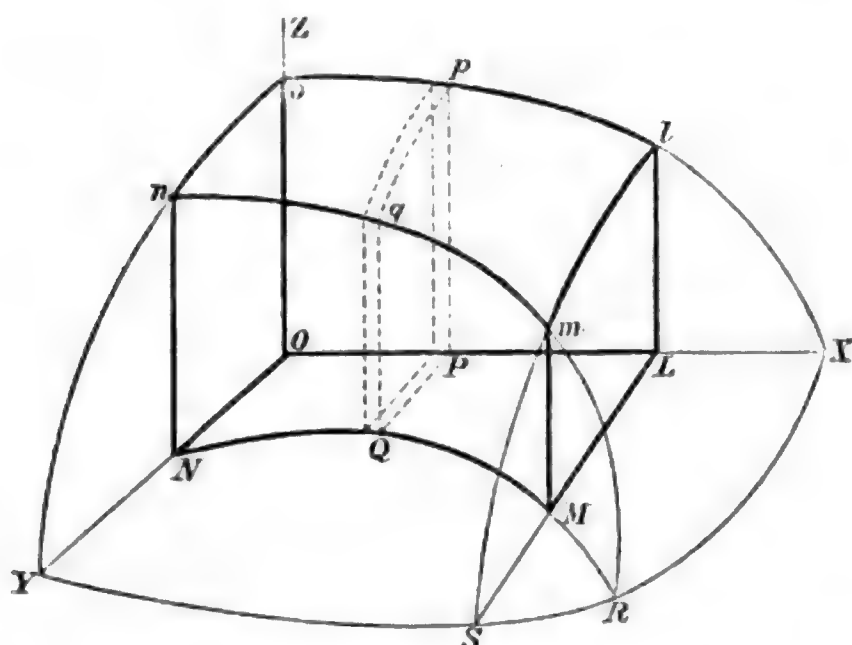
is used to denote the following operations.

Integrate the function for  $z$  between the limits  $z=z_1, z=z_2$ , considering the remaining variables  $x$  and  $y$  constant. Then, whether the limits  $z_1, z_2$  are constants or functions of  $x$  and  $y$ , the result will be a function of  $x$  and  $y$  only. Next, considering  $x$  constant, integrate this function for  $y$  between the limits  $y_1$  and  $y_2$ , which may either be constants or functions of  $x$ . The result will now be a function of  $x$  only. Lastly, integrate this function for  $x$  between the limits  $x_1$  and  $x_2$ .

Similarly for a function of any number of variables.

**1906** The clearest view of the nature of a multiple integral is afforded by the geometrical interpretation of a triple integral.

\* The integral may be read "Sum  $a$  to  $b$ ,  $\int (x) dx$ ";  $\int$  signifying "sum."



Taking rectangular coordinate axes, let the surface  $z = \phi(x, y)$  ( $XYom$  in the figure) be drawn, intersected by the cylindrical surface  $y = \psi(x)$  ( $RMNnm$ ), and by the plane  $x = x_1$  ( $LSml$ ). The volume of the solid  $OLMNolmn$  bounded by these surfaces and the coordinate planes will be

$$\int_0^{x_1} \int_0^{\psi(x)} \phi(x, y) dx dy = \int_0^{x_1} \int_0^{\psi(x)} \int_0^{\phi(x, y)} dx dy dz.$$

PROOF.—Since the volume cut off by any plane parallel to  $OYZ$ , and at a distance  $x$  from it, varies continuously with  $x$ , it must be *some function of  $x$* . Let  $V$  be this volume, and let  $dV$  be the small change in its value due to a change  $dx$  in  $x$ . Then, in the limit,  $dV = PQqp \times dx$ , an element of the solid shown by dotted lines in the figure. Therefore

$$\frac{dV}{dx} = PQqp = \int_0^{\psi(x)} \phi(x, y) dy, \text{ by (1902),}$$

$z$  being constant throughout the integration for  $y$ . The result will be a function of  $x$  only. Making  $x$  then vary from 0 to  $x$ , we have, for the whole

$$\text{volume, } \int_0^{x_1} \left\{ \int_0^{\psi(x)} \phi(x, y) dy \right\} dx = \int_0^{x_1} \left\{ \int_0^{\psi(x)} \left[ \int_0^{\phi(x, y)} dz \right] dy \right\} dx,$$

since  $\phi(x, y) = z$ . With the notation explained in (1905), the brackets are not required, and the integrals are written as above.

**1907** If the solid is bounded by two surfaces  $z_1 = \phi_1(x, y)$ ,  $z_2 = \phi_2(x, y)$ , two cylindrical surfaces  $y_1 = \psi_1(x)$ ,  $y_2 = \psi_2(x)$ , and two planes  $x = x_1$ ,  $x = x_2$ , the volume will then be arrived at by taking the difference of two similar integrals at each integration, and will be expressed by the integral in (1905).

If any limit is a constant, the corresponding boundary of the solid becomes a plane.

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## METHODS OF INTEGRATION.

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### INTEGRATION BY SUBSTITUTION.

**1908** The formula is  $\int \phi(x) dx = \int \phi(x) \frac{dx}{dz} dz$ ,

where  $z$  is equal to  $f(x)$ , some function chosen so as to facilitate the integration.

**RULE I.**—Put  $x$  in terms of  $z$  in the given function, and multiply the function also by  $x_z$ ; then integrate for  $z$ .

If the limits of the proposed integral are given by  $x = a$ ,  $x = b$ , these must be converted into limits of  $z$  by the equation  $x = f^{-1}(z)$ .

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The following rule presents another view of the method of substitution, and is useful in practice.

**1909** **RULE II.**—If  $\phi(x)$  can be expressed in the form  $F(z)z_x$ ;

then  $\int \phi(x) dx = \int F(z) z_x dx = \int F(z) dz$ .

---

**Ex. 1.**—To integrate  $\int \frac{dx}{\sqrt{(x^2+a^2)}}$ . Substitute  $z = x + \sqrt{(x^2+a^2)}$ ;

therefore  $\frac{dz}{dx} = 1 + \frac{x}{\sqrt{(x^2+a^2)}} = \frac{x + \sqrt{(x^2+a^2)}}{\sqrt{(x^2+a^2)}} = \frac{z}{\sqrt{(x^2+a^2)}}$ ,  
 $\int \frac{1}{\sqrt{(x^2+a^2)}} \frac{dx}{dz} dz = \int \frac{dz}{z} = \log z = \log \{x + \sqrt{(x^2+a^2)}\}.$

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**Ex. 2.**  $\int \frac{5+2x^3}{x+x^4} dx = \int \frac{5x^{-3}+2x^{-2}}{x^{-3}+x^{-1}} dx = -\log(x^{-3}+x^{-1}).$

Here  $z = x^{-3}+x^{-1}$ ,  $F(z) = -\frac{1}{z}$ ,  $z_x = -(5x^{-3}+2x^{-2}).$

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**Ex. 3.**  $\int \frac{1-cx^3}{1+cx^3} \frac{dx}{\sqrt{(1+ax^3+c^2x^6)}} = \int \frac{x^{-2}-c}{x^{-1}+cx} \frac{dx}{\sqrt{(c^2x^2+x^{-2}+a)}}$   
 $= -\int \frac{d_x(x^{-1}+cx) dx}{(x^{-1}+cx) \sqrt{\{(x^{-1}+cx)^2+a-2c\}}}$   
 $= \frac{1}{\sqrt{(a-2c)}} \log \frac{x\sqrt{(a-2c)} + \sqrt{(1+ax^3+c^2x^6)}}{1+cx^3},$   
 or  $-\frac{1}{\sqrt{(2c-a)}} \cos^{-1} \frac{x\sqrt{(2c-a)}}{1+cx^3}.$

By (1927) or (1926). Here  $z = x^{-1}+cx.$

In Examples (2) and (3) the process is analytical, and leads to the discovery of the particular function  $z$ , with respect to which the integration is effected. If  $z$  be known, Rule I. supplies the direct, though not always the simplest, method of integrating the function.

#### INTEGRATION BY PARTS.

**1910** By differentiating  $uv$  with respect to  $x$ , we obtain the general formula  $\int u_x v dx = uv - \int uv_x dx$ .

The value of the first integral is thus determined if that of the second is known.

**RULE.** — *Separate the quantity to be integrated into two factors. Integrate one factor, and differentiate the other with respect to  $x$ . If the integral of the resulting quantity is known, or more readily ascertained than that of the original one, the method by "Parts" is applicable.*

**1911** **NOTE.** — In subsequent examples, where integration by Parts is directed, the factor which is to be integrated will be indicated. Thus, in example (1951), "By Parts  $\int e^x dx$ " signifies that  $e^x$  is to be integrated and  $\sin bx$  differentiated afterwards in applying the foregoing rule. The factor 1 is more frequently integrated than any other, and this step will be denoted by  $\int dx$ .

#### INTEGRATION BY DIVISION.

**1912** A formula is

$$\int (a + bx^n)^p dx = a \int (a + bx^n)^{p-1} + b \int x^n (a + bx^n)^{p-1} dx :$$

The expression to be integrated is thus *divided* into two terms, the index  $p$  in each being diminished by unity, a step which often facilitates integration.

$$\begin{aligned} \text{Similarly,} \quad & \int (a + bx^n + cx^m)^p dx \\ &= a \int (a + bx^n + cx^m)^{p-1} + bx^n \int (a + bx^n + cx^m)^{p-1} + cx^m \int (a + bx^n + cx^m)^{p-1}. \end{aligned}$$

**1913** **Ex.** — To integrate  $\int \sqrt{x^2 + a^2} dx$ .

$$\text{By Parts } \int dx, \quad \int \sqrt{x^2 + a^2} dx = x \sqrt{x^2 + a^2} - \int \frac{x^2 dx}{\sqrt{x^2 + a^2}}.$$

$$\text{By Division,} \quad \int \sqrt{x^2 + a^2} dx = \int \frac{a^2 dx}{\sqrt{x^2 + a^2}} + \int \frac{x^2 dx}{\sqrt{x^2 + a^2}}.$$

Therefore, by addition,

$$\begin{aligned} \int \sqrt{x^2 + a^2} dx &= \frac{1}{2} x \sqrt{x^2 + a^2} + \frac{1}{2} \int \frac{a^2 dx}{\sqrt{x^2 + a^2}} \\ &= \frac{1}{2} x \sqrt{x^2 + a^2} + \frac{1}{2} a^2 \log \{x + \sqrt{x^2 + a^2}\}, \text{ by (1909, Ex. 1).} \end{aligned}$$

## INTEGRATION BY RATIONALIZATION.

**1914** In the following example, 6 is the least common denominator of the fractional indices. Hence, by substituting  $z = x^{\frac{1}{6}}$ , and therefore  $x = 6z^6$ , we have

$$\begin{aligned} \int \frac{x^{\frac{1}{6}} - 1}{x^{\frac{1}{6}} - 1} dx &= \int \frac{z^6 - 1}{z^2 - 1} \frac{dx}{dz} dz = 6 \int \frac{z^6 - z^5}{z^2 - 1} dz \\ &= 6 \int \left( z^4 + z^3 - z^3 + z^2 - z + 1 - \frac{1}{z+1} \right) dz. \end{aligned}$$

Each term of the result is directly integrable by (1922) and (1923). For other examples see (2110).

## INTEGRATION BY PARTIAL FRACTIONS.

**1915** Rational fractions can always be integrated by first resolving them into partial fractions. The theory of such resolutions will now be given.

**1916** If  $\phi(x)$  and  $F(x)$  are rational algebraic functions of  $x$ ,  $\phi(x)$  being of lowest dimensions, and if  $F(x)$  contains the factor  $(x-a)$  once, so that

$$F(x) = (x-a)\psi(x) \dots \dots \dots (1);$$

**1917** then  $\frac{\phi(x)}{F(x)} = \frac{A}{x-a} + \frac{\chi(x)}{\psi(x)}$  and  $A = \frac{\phi(a)}{F'(a)} \dots \dots (2).$

PROOF.—Multiply equation (2) by (1), thus

$$\phi(x) = A\psi(x) + (x-a)\chi(x).$$

Therefore, putting  $x=a$ ,  $\phi(a) = A\psi(a)$ . Also, by differentiating (1), and putting  $x=a$  afterwards,  $F'(a) = \psi(a)$ . Therefore  $A = \phi(a) \div F'(a)$ .

**1918** Again, if  $F(x)$  contains the factor  $(x-a)$ ,  $n$  times, so that

$$F(x) = (x-a)^n \psi(x).$$

Assume  $\frac{\phi(x)}{F(x)} = \frac{A_1}{(x-a)^n} + \frac{A_2}{(x-a)^{n-1}} + \dots + \frac{A_n}{x-a} + \frac{\chi(x)}{\psi(x)}.$

To determine  $A_1, A_2 \dots A_n$ . Multiply by  $(x-a)^n$ ; put  $x=a$  and differentiate, alternately.

**1919** If  $F(x) = 0$  has a single pair of imaginary roots

$a \pm i\beta$ ; then, applying (1917), let

$$\frac{\phi(a+i\beta)}{F'(a+i\beta)} = A-iB; \quad \therefore \frac{\phi(a-i\beta)}{F'(a-i\beta)} = A+iB;$$

and the partial fractions corresponding to these roots will be

$$\frac{A-iB}{x-a-i\beta} + \frac{A+iB}{x-a+i\beta} = \frac{2A(x-a)+2B\beta}{(x-a)^2+\beta^2}.$$

For practical methods of resolving a fraction into partial fractions in the different cases which occur, see (235-238).

### INTEGRATION BY INFINITE SERIES.

When other methods are not applicable, an integral may sometimes be evaluated by expanding the function in a converging series and integrating the separate terms.

$$\text{Ex.} \quad \int \frac{e^{ax}}{x} dx = \log x + ax + \frac{a^2 x^2}{1 \cdot 2^2} + \frac{a^3 x^3}{1 \cdot 2 \cdot 3^3} + \frac{a^4 x^4}{1 \cdot 2 \cdot 3 \cdot 4^3} + \&c. \quad (150)$$

### STANDARD INTEGRALS.

**1921** Some elementary integrals are obtained at once from the known derivatives of simple functions. Thus the derivatives (1422-38) furnish corresponding integrals. The following are in constant use:—

$$\text{1922} \quad \int x^m dx = \frac{x^{m+1}}{m+1}. \quad \int \frac{dx}{x} = \log x.$$

$$\text{1924} \quad \int a^x dx = \frac{a^x}{\log a}. \quad \int e^x = e^x.$$

$$\text{1926} \quad \int \frac{dx}{x\sqrt{(x^2-a^2)}} = \frac{1}{a} \cos^{-1} \frac{a}{x} \text{ or } -\frac{1}{a} \sin^{-1} \frac{a}{x}. \quad [\text{Subs. } \frac{1}{x}]$$

$$\text{1927} \quad \int \frac{dx}{x\sqrt{(a^2 \pm x^2)}} = \frac{1}{a} \log \frac{x}{a + \sqrt{(a^2 \pm x^2)}}. \quad [\text{Substitute } \frac{1}{x}]$$



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$$1928 \quad \int \frac{dx}{\sqrt{(x^2 \pm a^2)}} = \log \{x + \sqrt{(x^2 \pm a^2)}\}. \quad (1909, \text{Ex. 1})$$

$$1929 \quad \int \frac{dx}{\sqrt{(a^2 - x^2)}} = \sin^{-1} \frac{x}{a} \quad \text{or} \quad -\cos^{-1} \frac{x}{a}. \quad (1434)$$

$$1930 \quad \int \frac{x dx}{\sqrt{(a^2 \pm x^2)}} = \pm \sqrt{(a^2 \pm x^2)}.$$


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1931 By Parts, Division, and adding results (1913), we obtain

$$\int \sqrt{(x^2 \pm a^2)} dx = \frac{1}{2} x \sqrt{(x^2 \pm a^2)} \pm \frac{1}{2} a^2 \log \{x + \sqrt{(x^2 \pm a^2)}\}.$$


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1932 By Parts, Division, and difference of results,

$$\int \frac{x^2 dx}{\sqrt{(x^2 \pm a^2)}} = \frac{1}{2} x \sqrt{(x^2 \pm a^2)} \mp \frac{1}{2} a^2 \log \{x + \sqrt{(x^2 \pm a^2)}\}.$$


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$$1933 \quad \int \sqrt{(a^2 - x^2)} dx = \frac{1}{2} a^2 \sin^{-1} \frac{x}{a} + \frac{1}{2} x \sqrt{(a^2 - x^2)}. \quad [\text{As in (1931)}]$$

$$1934 \quad \int \frac{x^2 dx}{\sqrt{(a^2 - x^2)}} = \frac{1}{2} a^2 \sin^{-1} \frac{x}{a} - \frac{1}{2} x \sqrt{(a^2 - x^2)}. \quad [\text{As in (1932)}]$$


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$$1935 \quad \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} \quad \text{or} \quad -\frac{1}{a} \cot^{-1} \frac{x}{a} \quad (1436)$$

$$1936 \quad \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x-a}{x+a}. \quad [\text{By Partial fractions}]$$

$$1937 \quad \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a+x}{a-x}. \quad [\text{Do.}]$$


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$$1938 \quad \int \sin x dx = -\cos x. \quad \int \cos x dx = \sin x.$$

$$1940 \quad \int \tan x dx = -\log \cos x. \quad \int \cot x dx = \log \sin x.$$

$$1942 \quad \int \sec x dx = \log \tan \left( \frac{\pi}{4} + \frac{x}{2} \right). \quad \int \operatorname{cosec} x dx = \log \tan \frac{x}{2}.$$

METHOD.—(1940, '2), substitute  $\cos x$ . (1941, '3), substitute  $\sin x$ .

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$$1944 \quad \int \sin^{-1} x dx = x \sin^{-1} x + \sqrt{(1-x^2)}.$$

$$1945 \quad \int \cos^{-1} x dx = x \cos^{-1} x - \sqrt{(1-x^2)}.$$

$$1946 \quad \int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2} \log(1+x^2).$$

$$1947 \quad \int \cot^{-1} x \, dx = x \tan^{-1} x + \frac{1}{2} \log(1+x^2).$$

$$1948 \quad \int \sec^{-1} x \, dx = x \sec^{-1} x - \log \{x + \sqrt{(x^2-1)}\}.$$

$$1949 \quad \int \operatorname{cosec}^{-1} x \, dx = x \operatorname{cosec}^{-1} x + \log \{x + \sqrt{(x^2-1)}\}.$$

METHOD.—(1944) to (1949), integrate by Parts,  $\int dx$ .

$$1950 \quad \int \log x \, dx = x \log x - x. \quad [\text{By Parts, } \int dx]$$

$$1951 \quad \int \frac{dx}{a+b \cos x} = \frac{2}{\sqrt{(a^2-b^2)}} \tan^{-1} \left\{ \tan \frac{x}{2} \sqrt{\frac{a-b}{a+b}} \right\},$$

$$1952 \quad \text{or} \quad \frac{1}{\sqrt{(b^2-a^2)}} \log \frac{\sqrt{(b+a)} + \sqrt{(b-a)} \tan \frac{1}{2}x}{\sqrt{(b+a)} - \sqrt{(b-a)} \tan \frac{1}{2}x},$$

according as  $a$  is  $>$  or  $<$   $b$ .

[Subs.  $\tan \frac{1}{2}x$ , and integrate by (1935 or '37)]

## VARIOUS INDEFINITE INTEGRALS.

### GENERALIZED CIRCULAR FUNCTIONS.

$$1954 \quad \int \sin^n x \, dx. \quad \int \cos^n x \, dx. \quad \int \operatorname{cosec}^n x \, dx.$$

METHOD.—When  $n$  is integral, integrate the expansions in (772-4). Otherwise by successive reduction, see (2060). For  $\int \operatorname{cosec}^n x \, dx$ , see (2058).

$$1957 \quad \int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \frac{\tan^{n-3} x}{n-3} + \frac{\tan^{n-5} x}{n-5} - \&c.$$

PROOF.—By Division;  $\tan^n x = \tan^{n-2} x \sec^2 x - \tan^{n-2} x$ , the first term of which is integrable; and so on.

$$1958 \quad \int \frac{dx}{(a+b \cos x)^n} = \frac{1}{(a^2-b^2)^{n-1}} \int (a-b \cos z)^{n-1} dz.$$

METHOD.—By substituting  $\tan \frac{z}{2} = \tan \frac{x}{2} \sqrt{\frac{a-b}{a+b}}$ . Similarly with  $\sin x$  in the place of  $\cos x$ , substitute  $\frac{1}{2}\pi - x$ .

$$1959 \quad \int \frac{\cos^p x}{\cos nx} \, dx. \quad \int \frac{\cos^p x}{\sin nx} \, dx. \quad \int \frac{\sin^p x}{\cos nx} \, dx. \quad \int \frac{\sin^p x}{\sin nx} \, dx.$$

2 T

METHOD.—By (809 & 812), when  $p$  and  $n$  are integers, the first two functions can be resolved into partial fractions as under,  $p$  being  $< n$  in the first and  $< n-1$  in the second. The third and fourth integrals reduce to one or other of the former by substituting  $\frac{1}{2}\pi - x$ .

$$1963 \quad \frac{\cos^p x}{\cos nx} = \frac{1}{n} \sum_{r=1}^{r=n} (-1)^{r+1} \frac{\sin(2r-1)\theta \cos^p(2r-1)\theta}{\cos x - \cos(2r-1)\theta}, \text{ with } \theta = \frac{\pi}{2n}.$$

$$1964 \quad \frac{\cos^p x}{\sin nx} = \frac{1}{n \sin x} \sum_{r=1}^{r=n-1} (-1)^{r+1} \frac{\sin^2 r\theta \cos^p r\theta}{\cos x - \cos r\theta}, \text{ with } \theta = \frac{\pi}{n}.$$

The fractions in (1963) are integrated by (1952); those in (1964) by (1990).

*Formulae of Reduction.*

$$1965 \quad \int \frac{\cos nx}{\cos^p x} dx = 2 \int \frac{\cos(n-1)x}{\cos^{p-1} x} dx - \int \frac{\cos(n-2)x}{\cos^p x} dx$$

$$1966 \quad \int \frac{\cos nx}{\sin^p x} dx = -2 \int \frac{\sin(n-1)x}{\sin^{p-1} x} dx + \int \frac{\cos(n-2)x}{\sin^p x} dx$$

$$1967 \quad \int \frac{\sin nx}{\sin^p x} dx = 2 \int \frac{\cos(n-1)x}{\sin^{p-1} x} dx + \int \frac{\sin(n-2)x}{\sin^p x} dx$$

$$1968 \quad \int \frac{\sin nx}{\cos^p x} dx = 2 \int \frac{\sin(n-1)x}{\cos^{p-1} x} dx - \int \frac{\sin(n-2)x}{\cos^p x} dx$$

PROOF.—In (1965).  $2 \cos x \cos(n-1)x = \cos nx + \cos(n-2)x$ , &c. Similarly in (1966–8).

$$1969 \quad \int \sin^p x \sin nx dx \\ = -\frac{\sin^p x \cos nx}{p+n} + \frac{p}{p+n} \int \sin^{p-1} x \cos(n-1)x dx.$$

$$1970 \quad \int \cos^p x \sin nx dx \\ = -\frac{\cos^p x \cos nx}{p+n} + \frac{p}{p+n} \int \cos^{p-1} x \sin(n-1)x dx.$$

$$1971 \quad \int \sin^p x \cos nx dx \\ = \frac{\sin^p x \sin nx}{p+n} - \frac{p}{p+n} \int \sin^{p-1} x \cos(n-1)x dx.$$

$$1972 \quad \int \cos^p x \cos nx dx \\ = \frac{\cos^p x \sin nx}{p+n} + \frac{p}{p+n} \int \cos^{p-1} x \cos(n-1)x dx.$$

PROOF.—(1969). By Parts,  $\int \sin nx \, dx$ . In the new integral change  $\cos nx \cos x$  into  $\cos (n-1)x - \sin nx \sin x$ . By successive reduction in this way the integral may be found. Similarly in (1970-2).

Otherwise, expand  $\sin^p x$  or  $\cos^p x$  in multiple angles by (772-4), and integrate the terms by the following formulæ.

### 1973—1975

$$\int \sin px \sin nx \, dx = \frac{1}{2} \left( \frac{\sin (p-n)x}{p-n} - \frac{\sin (p+n)x}{p+n} \right).$$

and so with similar forms, by (666-9).

1976  $\int \frac{\cos px \, dx}{\cos nx}$  and  $\int \frac{\sin px \, dx}{\cos nx}$  are found from

$$\int \frac{i \cos px - \sin px}{\cos nx} \, dx = 2 \int \frac{z^{p+n-1} \, dz}{1+z^{2n}}$$

when  $p$  and  $n$  are integers, by equating real and imaginary parts after integrating the right side by (2023).

PROOF.—Put  $\cos x + i \sin x = z$ ; therefore  $iz \, dx = dz$ . Multiplying numerator and denominator of the fraction below by  $\cos nx + i \sin nx$ , we get

$$\frac{\cos px + i \sin px}{\cos nx} = 2 \frac{\cos (p+n)x + i \sin (p+n)x}{1 + \cos 2nx + i \sin 2nx} = 2 \frac{z^{p+n}}{1+z^{2n}};$$

therefore 
$$\int \frac{\cos px + i \sin px}{\cos nx} \, dx = -2i \int \frac{z^{p+n-1} \, dz}{1+z^{2n}}.$$

1978  $\int \frac{\cos px \, dx}{\sin nx}$  and  $\int \frac{\sin px \, dx}{\sin nx}$  are found in the same

way from 
$$\int \frac{\cos px + i \sin px}{\sin nx} \, dx = 2 \int \frac{z^{p+n-1} \, dz}{1-z^{2n}}.$$

PROOF.—As in (1976), by multiplying numerator and denominator of  $\frac{\cos px + i \sin px}{\sin nx}$  by  $\cos nx - i \sin nx$ .

1980  $\int \frac{\cos x \, dx}{\sqrt[n]{\cos nx}}$  and  $\int \frac{\sin x \, dx}{\sqrt[n]{\cos nx}}$

Putting  $y = \frac{\cos x + i \sin x}{\sqrt[n]{\cos nx}}$ , we find  $\tan nx = i(1-y^n)$ , and therefore  $y \, dx = \frac{-i \, dy}{2-y^n}$ . Hence, multiplying by  $i$ , we have

$$\int \frac{i \cos x - \sin x}{\sqrt[n]{\cos nx}} \, dx = \int \frac{dy}{2-y^n}.$$

The real part and the coefficient of  $i$  in the expansion of the integral on the right by (2021, '2), are the values required.

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$$1982 \quad \int \frac{dx}{a \cos^2 x + b \sin^2 x} = \frac{1}{\sqrt{ab}} \tan^{-1} \left( \tan x \sqrt{\frac{b}{a}} \right). \quad [\text{Subs. } \tan x]$$

$$1983 \quad \int \frac{dx}{a + b \tan x} = \frac{1}{a^2 + b^2} \{ b \log (a \cos x + b \sin x) + ax \}. \quad [\text{Subs. } \tan x]$$

$$1984 \quad \int \frac{dx}{a + b \sin^2 x} = \frac{1}{\sqrt{(a^2 + ab)}} \tan^{-1} \frac{\sqrt{(a+b)}}{\cot x \sqrt{a}}. \quad [\text{Substitute } \cot x]$$


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$$1985 \quad \int \frac{\sin x \cos^2 x}{1 + a^2 \cos^2 x} dx = \frac{1}{a^3} \tan^{-1} (a \cos x) - \frac{\cos x}{a^3}. \quad [\text{Substitute } a \cos x]$$

$$1986 \quad \int \frac{\cos^3 x dx}{1 - a^2 \cos^2 x} = \frac{1}{a^3 \sqrt{(1-a^2)}} \tan^{-1} \frac{a \sin x}{\sqrt{(1-a^2)}} - \frac{\sin x}{a^3}. \quad [\text{Subs. } \sin x]$$


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$$1987 \quad \int \cos x \sqrt{(1-a^2 \sin^2 x)} dx = \frac{1}{2} \sin x \sqrt{(1-a^2 \sin^2 x)} + \frac{1}{2a} \sin^{-1} (a \sin x). \\ [\text{Substitute } a \sin x]$$

$$1988 \quad \int \sin x \sqrt{(1-a^2 \sin^2 x)} dx = -\frac{1}{2} \cos x \sqrt{(1-a^2 \sin^2 x)} \\ - \frac{1-a^2}{2a} \log \{ a \cos x + \sqrt{(1-a^2 \sin^2 x)} \}. \quad [\text{Subs. } a \cos x]$$

$$1989 \quad \int \sin x (1-a^2 \sin^2 x)^{\frac{3}{2}} dx = -\frac{1}{4} \cos x (1-a^2 \sin^2 x)^{\frac{3}{2}} \\ + \frac{3}{4} (1-a^2) \int \sin x \sqrt{(1-a^2 \sin^2 x)} dx. \quad [\text{Subs. } a \cos x]$$


---

$$1990 \quad \int \frac{dx}{\sin x (a + b \cos x)} = \frac{\log \{ (a \operatorname{cosec} x + b \cot x)^b \tan^a \frac{1}{2} x \}}{a^2 - b^2}.$$

$$\text{By } \frac{1}{\sin x (a + b \cos x)} = \frac{a - b \cos x}{(a^2 - b^2) \sin x} - \frac{b^2 \sin x}{(a^2 - b^2)(a + b \cos x)}.$$


---

$$1991 \quad \int \frac{\tan x dx}{\sqrt{(a + b \tan^2 x)}} = \frac{1}{\sqrt{(b-a)}} \cos^{-1} \frac{\cos x \sqrt{(b-a)}}{\sqrt{b}}. \\ [\text{Subs. } \cos x \sqrt{(b-a)}]$$


---

$$1992 \quad \int \frac{\sqrt{(a + b \sin^2 x)}}{\sin x} dx = \sqrt{b} \cos^{-1} \frac{\sqrt{b} \cos x}{\sqrt{(a+b)}} \\ - \sqrt{a} \log \{ \sqrt{a} \cot x + \sqrt{(a \operatorname{cosec}^2 x + b)} \}.$$

METHOD.—By Division (1912), making the numerator rational, and integrating the two fractions by substituting  $\cot x$  and  $\cos x$  respectively.

---

$$1993 \quad \int \frac{dx}{a + 2b \cos x + c \cos 2x} = \frac{c}{m} \left\{ \int \frac{dx}{2c \cos x + b - m} - \int \frac{dx}{2c \cos x + b + m} \right\},$$

where  $m = \sqrt{\{b^2 - 2c(a-c)\}}$ . Then integrate by (1953).

---

$$1994 \quad \int \frac{dx}{a \cos x + b \sin x + c} = \int \frac{d\theta}{\sqrt{(a^2 + b^2)} \cos \theta + c}. \quad (1953)$$

METHOD.—Substitute  $\theta = x - \alpha$ , where  $\tan \alpha = \frac{b}{a}$ .

$$1995 \quad \int \frac{F(\sin x, \cos x) dx}{a \cos x + b \sin x + c},$$

$F$  being an integral algebraic function of  $\sin x$  and  $\cos x$ .

METHOD.—Substitute  $\theta = x - \alpha$  as in (1994), and the resulting integral takes the form  $\int \frac{f(\sin \theta, \cos \theta) d\theta}{A \cos \theta + B} = \int \frac{\phi(\cos \theta) d\theta}{A \cos \theta + B} + \int \frac{\sin \theta \psi(\cos \theta)}{A \cos \theta + B},$

since  $f$  contains only integral powers of the sine and cosine, and may therefore be resolved into the two terms as indicated.

To find the first integral on the right, divide by the denominator and integrate each term separately. To find the second integral, substitute the denominator.

$$1996 \quad \int \frac{F(\cos x) dx}{(a_1 + b_1 \cos x)(a_2 + b_2 \cos x) \dots (a_n + b_n \cos x)},$$

where  $F$  is an integral function of  $\cos x$ .

METHOD.—Resolve into partial fractions. Each integral will be of the form  $\int \frac{dx}{A + B \cos x}$  (1951).

$$1997 \quad \int \frac{A \cos x + B \sin x + C}{a \cos x + b \sin x + c} dx.$$

METHOD.—Let  $\phi(x) = a \cos x + b \sin x + c$ ;  $\therefore \phi'(x) = -a \sin x + b \cos x$ . Assume  $A \cos x + B \sin x + C = \lambda \phi(x) + \mu \phi'(x) + \nu$ . Substitute the values of  $\phi(x)$  and  $\phi'(x)$ , and equate the coefficients to zero to determine  $\lambda, \mu, \nu$ . The integral becomes

$$\int \left\{ \lambda + \mu \frac{\phi'(x)}{\phi(x)} + \frac{\nu}{\phi(x)} \right\} dx = \lambda x + \mu \log \phi(x) + \int \frac{\nu}{\phi(x)} dx,$$

and the last integral is found by (1994).

## EXPONENTIAL AND LOGARITHMIC FUNCTIONS.

1998  $\int e^x F(x) dx$  can be found at once when  $F(x)$  can be expressed as the sum of two functions, one of which is the derivative of the other, for

$$\int e^x \{ \phi(x) + \phi'(x) \} dx = e^x \phi(x).$$

$$1999 \quad \int e^{ax} \cos^n bx dx \text{ and } \int e^{ax} \sin^n bx dx \text{ are respectively } = \frac{a \cos bx + nb \sin bx}{a^2 + n^2 b^2} e^{ax} \cos^{n-1} bx + \frac{n(n-1)b^2}{a^2 + n^2 b^2} \int e^{ax} \cos^{n-2} bx dx.$$



and

$$\frac{a \sin bx - nb \cos bx}{a^2 + n^2 b^2} e^{ax} \sin^{n-1} bx + \frac{n(n-1)b^2}{a^2 + n^2 b^2} \int e^{ax} \sin^{n-2} bx dx.$$

PROOF.—In either case, integrate twice by Parts,  $\int e^{ax} dx$ .

Otherwise, these integrals may be found in terms of multiple angles by expanding  $\sin^n x$  and  $\cos^n x$  by (772-4), and integrating each term by (1951-2).

**2000**  $\int e^x \sin^n x \cos^m x dx$  is found by expressing  $\sin^n x$  and  $\cos^n x$  in terms of multiple angles.

Ex.:  $\int e^x \sin^5 x \cos^3 x dx$ . Put  $e^{i\theta} = z$  in (768),

$$\begin{aligned} (2i \sin x)^5 (2 \cos x)^3 &= (z - z^{-1})^5 (z + z^{-1})^3 \\ &= (z - z^{-1})^3 (z^2 - z^{-2})^2 = (z^7 - z^{-7}) - 3(z^5 - z^{-5}) + (z^3 - z^{-3}) + 5(z - z^{-1}); \\ \therefore 2^8 e^x \sin^5 x \cos^3 x &= e^x (\sin 7x - 3 \sin 5x + \sin 3x + 5 \sin x). \end{aligned}$$

Then integrate by (1999).

**2001** THEOREM.—Let  $P, Q$  be functions of  $x$ ; and let

$$\int P dx = P_1, \quad \int P_1 Q_x dx = P_2, \quad \int P_2 Q_x dx = P_3, \quad \&c. \quad \text{Then} \\ \int P Q^n dx = P_1 Q^n - n P_2 Q^{n-1} + n(n-1) P_3 Q^{n-2} - \dots \pm \lfloor n \rfloor P_{n+1}.$$

PROOF.—Integrate successively by Parts,  $\int P dx$ , &c.

**2002** THEOREM.—Let  $P, Q$ , as before, be functions of  $x$ ; and

let  $P_1 = \left(\frac{P}{Q_x}\right)_x, \quad P_2 = \left(\frac{P_1}{Q_x}\right)_x, \quad P_3 = \left(\frac{P_2}{Q_x}\right)_x, \quad \&c. \quad \text{Then}$

$$\begin{aligned} \int \frac{P}{Q^n} dx &= -\frac{P}{(n-1) Q_x Q^{n-1}} - \frac{P_1}{(n-1)(n-2) Q_x Q^{n-2}} \\ &\quad - \frac{P_2}{(n-1)(n-2)(n-3) Q_x Q^{n-3}} \dots\dots + \frac{1}{\lfloor n-1 \rfloor} \int \frac{P_{n-1}}{Q} dx. \end{aligned}$$

PROOF.—Integrate successively by Parts,  $\int -\frac{(n-1) Q_x}{Q^n} dx$ .

#### EXAMPLES.

**2003** 
$$\int x^{m-1} (\log x)^n dx$$

$$= \frac{x^m}{m} \left( l^n - \frac{n}{m} l^{n-1} + \frac{n(n-1)}{m^2} l^{n-2} - \dots + (-1)^n \frac{n!}{m^n} \right).$$

METHOD.—By (2001).  $P_1 = \frac{x^m}{m}, \quad P_2 = \frac{x^m}{m^2}, \quad P_3 = \frac{x^m}{m^3}, \quad \&c.$

2004

$$\int x^{m+nx} dx = \int x^m \left\{ 1 + nx \log x + \frac{1}{2} (nx \log x)^2 + \&c. \right\} dx, \quad [\text{By (149)}]$$

and each term of this result can be integrated by (2003).

2005

$$\int \frac{x^{m-1} dx}{(\log x)^n} = -x^m \left( \frac{l^{-n+1}}{n-1} + \frac{ml^{-n+2}}{(n-1)(n-2)} + \frac{m^2 l^{-n+3}}{(n-1)(n-2)(n-3)} + \&c. \right) + \frac{m^{n-1}}{n-1} \int \frac{x^{m-1} dx}{\log x}.$$

METHOD.—By (2002).  $Px = x^m$ ,  $P_1 = mx^{m-1}$ ,  $P_2 = m^2 x^{m-2}$ , &c.

The last method is not applicable when  $n = 1$ . In this case, writing  $l$  for  $\log x$ ,

$$2006 \quad \int \frac{x^{m-1} dx}{\log x} = \log l + ml + \frac{m^2 l^2}{1 \cdot 2^2} + \frac{m^3 l^3}{1 \cdot 2 \cdot 3^2} + \frac{m^4 l^4}{1 \cdot 2 \cdot 3 \cdot 4^2} + \&c.$$

METHOD:  $\frac{x^{m-1}}{\log x} = \frac{e^{m \log x}}{x \log x}$ . Expand the numerator by (150), and integrate.

See also (2161–6) for similar developments of the exponential forms of the same functions.

## PARTICULAR ALGEBRAIC FUNCTIONS.

2007

$$\int \frac{dx}{x^n (x-1)^n} = \frac{1}{n-1} \left\{ \frac{1}{(1-x)^{n-1}} - \frac{1}{x^{n-1}} \right\} + \frac{n}{n-2} \left\{ \frac{1}{(1-x)^{n-2}} - \frac{1}{x^{n-2}} \right\} + \dots + \frac{n(n+1) \dots 2(n-1)}{1 \cdot 2 \dots (n-1)} \log \frac{x}{1-x}.$$

$n$  being even. (1918)

2008

$$\int \frac{dx}{(1+a^2 x^2) \sqrt{1-x^2}} = \frac{1}{\sqrt{1+a^2}} \cos^{-1} \sqrt{\left( \frac{1-x^2}{1+a^2 x^2} \right)}.$$

[Subs.  $\frac{\sqrt{1-x^2}}{x}$ ]

2009

$$\int \frac{dx}{(1-x^2) \sqrt{1+x^2}} = \frac{1}{\sqrt{2}} \log \frac{x\sqrt{2} + \sqrt{1+x^2}}{\sqrt{1-x^2}}.$$

2010

$$\int \frac{dx}{(1+x^2) \sqrt{x^2-1}} = \frac{1}{\sqrt{2}} \log \frac{x\sqrt{2} + \sqrt{x^2-1}}{\sqrt{1+x^2}}.$$

---


$$\begin{aligned}
 2011 \quad \frac{dx}{(c+ex^2)\sqrt{(a+bx^2)}} &= \frac{1}{\sqrt{(ace-bc^2)}} \sin^{-1} \sqrt{\left(\frac{x^2(ae-bc)}{aex^2+ac}\right)}, \quad ae > bc \\
 &= \frac{1}{\sqrt{(bc^2-ace)}} \log \frac{c\sqrt{(a+bx^2)}+x\sqrt{(bc^2-ace)}}{\sqrt{(c+ex^2)}}, \quad ae < bc
 \end{aligned}$$


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$$2012 \quad \int \frac{dx}{x\sqrt{(a+bx^n)}} = \frac{2}{n\sqrt{a}} \log \frac{\sqrt{(a+bx^n)} - \sqrt{a}}{\sqrt{x^n}}. \quad [\text{Subs. } \frac{1}{\sqrt{x^n}}]$$


---

$$2013 \quad \int \frac{(1+x^2) dx}{(1-x^2)\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}} \log \frac{x\sqrt{2} + \sqrt{(1+x^4)}}{1-x^2}. \quad [\text{Subs. } \frac{x\sqrt{2}}{1-x^2}]$$

$$2014 \quad \int \frac{(1-x^2) dx}{(1+x^2)\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}} \sin^{-1} \frac{x\sqrt{2}}{1+x^2}. \quad [\text{Subs. } \frac{x\sqrt{2}}{1+x^2}]$$


---

$$2015 \quad \int \frac{\sqrt{(1+x^4)}}{1-x^4} dx = \frac{1}{\sqrt{8}} \left\{ \log \frac{\sqrt{(1+x^4)} + x\sqrt{2}}{1-x^2} + \sin^{-1} \frac{x\sqrt{2}}{1+x^2} \right\}.$$

$$\begin{aligned}
 2016 \quad \int \frac{x^2 dx}{(1-x^4)\sqrt{(1+x^4)}} &= \frac{1}{4\sqrt{2}} \left\{ \log \frac{\sqrt{(1+x^4)} + x\sqrt{2}}{1-x^2} - \sin^{-1} \frac{x\sqrt{2}}{1+x^2} \right\}. \\
 &[\text{Substitute } z\sqrt{(1+x^4)} = x\sqrt{2} \text{ in (2015-6)}]
 \end{aligned}$$


---

$$\begin{aligned}
 2017 \quad \int \frac{dx}{(1-x^2)(2x^2-1)^{\frac{1}{2}}} &= \frac{1}{4} \log \frac{(2x^2-1)^{\frac{1}{2}}-x}{(2x^2-1)^{\frac{1}{2}}+x} - \frac{1}{2} \tan^{-1} \frac{x}{(2x^2-1)^{\frac{1}{2}}}. \\
 &[\text{Substitute } x = z(2x^2-1)^{\frac{1}{2}}]
 \end{aligned}$$


---

$$\begin{aligned}
 2018 \quad \int \frac{dx}{(1+x^4)\{\sqrt{(1+x^4)}-x^2\}^{\frac{1}{2}}} &= \tan^{-1} \frac{x}{\{\sqrt{(1+x^4)}-x^2\}^{\frac{1}{2}}}. \\
 &[\text{Substitute } \frac{x}{\sqrt{\{\sqrt{(1+x^4)}-x^2\}}}
 \end{aligned}$$


---

$$2019 \quad \int \frac{dx}{\sqrt[3]{(1-x^3)}} = \log \sqrt{x + \sqrt[3]{1-x^3}} - \frac{1}{\sqrt{3}} \tan^{-1} \frac{2\sqrt[3]{1-x^3}-x}{x\sqrt{3}}. \quad [\text{Subs. } \frac{\sqrt[3]{(1-x^3)}}{x}]$$


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$$2020 \quad \int \frac{dx}{(1+x)\sqrt[3]{(1+3x+3x^2)}} \text{ reduces to (2019) by substituting } \frac{x}{1+x}.$$


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### INTEGRATION OF $\frac{x^{l-1}}{x^n \pm 1}$ .

If  $l$  and  $n$  are positive integers, and  $l-1 < n$ ;<sup>\*</sup> then,  $n$  being even,

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\* If  $l=n$ , the value of the integral is simply  $\frac{1}{n} \log (x^n-1)$ .

$$2021 \quad \int \frac{x^{l-1} dx}{x^n - 1} = \frac{1}{n} \log(x-1) + \frac{(-1)^l}{n} \log(x+1) \\ + \frac{1}{n} \Sigma \cos rl\beta \log(x^2 - 2x \cos r\beta + 1) - \frac{2}{n} \Sigma \sin rl\beta \tan^{-1} \frac{x - \cos r\beta}{\sin r\beta}$$

where  $\beta = \frac{\pi}{n}$ , and  $\Sigma$  denotes that the sum of all the terms obtained by making  $r = 2, 4, 6 \dots n-2$  successively, is to be taken.

If  $n$  be odd,

$$2022 \quad \int \frac{x^{l-1} dx}{x^n - 1} = \frac{1}{n} \log(x-1) \\ + \frac{1}{n} \Sigma \cos rl\beta \log(x^2 - 2x \cos r\beta + 1) - \frac{2}{n} \Sigma \sin rl\beta \tan^{-1} \frac{x - \cos r\beta}{\sin r\beta}$$

with  $r = 2, 4, 6 \dots n-1$  successively.

If  $n$  be even,

$$2023 \quad \int \frac{x^{l-1} dx}{x^n + 1} = -\frac{1}{n} \Sigma \cos rl\beta \log(x^2 - 2x \cos r\beta + 1) \\ + \frac{2}{n} \Sigma \sin rl\beta \tan^{-1} \frac{x - \cos r\beta}{\sin r\beta},$$

with  $r = 1, 3, 5 \dots n-1$  successively.

If  $n$  be odd,

$$2024 \quad \int \frac{x^{l-1} dx}{x^n + 1} = \frac{(-1)^{l-1}}{n} \log(x+1) \\ - \frac{1}{n} \Sigma \cos rl\beta \log(x^2 - 2x \cos r\beta + 1) + \frac{2}{n} \Sigma \sin rl\beta \tan^{-1} \frac{x - \cos r\beta}{\sin r\beta}$$

with  $r = 1, 3, 5 \dots n-2$  successively.

PROOF.—(2021-4). Resolve  $\frac{x^{l-1}}{x^n \pm 1}$  into partial fractions by the method of (1917). We have  $\frac{\phi(a)}{F'(a)} = \frac{a^{l-1}}{na^{n-1}} = \frac{a^m}{n}$ , since  $a^n = \mp 1$ . The different values of  $a$  are the roots of  $x^n \pm 1 = 0$ , and these are given by  $x = \cos r\beta \pm i \sin r\beta$ , with odd or even integral values of  $r$ . (See 480, 481;  $2r$  and  $2r+1$  of those articles being in each case here represented by  $r$ .) The first two terms on the right in (2021) arise from the factors  $x \pm 1$ ; the remaining terms from quadratic factors of the type

$$(x - \cos r\beta - i \sin r\beta)(x - \cos r\beta + i \sin r\beta) = (x - \cos r\beta)^2 + \sin^2 r\beta.$$

These last terms are integrated by (1923) and (1935). Similarly for the cases (2022-4).

**2025** If, in formulæ (2021-4),  $\mp \frac{2}{n} \Sigma (\frac{1}{2}\pi - r\beta) \sin rl\beta$  be added to the last term for the constant of integration, the integral vanishes with  $x$ , and the last term becomes

$$\mp \frac{2}{n} \Sigma \sin rl\beta \tan^{-1} \frac{x \sin r\beta}{1 - x \cos r\beta},$$

reading  $-$  in (2021-2), and  $+$  in (2023-4).

**2026** 
$$\int \frac{x^m dx}{a + bx^n} = \frac{1}{a} \left( \frac{a}{b} \right)^{\frac{m+1}{n}} \int \frac{z^m}{1+z^n} dz,$$

where  $az^n = bx^n$ . Then integrate by (2023-4).

**2027** 
$$\int \frac{x^{m-1} + x^{n-m-1}}{x^n + 1} dx = \frac{4}{n} \Sigma \sin mr\beta \tan^{-1} \frac{x - \cos r\beta}{\sin r\beta},$$

where  $\beta = \pi \div n$ , and  $r = 1, 3, 5, \dots$  successively up to  $n-1$  or  $n-2$ , according as  $n$  is even or odd.

**2028**

$$\int \frac{x^{m-1} - x^{n-m-1}}{x^n + 1} dx = -\frac{2}{n} \Sigma \cos mr\beta \cdot \log (x^2 - 2x \cos r\beta + 1),$$

with the same values of  $r$ ; but when  $n$  is odd, supply the additional term  $(-1)^m 2 \log (x+1) \div n$ .

PROOF.—Follow the method of (2024, *Proof*).

Similar forms are obtainable when the denominator is  $x^n - 1$ .

**2029**

$$\int \frac{x^{l-1} dx}{x^{2n} - 2x^n \cos n\theta + 1} = \frac{1}{n \sin n\theta} \Sigma \cos (n-l)\phi \cdot \tan^{-1} \frac{x - \cos \phi}{\sin \phi} \\ - \frac{1}{2} \Sigma \sin (n-l)\phi \cdot \log (x^2 - 2x \cos \phi + 1),$$

where  $\phi = \theta + \frac{r\pi}{n}$  and  $r = 0, 2, 4, \dots, 2(n-1)$  successively.

But if the integral is to vanish with  $x$ , write  $\tan^{-1} \frac{x \sin \phi}{1 - x \cos \phi}$ , as in (2025).

PROOF.—By the method of (2024).—The factors of the denominator are given in (607), putting  $y$  equal to unity.

$$2030 \quad \int \frac{x^{l-1} (\log x)^m}{x^n \pm 1} dx = \mp \Sigma \frac{(r\beta)^m}{n} \left\{ \cos (rl\beta + \tfrac{1}{2}m\pi) \right. \\ \left. \times \log (x^2 - 2x \cos r\beta + 1) - 2 \sin (rl\beta + \tfrac{1}{2}m\pi) \tan^{-1} \frac{x - \cos r\beta}{\sin r\beta} \right\},$$

with the values of  $\beta$  and  $r$  in (2021-4).

PROOF.—Differentiate the equations (2021-4)  $m$  times with respect to  $l$ , by (1427) and (1461-2). If  $m$  be negative, integrate  $m$  times with respect to  $l$ , and the same formula is obtained by (2155-6).

In a similar manner, from (2027-8) and (2029), the general terms may be found for the integrals

2032

$$\int \frac{\{x^{m-1} \pm (-1)^p x^{n-m-1}\} (\log x)^p}{x^n \pm 1} dx \quad \text{and} \quad \int \frac{x^{l-1} (\log x)^m dx}{x^{2n} - 2x^n \cos n\theta + 1}.$$

INTEGRATION OF  $\int x^m (ax + bx^n)^{\frac{p}{q}} dx$ .

2035 RULE I.—When  $\frac{m+1}{n}$  is a positive integer, integrate by substituting  $z = (a + bx^n)^{\frac{1}{q}}$ . Thus

$$\int x^m (a + bx^n)^{\frac{p}{q}} dx = \frac{q}{nb} \int z^{p+q-1} \left( \frac{z^q - a}{b} \right)^{\frac{m+1}{n}-1} dz.$$

Expand the binomial, and integrate the separate terms by (1922).

2036 But if the positive integer be 1, the integral is known at sight, since  $m$  then becomes  $= n-1$ .

2037 RULE II.—When  $\frac{m+1}{n} + \frac{p}{q}$  is a negative integer, substitute  $z = (ax^{-n} + b)^{\frac{1}{q}}$ . Thus

$$\int x^m (a + bx^n)^{\frac{p}{q}} dx = -\frac{q}{na} \int z^{p+q-1} \left( \frac{z^q - b}{a} \right)^{-\frac{m+1}{n} - \frac{p}{q} - 1} dz.$$

Expand and integrate as before.

2038 But, if the negative integer be  $-1$ , the integral is found immediately by writing it in the form

$$\int x^{m+\frac{np}{q}} (ax^{-n} + b)^{\frac{p}{q}} dx = \int x^{-n-1} (ax^{-n} + b)^{\frac{p}{q}} dx \\ = \frac{q}{na(p+q)} (ax^{-n} + b)^{\frac{p}{q}+1}.$$



## EXAMPLES.

**2039** To find  $\int x^{\frac{1}{2}}(1+x^{\frac{1}{2}})^{\frac{1}{2}} dx$ . Here  $m=\frac{1}{2}$ ,  $n=\frac{1}{2}$ ,  $p=2$ ,  $q=3$ ,  $\frac{m+1}{n}=3$ , a positive integer. Therefore, substituting  $y=(1+x^{\frac{1}{2}})^{\frac{1}{2}}$ ,  $x=(y^2-1)^2$ ,  $x_y=6y^3(y^2-1)$ , and the integral becomes

$$\int (y^2-1)y^3 \frac{dx}{dy} dy = 6 \int y^4 (y^2-1)^2 dy,$$

the value of which can be found immediately by expanding and integrating the separate terms.

**2040** 
$$\int x^3 (a+bx^4)^{\frac{1}{2}} dx = \frac{3}{16b} (a+bx^4)^{\frac{1}{2}}.$$

For  $\frac{m+1}{n}=1$  (2036); that is,  $m+1=n$ , and the factor  $x^3$  is the derivative of  $\frac{1}{4}x^4$ .

**2041**  $\int \frac{(1+\sqrt{x})^{\frac{1}{2}}}{x^{\frac{1}{2}}} dx$ , or  $\int x^{-\frac{1}{2}}(1+x^{\frac{1}{2}})^{\frac{1}{2}} dx$ . Here  $m=-\frac{1}{2}$ ,  $n=\frac{1}{2}$ ,  $p=2$ ,  $q=3$ ,  $\frac{m+1}{n} + \frac{p}{q} = -2$ , a negative integer. Therefore, substitute  $y=(x^{-\frac{1}{2}}+1)^{\frac{1}{2}}$ ,  $x=(y^2-1)^{-2}$ ,  $x_y=-6y^3(y^2-1)^{-3}$ . Writing the integral in the form below, and then substituting the values, we have

$$\int x^{-2}(x^{-\frac{1}{2}}+1)^{\frac{1}{2}} x_y dy = -6 \int y^4 (y^2-1) dy,$$

which can be integrated at once.

**2042** 
$$\int \frac{dx}{x(a+bx^n)} = \int x^{-1}(a+bx^n)^{-1} dx. \text{ Here } \frac{m+1}{n} + \frac{p}{q} = -1;$$

therefore, by (2038), the integral

$$= \int x^{-n-1}(ax^{-n}+b)^{-1} dx = -\frac{1}{na} \log(ax^{-n}+b).$$

REDUCTION OF  $\int x^m (a+bx^n)^p dx$ .

When neither of the conditions in (2035, 2037) are fulfilled, the integral may be reduced by any of the six following rules, so as to alter the indices  $m$  and  $p$ , those indices having any algebraic values.

**2043** I. To change  $m$  and  $p$  into  $m+n$  and  $p-1$ .

Integrate by Parts,  $\int x^m dx$ .

**2044** II. To change  $m$  and  $p$  into  $m-n$  and  $p+1$ .

Integrate by Parts,  $\int x^{n-1}(a+bx^n)^p dx$ .

**2045** III. To change  $m$  into  $m+n$ .

Add 1 to  $p$ . Then integrate by Parts,  $\int x^m dx$ ; and also by Division, and equate the results.

**2046** IV. To change  $m$  into  $m - n$ .

Add 1 to  $p$ , and subtract  $n$  from  $m$ . Then integrate by Parts,  $\int x^m dx$ ; and also by Division, and equate the results.

**2047** V. To change  $p$  into  $p + 1$ .

Add 1 to  $p$ . Then integrate by Division, and the new integral by Parts,  $\int x^{n-1} (a + bx^n)^p$ .

**2048** VI. To change  $p$  into  $p - 1$ .

Integrate by Division, and the new integral by Parts,  $\int x^{n-1} (a + bx^n)^p$ .

**2049** MNEMONIC TABLE FOR THE SAME RULES.

I.	$m + n, \quad p - 1$	By Parts ( $m$ ).
II.	$m - n, \quad p + 1$	By Parts ( $p$ ).
III.	$m + n$	$(p + 1)$ , Parts ( $m$ ) and Division.
IV.	$m - n$	$(p + 1, m - n)$ , Parts ( $m$ ) and Division.
V.	$p + 1$	$(p + 1)$ , Division, and the new integral by Parts ( $p$ ).
VI.	$p - 1$	Division, and the new integral by Parts ( $p$ ).

By applying the rules, *Formulæ of reduction* are obtained. Thus, any of the six values below may be substituted for the integral

$$\int x^m (a + bx^n)^p dx.$$

**2050—2055**

- I.  $\frac{x^{m+1} (a + bx^n)^p}{m+1} - \frac{bnp}{m+1} \int x^{m+n} (a + bx^n)^{p-1} dx.$
- II.  $\frac{x^{m-n+1} (a + bx^n)^{p+1}}{bn(p+1)} - \frac{m-n+1}{bn(p+1)} \int x^{m-n} (a + bx^n)^{p+1} dx.$
- III.  $\frac{x^{m+1} (a + bx^n)^{p+1}}{a(m+1)} - \frac{b(m+n+np+1)}{a(m+1)} \int x^{m+n} (a + bx^n)^p dx.$
- IV.  $\frac{x^{m-n+1} (a + bx^n)^{p+1}}{b(m+np+1)} - \frac{a(m-n+1)}{b(m+np+1)} \int x^{m-n} (a + bx^n)^p dx.$
- V.  $-\frac{x^{m+1} (a + bx^n)^{p+1}}{an(p+1)} + \frac{m+n+np+1}{an(p+1)} \int x^m (a + bx^n)^{p+1} dx.$
- VI.  $\frac{x^{m+1} (a + bx^n)^p}{m+np+1} + \frac{anp}{m+np+1} \int x^m (a + bx^n)^{p-1} dx.$

## EXAMPLES.

**2056** To find  $\int \frac{\sqrt{(a^2-x^2)}}{x^3} dx$ . Apply Rule I. or Formula I.; thus  

$$\int x^{-3}(a^2-x^2)^{\frac{1}{2}} dx = -\frac{1}{2}x^{-2}(a^2-x^2)^{\frac{1}{2}} + \frac{1}{2} \int x^{-1}(a^2-x^2)^{-\frac{1}{2}} dx \quad (1927).$$

**2057** To find  $\int \frac{x^4}{(a^2-x^2)^{\frac{1}{2}}} dx$ . Apply Rule II. or Formula II.; thus  

$$\int x^4(a^2-x^2)^{-\frac{1}{2}} dx = x^3(a^2-x^2)^{-\frac{1}{2}} - 3 \int x^2(a^2-x^2)^{-\frac{1}{2}} dx \quad (1934).$$

**2058**  $\int \operatorname{cosec}^m \theta d\theta$ . Substituting  $\sin \theta = x$ , the integral becomes  

$$\int \sin^{-m} \theta \frac{d\theta}{dx} dx = \int x^{-m} (1-x^2)^{-\frac{1}{2}} dx.$$

Apply Rule III.; thus, increasing  $p$  by 1 and integrating, first by Parts  $\int x^{-m} dx$ , and again by Division;

$$\begin{aligned} \int x^{-m} (1-x^2)^{\frac{1}{2}} dx &= \frac{x^{1-m} (1-x^2)^{\frac{1}{2}}}{1-m} + \frac{1}{1-m} \int x^{2-m} (1-x^2)^{-\frac{1}{2}} dx, \\ \int x^{-m} (1-x^2)^{\frac{1}{2}} dx &= \int x^{-m} (1-x^2)^{-\frac{1}{2}} dx - \int x^{2-m} (1-x^2)^{-\frac{1}{2}} dx. \end{aligned}$$

Equating the results, we obtain

**2059** 
$$\int x^{-m} (1-x^2)^{-\frac{1}{2}} dx = \frac{x^{1-m}}{1-m} (1-x^2)^{\frac{1}{2}} + \frac{2-m}{1-m} \int x^{2-m} (1-x^2)^{-\frac{1}{2}} dx.$$

By repeating the process, the integral is made to depend finally upon

$$\int x^{-1} (1-x^2)^{-\frac{1}{2}} dx \quad \text{or} \quad \int (1-x^2)^{-\frac{1}{2}} dx,$$

according as  $m$  is an odd or even integer (1927, '29).

**2060**  $\int \sin^m \theta d\theta$  is found in a similar manner by Rule IV. The integral to be evaluated is  $\int x^m (1-x^2)^{-\frac{1}{2}} dx$ ; and the integral operated upon is  $\int x^{m-2} (1-x^2)^{\frac{1}{2}} dx$ . Otherwise apply Formula IV. See also (1954).

**2061** To find  $\int \frac{dx}{(x^2+a^2)^r}$ . Apply Rule V.  $p = -r$ , and increasing  $p$  by 1, we have, first, by Division,

$$\int (x^2+a^2)^{1-r} dx = \int x^2 (x^2+a^2)^{-r} dx + a^2 \int (x^2+a^2)^{-r} dx.$$

Integrating the new form by Parts,  $\int x (x^2+a^2)^{-r} dx$ , we next obtain

$$\int x^2 (x^2+a^2)^{-r} dx = \frac{x (x^2+a^2)^{1-r}}{2(1-r)} - \frac{1}{2(1-r)} \int (x^2+a^2)^{1-r} dx.$$

Substituting this value in the previous equation, we have, finally,

**2062**

$$\int \frac{dx}{(x^2+a^2)^r} = \frac{x}{2(r-1)a^2(x^2+a^2)^{r-1}} + \frac{2r-3}{a^2(2r-2)} \int \frac{dx}{(x^2+a^2)^{r-1}}.$$

This equation is given at once by Formula V. Thus  $r$  is changed into  $r-1$ , and by repeating the process of reduction, the original integral is ultimately made to depend upon (1935) for its value if  $r$  be an integer.

Another formula for this integral is

$$2063 \quad \int \frac{dx}{(x^2 + \beta)^r} = \frac{(-1)^{r-1}}{1 \cdot 2 \dots (r-1)} \frac{d^{r-1}}{d\beta^{r-1}} \left( \frac{1}{\sqrt{\beta}} \tan^{-1} \frac{x}{\sqrt{\beta}} \right).$$

PROOF.—Write  $\beta$  for  $a^2$  in (1935), and differentiate the equation  $r-1$  times for  $\beta$  by the principle in (2255).

2064 To find  $\int (a^2 + x^2)^{\frac{1}{2}n} dx$ . Apply Rule VI. By Division, we have

$$\int (a^2 + x^2)^{\frac{1}{2}n} dx = a^2 \int (a^2 + x^2)^{\frac{1}{2}n-1} dx + \int x^2 (a^2 + x^2)^{\frac{1}{2}n-1} dx.$$

The last integral, by Parts, becomes

$$\int x^2 (a^2 + x^2)^{\frac{1}{2}n-1} dx = \frac{1}{n} x (a^2 + x^2)^{\frac{1}{2}n} - \frac{1}{n} \int (a^2 + x^2)^{\frac{1}{2}n} dx.$$

Substituting this value in the previous equation, we obtain

$$2065 \quad \int (a^2 + x^2)^{\frac{1}{2}n} dx = \frac{x(a^2 + x^2)^{\frac{1}{2}n}}{n+1} + \frac{na^2}{n+1} \int (a^2 + x^2)^{\frac{1}{2}n-1} dx,$$

a result given at once by Formula VI.

If  $n$  be an odd integer, we arrive, finally, by successive reduction in this manner, at  $\int (a^2 + x^2)^{\frac{1}{2}} dx$  (1931).

2066 The integral  $\int \sin^m \theta \cos^p \theta d\theta$  is reducible by the foregoing Rules I. to VI., if, in applying them,  $n$  be always put equal to 2; if  $p$  be changed into  $p \pm 2$  instead of  $p \pm 1$ ; and if Division be always effected by separating the factor  $\cos^2 \theta = 1 - \sin^2 \theta$ .

PROOF.  $\int \sin^m \theta \cos^p \theta d\theta = \int x^m (1-x^2)^{\frac{1}{2}(p-1)} dx$ , where  $x = \sin \theta$ . Thus  $n = 2$  always, and the index  $\frac{1}{2}(p-1)$  is increased by 1 by adding 2 to  $p$ .

Thus, Rule I. gives the formula of reduction

2067

$$\int \sin^m \theta \cos^p \theta d\theta = \frac{\sin^{m+1} \theta \cos^{p-1} \theta}{m+1} + \frac{p-1}{m+1} \int \sin^{m+2} \theta \cos^{p-2} \theta d\theta.$$

But the integral can be found by substitution in the following cases:—

If  $r$  be a positive integer,

$$2068 \quad \int \cos^{2r+1} x \sin^p x dx = \int (1-z^2)^r z^p dz, \text{ where } z = \sin x.$$

$$2069 \quad \int \sin^{2r+1} x \cos^p x dx = -\int (1-z^2)^r z^p dz, \text{ where } z = \cos x.$$

If  $m+p = -2r$ ,

$$2070 \quad \int \sin^m x \cos^p x dx = \int (1+z^2)^{r-1} z^m dz, \text{ where } z = \tan x.$$

#### FUNCTIONS OF $a+bx \pm cx^2$ .

The seven following integrals are found either by writing

$$2071 \quad a+bx+cx^2 = \{(2cx+b)^2 + 4ac - b^2\} \div 4c,$$

and substituting  $2cx+b$ ; or by writing

$$2072 \quad a+bx-cx^2 = \{4ac+b^2 - (2cx-b)^2\} \div 4c,$$

and substituting  $2cx-b$ .

$$2073 \quad \int \frac{dx}{a+bx+cx^2} = \frac{1}{\sqrt{(b^2-4ac)}} \log \frac{2cx+b-\sqrt{(b^2-4ac)}}{2cx+b+\sqrt{(b^2-4ac)}}$$

$$\text{or } \frac{2}{\sqrt{(4ac-b^2)}} \tan^{-1} \frac{2cx+b}{\sqrt{(4ac-b^2)}},$$

according as  $b^2 >$  or  $< 4ac$  (2071, 1935-6).

$$2074 \quad \int \frac{dx}{a+bx-cx^2} = \frac{1}{b^2+4ac} \log \frac{\sqrt{(b^2+4ac)} + (2cx-b)}{\sqrt{(b^2+4ac)} - (2cx-b)}.$$

(2072, 1937)

2075

$$\int \frac{dx}{\sqrt{(a+bx+cx^2)}} = \frac{1}{\sqrt{c}} \log \{2cx+b+2\sqrt{c} \sqrt{(a+bx+cx^2)}\}.$$

$$2076 \quad \int \frac{dx}{\sqrt{(a+bx-cx^2)}} = \frac{1}{\sqrt{c}} \sin^{-1} \frac{2cx-b}{\sqrt{(4ac+b^2)}}.$$

(2071-2, 1928-9)

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$$2077 \quad \int \sqrt{a+bx+cx^2} \, dx = \frac{1}{4}c^{-\frac{1}{2}} \int \sqrt{y^2+4ac-b^2} \, dy.$$

$$2078 \quad \int \sqrt{a+bx-cx^2} \, dx = \frac{1}{4}c^{-\frac{1}{2}} \int \sqrt{4ac+b^2-y^2} \, dy,$$

where  $y = 2cx \pm b$ . The integrals are given at (1931-3).

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$$2079 \quad \int \frac{dx}{(a+bx+cx^2)^p} = 2^{2p-1} c^{p-1} \int \frac{dy}{(y^2+4ac-b^2)^p}.$$

[By (2071), the integral being reduced by (2062-3).]

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$$2080 \quad \int \frac{(lx+m) \, dx}{(a+bx+cx^2)^p} \\ = \frac{l}{2c} \int \frac{(2cx+b) \, dx}{(a+bx+cx^2)^p} + \left(m - \frac{bl}{2c}\right) \int \frac{dx}{(a+bx+cx^2)^p}.$$

The value of the second integral is  $(a+bx+cx^2)^{1-p} \div (1-p)$ , unless  $p=1$ , when the value is  $\log(a+bx+cx^2)$ . For the third, see (2079).

METHOD.—Decompose into two fractions, making the numerator of the first  $2cx+b$ ; that is, the derivative of  $a+bx+cx^2$ .

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$$2081 \quad \int \frac{px^2+q}{a+bx^2+cx^4} \, dx \text{ may be integrated as follows:—}$$

I. If  $b^2 > 4ac$ , put  $\alpha$  and  $\beta$  for  $\frac{-b \pm \sqrt{b^2-4ac}}{2c}$ , and, by Partial Fractions, the integral is resolved into

$$\frac{1}{c(\alpha-\beta)} \left\{ \int \frac{p\alpha+q}{x^2-\alpha} \, dx - \int \frac{p\beta+q}{x^2-\beta} \, dx \right\}. \quad (1936)$$


---

II. If  $b^2 < 4ac$ , put  $\frac{a}{c} = n^2$  and  $\frac{2\sqrt{(ac)-b}}{c} = m^2$ , and the integral may be decomposed into

$$2082 \quad \frac{1}{2mnc} \left\{ \int \frac{(q-pn)x+qm}{x^2+mx+n} \, dx - \int \frac{(q-pn)x-qm}{x^2-mx+n} \, dx \right\},$$

the value of which is found by (2080).

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III. If  $b^2 = 4ac$ ,

$$2083 \quad \int \frac{dx}{a+bx^2+cx^4} = \frac{x}{2a+bx^2} + \frac{1}{\sqrt{(2ab)}} \tan^{-1}\left(x\sqrt{\frac{b}{2a}}\right). \quad (2062)$$

2 x



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$$2084 \quad \int \frac{x \, dx}{a+bx^2+cx^4} = -\frac{2a}{b(2a+bx^2)}.$$

$$2085 \quad \int \frac{x^2 \, dx}{a+bx^2+cx^4} = \frac{2a}{b} \left\{ \frac{1}{\sqrt{(2ab)}} \tan^{-1} \left( x \sqrt{\frac{b}{2a}} \right) - \frac{x}{2a+bx^2} \right\}$$


---

REDUCTION OF  $\int x^m (a+bx^n+cx^{2n})^p \, dx$ .

NOTE.—In the following *Formulae of Reduction*, for the sake of clearness,  $x^m (a+bx^n+cx^{2n})^p$  is denoted by  $(m, p)$ , and the integral merely by  $\int (m, p)$ .

---

$$2086 \quad (m+1) \int (m, p) = (m+1, p) \\ -bnp \int (m+n, p-1) - 2cnp \int (m+2n, p-1) \dots\dots\dots(1).$$


---

$$2087 \quad bn(p+1) \int (m, p) = (m-n+1, p+1) \\ -(m-n+1) \int (m-n, p+1) - 2cn(p+1) \int (m+n, p) \dots(2).$$


---

$$2088 \quad 2cn(p+1) \int (m, p) = (m-2n+1, p+1) \\ -(m-2n+1) \int (m-2n, p+1) - bn(p+1) \int (m-n, p) \dots(3).$$


---

$$2089 \quad (m+np+1) \int (m, p) = (m+1, p) \\ +anp \int (m, p-1) - cnp \int (m+2n, p-1) \dots\dots\dots(4).$$


---

$$2090 \quad (m+2np+1) \int (m, p) = (m+1, p) \\ +2anp \int (m, p-1) + bnp \int (m+n, p-1) \dots\dots\dots(5).$$


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$$2091 \quad b(m+np+1) \int (m, p) = (m-n+1, p+1) \\ -a(m-n+1) \int (m-n, p) - c(m+2np+n+1) \int (m+n, p) \\ \dots\dots\dots(6).$$


---

$$2092 \quad bn(p+1) \int (m, p) = -(m-n+1, p+1) \\ +(m+2np+n+1) \int (m-n, p+1) - 2an(p+1) \int (m-n, p) \\ \dots\dots\dots(7).$$

---


$$\begin{aligned} 2093 \quad cn(p+1) \int (m, p) &= (m-2n+1, p+1) \\ &+ an(p+1) \int (m-2n, p) - (m+np-n+1) \int (m-2n, p+1) \\ &\dots\dots(8). \end{aligned}$$


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$$\begin{aligned} 2094 \quad an(p+1) \int (m, p) &= -(m+1, p+1) \\ &+ (m+np+n+1) \int (m, p+1) + cn(p+1) \int (m+2n, p) \dots(9). \end{aligned}$$


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$$\begin{aligned} 2095 \quad 2an(p+1) \int (m, p) &= -(m+1, p+1) \\ &+ (m+2np+2n+1) \int (m, p+1) - bn(p+1) \int (m+n, p) \\ &\dots\dots(10). \end{aligned}$$


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$$\begin{aligned} 2096 \quad a(m+1) \int (m, p) &= (m+1, p+1) \\ &- b(m+np+n+1) \int (m+n, p) - c(m+2np+2n+1) \int (m+2n, p) \\ &\dots\dots(11). \end{aligned}$$


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$$\begin{aligned} 2097 \quad c(m+2np+1) \int (m, p) &= (m-2n+1, p+1) \\ &- b(m+np-n+1) \int (m-n, p) - a(m-2n+1) \int (m-2n, p) \\ &\dots\dots(12). \end{aligned}$$


---

PROOF.—By differentiation, we have

$$2098 \quad \int (m, p) = m \int (m-1, p) + bnp \int (m+n-1, p-1) + 2cnp \int (m+2n-1, p-1).$$

Formulae (1), (2), and (3) are obtained from this equation by altering the indices  $m$  and  $p$ , so that each integral on the right, in turn, becomes  $\int (m, p)$ .

Again, by division,

$$2099 \quad \int (m, p) = a \int (m, p-1) + b \int (m+n, p-1) + c \int (m+2n, p-1) \dots(A).$$

And, by changing  $m$  into  $m-n$ , and  $p$  into  $p+1$ ,

$$2100 \quad \int (m-n, p+1) = a \int (m-n, p) + b \int (m, p) + c \int (m+n, p) \dots(B).$$

Formulae (4) to (12) may now be found as follows:—

(4), by eliminating  $\int (m+n, p-1)$  between (1) and (A);

(5), by eliminating  $\int (m+2n, p-1)$  between (1) and (A);

(6), by eliminating  $\int (m-n, p+1)$  between (2) and (B);

(7), by eliminating  $\int (m+n, p)$  between (2) and (B);

- (8), from (4), by changing  $m$  into  $m-2n$ , and  $p$  into  $p+1$ ;  
 (9), from (4), by changing  $p$  into  $p+1$ ;  
 (10), from (5), by changing  $p$  into  $p+1$ ;  
 (11), from (6), by changing  $m$  into  $m+n$ ;  
 (12), from (6), by changing  $m$  into  $m-n$ .

If  $a$  and  $\beta$  are real roots of the quadratic equation  $a+bx^n+cx^{2n}=0$ , then, by Partial Fractions,

$$2101 \quad \int \frac{x^m dx}{a+bx^n+cx^{2n}} = \frac{1}{c(a-\beta)} \left\{ \int \frac{x^m dx}{x^n-a} - \int \frac{x^m dx}{x^n-\beta} \right\},$$

and the integrals are obtained by (2021-2).

But, if the roots are imaginary,

$$2102 \quad \int \frac{x^m dx}{a+bx^{2n}+cx^{4n}} = \frac{1}{a} \left( \frac{a}{c} \right)^{\frac{m+1}{2n}} \int \frac{z^m dz}{1-2z^n \cos n\theta + z^{2n}}, \quad (2029)$$

where  $\cos n\theta = -\frac{b}{2\sqrt{ac}}$  and  $z = \left( \frac{c}{a} \right)^{\frac{1}{2n}} x$ .

$$2103 \quad \int \frac{x^m dx}{(a+bx+cx^2)^p} \text{ is reduced to (2079-80) by (2097).}$$

$$2104 \quad \int \frac{dx}{(x+h)\sqrt{(a+bx+cx^2)}} = -\int \frac{dy}{\sqrt{(A+By+Cy^2)}}, \quad (2075)$$

where  $y = (x+h)^{-1}$ ,  $A = c$ ,  $B = b-2ch$ ,  $C = a-bh+ch^2$ .

$$2105 \quad \int \frac{dx}{(x+h)\sqrt{x^2-1}} = \frac{1}{\sqrt{1-h^2}} \cos^{-1} \frac{1+hx}{x+h} = \frac{1}{\sqrt{h^2-1}} \cosh^{-1} \frac{1+hx}{x+h}.$$

$$2106 \quad \int \frac{dx}{(x+h)\sqrt{1-x^2}} = \frac{1}{\sqrt{h^2-1}} \sin^{-1} \frac{1+hx}{x+h}. \quad [\text{By (2181).}]$$

METHOD.—Substitute  $(x+h)^{-1}$ , as in (2104). Observe the cases in which  $h=1$ .

$$2107 \quad \int \frac{dx}{(x+h)^r \sqrt{(a+bx+cx^2)}} = -\int \frac{y^{r-1} dy}{\sqrt{(A+By+Cy^2)}},$$

with the same values for  $A$ ,  $B$ ,  $C$ , and  $y$  as in (2104). The integral is reduced by (2097).

$$2108 \quad \int \frac{(lx+m) dx}{(x^2+\beta^2) \sqrt{(a+bx+cx^2)}}.$$

METHOD.—Substitute  $\theta$  by putting  $x = \beta \tan(\theta + \gamma)$ , and determine the constant  $\gamma$  by equating to zero the coefficient of  $\sin 2\theta$  in the denominator.

The resulting integral is of the form  $\int \frac{L \cos \theta + M \sin \theta}{\sqrt{(P + Q \cos 2\theta)}} d\theta$ .

Separate this into two terms, and integrate by substituting  $\sin \theta$  in the first and  $\cos \theta$  in the second.

$$2109 \quad \int \frac{\phi(x) dx}{F(x) \sqrt{(a + bx + cx^2)}},$$

where  $\phi(x)$  and  $F(x)$  are rational algebraic functions of  $x$ , the former being of the lowest dimensions.

METHOD.—Resolve  $\frac{\phi(x)}{F(x)}$  into partial fractions. The resulting integrals are either of the form (2107), or else they arise from a pair of imaginary roots of  $F(x) = 0$ , and are of the type  $\int \frac{(Ax + B) dx}{\{(x - a)^2 + \rho^2\} \sqrt{(a + bx + cx^2)}}$ . Substitute  $x - a$  in this, and the integral (2108) is obtained.

## INTEGRATION BY RATIONALIZATION.

In the following articles,  $F$  denotes a rational algebraic function. In each case, an integral involving an irrational function of  $x$  is, by substitution, made to take the form  $\int F(z) dz$ . This latter integral can always be found by the method of Partial Fractions (1915).

$$2110 \quad \int F \left\{ x^n, \left( \frac{a + bx}{f + gx} \right)^{\frac{p}{q}}, \left( \frac{a + bx}{f + gx} \right)^{\frac{r}{s}}, \&c. \right\} dx.$$

Substitute  $\frac{a + bx}{f + gx} = z^l$ , where  $l$  is the least common denominator of the fractional indices; thus,

$$x = \frac{fz^l - a}{b - gz^l}, \quad \frac{dx}{dz} = \frac{lz^{l-1}(bf - ag)}{(b - gz^l)^2}, \quad \left( \frac{a + bx}{f + gx} \right)^{\frac{p}{q}} = z^{\frac{lp}{q}}, \&c.,$$

the powers of  $z$  being now all integral.

$$2111 \quad \int x^{n-1} F \left\{ x^{mn}, \left( \frac{a + bx^n}{f + gx^n} \right)^{\frac{p}{q}}, \left( \frac{a + bx^n}{f + gx^n} \right)^{\frac{r}{s}}, \&c. \right\} dx.$$

Reduce to the form of (2110) by substituting  $x^n$ .

$$2112 \quad \int F \{ \sqrt{(a + \sqrt{mx + n})} \} dx. \quad \text{Subs. } \sqrt{(a + \sqrt{mx + n})}.$$

$$2113 \quad \int F \{ x, \sqrt{(bx \pm cx^2)} \} dx. \quad \text{Substitute } x = \frac{b}{z^2 \mp c}.$$

And therefore  $\sqrt{(bx \pm cx^2)} = \frac{bz}{z^2 \mp c}, \quad \frac{dx}{dz} = -\frac{2bz}{(z^2 \mp c)^2}.$

$$2114 \quad \int F \{ x, \sqrt{(a + bx + cx^2)} \} dx.$$

Writing  $Q$  for  $a + bx + cx^2$ ,  $F$  may always be reduced to the form  $\frac{A + B\sqrt{Q}}{C + D\sqrt{Q}}$ , in which  $A, B, C, D$  are constants or rational functions of  $x$ . Rationalizing this fraction, it takes the form  $L + M\sqrt{Q}$ . Thus the integral becomes  $\int L dx + \int M\sqrt{Q} dx$ , the first of which two integrals is rational, while the second is equivalent to  $\int \frac{MQ}{\sqrt{Q}} dx$ , which is of the form in (2075).

**2115** OTHERWISE.—(i.) When  $c$  is positive, the integral may be made rational by substituting

$$x = \frac{a - cz^2}{2cz - b}, \quad \frac{dx}{dz} = \frac{2c(bz - cz^2 - a)}{(2cz - b)^2}, \quad \sqrt{(a + bx + cx^2)} = \sqrt{c} \left( \frac{a - cz^2}{2cz - b} + z \right).$$

(ii.) When  $c$  is negative, let  $\alpha, \beta$  be the roots of the equation  $a + bx - cx^2 = 0$ , which are necessarily real ( $a, b$ , and  $c$  being now all positive), so that  $a + bx - cx^2 = c(x - \alpha)(\beta - x)$ . The integral is now made rational by substituting

$$x = \frac{c\alpha z^2 + \beta}{cz^2 + 1}, \quad \frac{dx}{dz} = \frac{2(\alpha - \beta)cz}{(cz^2 + 1)^2}, \quad \sqrt{(a + bx - cx^2)} = \frac{(\beta - \alpha)cz}{cz^2 + 1}.$$

In each case the result is of the form  $\int F(z) dz$ .

$$2116 \quad \int x^m F \{ x^n, \sqrt{a + bx^n + cx^{2n}} \} dx,$$

when  $\frac{m+1}{n}$  is an integer, is reduced to the form (2114) by substituting  $x^n$ .

**2117**

$$\int F \{ x, \sqrt{(a + bx)}, \sqrt{(f + gx)} \} dx. \quad \text{Substitute } z^2 = \frac{a + bx}{f + gx},$$

and, therefore,  $x = \frac{a-fz^2}{gz^2-b}$ ,  $\sqrt{(a+bx)} = \frac{z\sqrt{(ag-bf)}}{\sqrt{(gz^2-b)}}$ ,

$$\sqrt{(f+gx)} = \frac{\sqrt{(ag-bf)}}{\sqrt{(gz^2-b)}}, \quad \frac{dx}{dz} = \frac{2(bf-ag)z}{(gz^2-b)^2}.$$

The form  $\int F\{z, \sqrt{(gz^2-b)}\} dz$  is obtained, which is comprehended in (2114).

**2118**  $\int x^m F\{x^n, \sqrt{(a+b^2x^{2n})}, bx^n \pm \sqrt{(a+b^2x^{2n})}\} dx$ ,

when  $\frac{m+1}{n}$  is an integer, is reduced to the form  $\int F(z) dz$

by substituting  $z = bx^n \pm \sqrt{(a+b^2x^{2n})}$ , and therefore

$$x^n = \frac{z^2-a}{2bz}; \quad \sqrt{(a+b^2x^{2n})} = \frac{z^2+a}{2z};$$

$$x^m \frac{dx}{dz} = \frac{1}{2nb} \left( \frac{z^2-a}{2bz} \right)^{\frac{m+1}{n}-1} \left( \frac{z^2+a}{z^2} \right).$$

**2119**  $\int x^m (a+bx^n)^{\frac{p}{q}} F(x^n) dx$  is rationalized by substituting either  $(a+bx^n)^{\frac{1}{q}}$  or  $(ax^{-n}+b)^{\frac{1}{q}}$  according as  $\frac{m+1}{n}$  or  $\frac{m+1}{n} + \frac{p}{q}$  is integral, whether positive or negative.

**2120**  $\int x^{m-1} F\{x^m, x^n, (a+bx^n)^{\frac{p}{q}}\} dx$ , when  $\frac{m}{n}$  is either a positive or negative integer, is rationalized by substituting  $(a+bx^n)^{\frac{1}{q}}$ .

## INTEGRALS REDUCIBLE TO ELLIPTIC INTEGRALS.

**2121**  $\int F\{x, \sqrt{(a+bx+cx^2+dx^3+ex^4)}\} dx$ .

Writing  $X$  for the quartic, the rational function  $F$  may always be brought to the form  $\frac{P+Q\sqrt{X}}{P'+Q'\sqrt{X}}$ ,

and this again, by rationalizing the denominator, to the form



$M+N\sqrt{X}$ , where  $P, Q, P', Q', M, N$  are all rational functions of  $x$ .  $\int Mdx$  has already been considered (1915).

$$\int N\sqrt{X} dx = \int \frac{NX}{\sqrt{X}} dx \text{ or } \int \frac{R dx}{\sqrt{X}}, \text{ where } R \text{ is rational.}$$

By substituting  $x = \frac{p+qy}{1+y}$ , and determining  $p$  and  $q$  so that the odd powers of  $y$  in the denominator may vanish, the last integral is brought to the form

$$2122 \quad \int \frac{R dy}{\sqrt{(a+by^2+cy^4)}}.$$

$R$  being a rational function of  $y$ , may be expressed as the sum of an odd and an even function; thus the integral is equivalent to the two

$$2123 \quad \int \frac{yF_1(y^2) dy}{\sqrt{(a+by^2+cy^4)}} + \int \frac{F_2(y^2) dy}{\sqrt{(a+by^2+cy^4)}}.$$

The first integral can be found by substituting  $\sqrt{y}$ .

The second, by substituting  $\frac{a+bx^2}{c+dx^2}$  for  $y^2$ , can be made to depend upon three integrals of the forms

$$2124 \quad \int \frac{dx}{\sqrt{1-x^2} \cdot 1-k^2x^2}, \quad \int \frac{\sqrt{1-k^2x^2} dx}{\sqrt{1-x^2}},$$

$$\int \frac{dx}{(1+nx^2) \sqrt{1-x^2} \cdot 1-k^2x^2}.$$

By substituting  $\phi = \sin^{-1} x$ , the above become

$$2125 \quad \int \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}}, \quad \int \sqrt{1-k^2 \sin^2 \phi} d\phi,$$

$$\int \frac{d\phi}{(1+n \sin^2 \phi) \sqrt{1-k^2 \sin^2 \phi}}.$$

These are the transcendental functions known as *Elliptic Integrals*. They are denoted respectively by

$$2126 \quad F(k, \phi), \quad E(k, \phi), \quad \Pi(n, k, \phi).$$

APPROXIMATIONS TO  $F(k, \phi)$  AND  $E(k, \phi)$  IN SERIES.

When  $k$  is less than unity, the values of  $F(k, \phi)$  and  $E(k, \phi)$ , from the origin  $\phi = 0$ , in converging series, are

$$2127 \quad F(k, \phi) = \phi - \frac{k^2}{2^3} A_2 + \frac{1.3.k^4}{2.4.2^3} A_4 - \frac{1.3.5.k^6}{2.4.6.2^5} A_6 + \dots$$

$$\dots + (-1)^{\frac{1}{2}n} \frac{1.3.5 \dots n-1}{2.4.6 \dots n} \frac{k^n}{2^{n-1}} A_n, \text{ \&c.}$$

$$2128 \quad E(k, \phi) = \phi + \frac{k^2}{2^3} A_2 - \frac{1.k^4}{2.4.2^3} A_4 + \frac{1.3.k^6}{2.4.6.2^5} A_6 - \dots$$

$$+ (-1)^{\frac{1}{2}n+1} \frac{1.3.5 \dots n-3}{2.4.6 \dots n} \frac{k^n}{2^{n-1}} A_n, \text{ \&c.,}$$

[ $n$  being an even integer.]

where

$$A_2 = \frac{\sin 2\phi}{2} - \phi,$$

$$A_4 = \frac{\sin 4\phi}{4} - \frac{4 \sin 2\phi}{2} + 3\phi,$$

$$A_6 = \frac{\sin 6\phi}{6} - \frac{6 \sin 4\phi}{4} + \frac{15 \sin 2\phi}{2} - 10\phi,$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$A_n = \frac{\sin n\phi}{n} - \frac{n \sin (n-2)\phi}{n-2} + \frac{C(n, 2) \sin (n-4)\phi}{n-4}$$

$$- \frac{C(n, 3) \sin (n-6)\phi}{n-6} + \dots + (-1)^{\frac{1}{2}n} \frac{1}{2} C(n, \frac{1}{2}n) \phi.$$

PROOF.—In each case expand by the Binomial Theorem; substitute from (773) for the powers of  $\sin \phi$ , and integrate the separate terms.

The values of  $F(k, \phi)$  and  $E(k, \phi)$ , between the limits  $\phi = 0$ ,  $\phi = \frac{1}{2}\pi$ , are therefore

$$2129 \quad F\left(k, \frac{\pi}{2}\right) = \frac{\pi}{2} \left\{ 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \left(\frac{1.3.5}{2.4.6}\right)^2 k^6 + \&c. \right\}.$$

$$2130 \quad E\left(k, \frac{\pi}{2}\right) = \frac{\pi}{2} \left\{ 1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1.3}{2.4}\right)^2 \frac{k^4}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{k^6}{5} - \&c. \right\}.$$

But series which converge more rapidly are

$$2131 \quad F\left(k, \frac{\pi}{2}\right) = \frac{\pi(1+n)}{2} \left\{ 1 + \left(\frac{1}{2}\right)^2 n^2 + \left(\frac{1.3}{2.4}\right)^2 n^4 + \left(\frac{1.3.5}{2.4.6}\right)^2 n^6 + \&c. \right\}$$

2 Y

**2132**

$$E\left(k, \frac{\pi}{2}\right) = \frac{\pi}{2(1+n)} \left\{ 1 + \left(\frac{1}{2}\right)^2 n^2 + \left(\frac{1}{2.4}\right)^2 n^4 + \left(\frac{1.3}{2.4.6}\right)^2 n^6 + \&c. \right\}$$

where 
$$n = \frac{1 - \sqrt{(1-k^2)}}{1 + \sqrt{(1-k^2)}}.$$

**2133**  $\int \frac{F(x) dx}{\sqrt{(a+bx+cx^2+bx^3+ax^4)}}$ , when  $F(x)$  can be expressed in the form  $\left(x - \frac{1}{x}\right)f\left(x + \frac{1}{x}\right)$ , is integrated by substituting  $x + \frac{1}{x}$ .

If  $b$  is negative, and  $F(x)$  of the form  $\left(x + \frac{1}{x}\right)f\left(x - \frac{1}{x}\right)$ ; substitute  $x - \frac{1}{x}$ .

**2134**

$$\frac{F(x) dx}{\sqrt{(a+bx+cx^2+dx^3+cx^4+bx^5+ax^6)}}.$$

Substitute 
$$x + \frac{1}{x} = z.$$

Hence 
$$\frac{dx}{\sqrt{x^3}} = \left( \frac{1}{2\sqrt{z-2}} - \frac{1}{2\sqrt{z+2}} \right) dz,$$

and the integral takes the form

$$\int \frac{P + Q \sqrt{z^2 - 4}}{\sqrt{\{a(z^3 - 3z) + b(z^2 - 2) + cz + d\}}} \cdot \frac{\sqrt{z+2} - \sqrt{z-2}}{2\sqrt{z^2 - 4}} dz,$$

where  $P, Q$  are rational functions of  $z$ . Writing  $Z$  for the cubic in  $z$ , we see that the integral depends upon

$$\int \frac{P dz}{\sqrt{Z(z \pm 2)}} \quad \text{and} \quad \int \frac{Q(z \pm 2) dz}{\sqrt{Z(z \pm 2)}},$$

the radicals in which contain no higher power of  $z$  than the fourth. The integrals therefore fall under (2121).

**2135**

$$\int \frac{F(x) dx}{\sqrt{(a+bx^2+cx^4+bx^6+ax^8)}}.$$

Expressing  $F(x)$  as the sum of an odd and an even function, as in (2123), the integral is divided into two; and, by substituting  $x^2$ , the first of these is reduced to the form in (2121), and the second to the form in (2134) with  $a = 0$ .

2136

$$\int \frac{F(x) dx}{\sqrt[3]{(a+bx+cx^2+dx^3)}}.$$

Put  $x = y + a$ ,  $a$  being a root of the equation  $a + bx + cx^2 + dx^3 = 0$ ; and, in the resulting integral, substitute  $zy$  for the denominator. The form finally obtained will be

$$\int (P + Q \sqrt{\alpha + \beta z^3}) dz,$$

which falls under (2134),  $P$  and  $Q$  being rational functions of  $z$ .

2137

$$\int \frac{F(x) dx}{\sqrt[4]{(a+bx^2+cx^4)}}.$$

Expressing  $F(x)$  as the sum of an odd and an even function, as in (2123), two integrals are obtained. By putting the denominator equal to  $z$  in the first, and equal to  $xz$  in the second, each is reducible to an integral of the form

$$\int (P + Q \sqrt{\alpha + \beta z^4}) dz,$$

which falls under (2121).

2138

$$\int \frac{dx}{\sqrt{(1-x^4)}} = -\frac{1}{\sqrt{2}} F\left(\frac{1}{\sqrt{2}}, \phi\right). \quad [\text{Expand by (2127).}]$$

2139

$$\int \frac{dx}{\sqrt{1+x^4}} = \frac{1}{2} F\left(\frac{1}{\sqrt{2}}, \phi\right).$$

PROOF.—Substitute  $\cos^{-1}x$  in (2138), and  $2 \tan^{-1}x$  in (2139).

2140

$$\int \frac{dx}{\sqrt{(2ax-x^2)(b-x)}} = \frac{2}{\sqrt{b}} F\left(\sqrt{\frac{2a}{b}}, \phi\right) \text{ or } \sqrt{\frac{2}{a}} F\left(\sqrt{\frac{b}{2a}}, \phi\right),$$

according as  $b$  is  $>$  or  $<$   $2a$ .

PROOF.—Substitute accordingly,  $x = 2a \sin^2 \phi$  or  $x = b \sin^2 \phi$ .

2141

$$\int \frac{dx}{\sqrt{(a-x)(x-b)(x+c)}} = -\frac{2}{\sqrt{(a+c)}} F\left(\sqrt{\frac{a-b}{a+c}}, \phi\right).$$

PROOF.—Substitute  $x = a - (a-b) \sin^2 \phi$ ,  $x$  being  $< a$  and  $> b$ .

### SUCCESSIVE INTEGRATION.

**2148** In conformity with the notation of (1487), let the operation of integrating a function  $v$ , once, twice, ...  $n$  times for  $x$ , be denoted either by

$$\int_x v, \int_{2x} v, \dots \int_{nx} v, \quad \text{or by} \quad d_{-x}, d_{-2x}, \dots d_{-nx},$$

the notation  $d_{-x}$  indicating an operation which is the inverse of  $d_x$ . Similarly, since  $y_x, y_{2x}, y_{3x}$ , &c. denote successive derivatives of  $y$ , so  $y_{-x}, y_{-2x}, y_{-3x}$ , &c. may be taken to represent the successive integrals of  $y$  with respect to  $x$ .

**2149** Since a constant is added to the result of each integration, every integral of the  $n^{\text{th}}$  order of a function of a single variable  $x$  must be supplemented by the quantity

$$\frac{a_1 x^{n-1}}{n-1} + \frac{a_2 x^{n-2}}{n-2} + \dots + a_{n-1} x + a_n = \int_{nx} 0,$$

where  $a_1, a_2, a_3 \dots a_n$  are arbitrary constants.

#### *Examples.*

The six following integrals are obtained from (1922) and (1923).

When  $p$  is any positive quantity,

$$\mathbf{2150} \quad \int_{nx} x^p = \frac{x^{p+n}}{(p+1)(p+2) \dots (p+n)} + \int_{nx} 0.$$

When  $p$  is any positive quantity not an integer, or any positive integer greater than  $n$ ,

$$\mathbf{2151} \quad \int_{nx} \frac{1}{x^p} = \frac{(-1)^n}{(p-1)(p-2) \dots (p-n) x^{p-n}} + \int_{nx} 0.$$

When  $p$  is a positive integer not greater than  $n$ , the following cases occur—

$$\mathbf{2152} \quad \int_{px} \frac{1}{x^p} = \frac{(-1)^{p-1}}{p-1} \log x + \int_{px} 0.$$

$$\mathbf{2153} \quad \int_{(p+1)x} \frac{1}{x^p} = \frac{(-1)^{p-1}}{p-1} (x \log x - x) + \int_{(p+1)x} 0,$$

$$2154 \quad \int_{(p+r)x} \frac{1}{x^p} = \frac{(-1)^{p-1}}{[p-1]} \left\{ \int_{(r-1)x} (x \log x) - \frac{x^r}{[r]} \right\} + \int_{(p+r)x} 0.$$

For the integral within the brackets, see (2166).

The following formula is analogous to (1461-2)

$$2155 \quad \int_{nx} \frac{\sin}{\cos} \} ax = \frac{1}{a^n} \frac{\sin}{\cos} \} (ax - \frac{1}{2}n\pi) + \int_{nx} 0,$$

#### SUCCESSIVE INTEGRATION OF A PRODUCT.

Leibnitz's Theorem (1460) and its analogue in the Integral Calculus are briefly expressed by the two equations

$$2157 \quad D_{nx}(uv) = (d_x + \delta_x)^n uv, \quad D_{-nx}(uv) = (d_x + \delta_x)^{-n} uv;$$

where  $D$  operates upon the product  $uv$ ,  $d$  only upon  $u$ , and  $\delta$  only upon  $v$ . Expanding the binomials, we get

$$2159 \quad D_{nx}(uv) = u_{nx}v + nu_{(n-1)x}v_x + \frac{n(n-1)}{1.2}u_{(n-2)x}v_{2x} + \&c.$$

$$2160 \quad D_{-nx}(uv) = u_{-nx}v - nu_{-(n+1)x}v_x + \frac{n(n+1)}{1.2}u_{-(n+2)x}v_{2x} - \&c.$$

PROOF.—The first equation is obtained in (1460). The second follows from the first by the operative law (1483); or it may be proved by Induction, independently, as follows—

Writing it in the equivalent form

$$\int_{nx} (uv) = \int_{nx} uv - n \int_{(n+1)x} uv_x + \frac{n(n+1)}{1.2} \int_{(n+2)x} uv_{2x} - \&c. \dots\dots (i),$$

make  $n = 1$ ; then

$$\int_x (uv) = \int_x uv - \int_{2x} uv_x + \int_{3x} uv_{2x} - \&c. \dots\dots\dots (ii),$$

a result which may be obtained directly by integrating the left member successively by Parts. Now integrate equation (i.) once more for  $x$ , integrating each term on the right as a product by formula (ii.), and equation (i.) will be reproduced with  $(n+1)$  in the place of  $n$ .

$$2161 \quad \int_{nx} e^{ax} x^m = e^{ax} (a + d_x)^{-n} x^m + \int_{nx} 0. \quad \text{Or, by expansion,}$$

$$2162 \quad \int_{nx} e^{ax} x^m = \frac{e^{ax}}{a^n} \left\{ x^m - \frac{nm}{a} x^{m-1} + \frac{n(n+1)}{1.2} \frac{m(m-1)}{a^2} x^{m-2} - \&c. \right\} + \int_{nx} 0.$$

If  $m$  be an integer, the series terminates with  $(-1)^m n^{(m)} \div a^m$ .



Similarly, by changing the sign of  $m$ ,

**2163**

$$\int_{nx} \frac{e^{ax}}{x^m} = \frac{e^{ax}}{a^n} \left\{ \frac{1}{x^m} + \frac{nm}{ax^{m+1}} + \frac{n(n+1)m(m+1)}{1 \cdot 2 a^2 x^{m+2}} + \&c. \right\} + \int_{nx} 0.$$

PROOF.—Putting  $u = e^{ax}$ ,  $v = x^m$  in (2158), the formula becomes

$$\int_{nx} e^{ax} x^m = (d_x + \delta_x)^{-n} e^{ax} x^m = (d_x e^{ax} + e^{ax} \delta_x)^{-n} x^m = e^{ax} (a + \delta_x)^{-n} x^m.$$

Here  $e^{ax}$  is written before  $\delta_x$  within the brackets, because  $\delta$  does not operate upon  $e^{ax}$ . Observe, also, that the index  $-n$  affects only the operative symbols  $d_x$  and  $\delta_x$ , but it therefore affects the *results* of those operations. Thus, since  $d_x e^{ax}$  produces  $ae^{ax}$ , the operation  $d_x$  is equivalent to  $aX$ , and is retained within the brackets, while the subject  $e^{ax}$ , being only now connected as a factor with each term in the expansion of  $(a + \delta_x)^{-n}$ , may be placed on the left.

**2164** 
$$\int e^{ax} x^m dx = \frac{e^{ax}}{a} \left\{ x^m - \frac{m}{a} x^{m-1} + \frac{m(m-1)}{a^2} x^{m-2} - \&c. \right\}$$

**2165** 
$$\int \frac{e^{ax}}{x^m} dx = \frac{e^{ax}}{a} \left\{ \frac{1}{x^m} + \frac{m}{ax^{m+1}} + \frac{m(m+1)}{a^2 x^{m+2}} + \&c. \right\}$$

PROOF.—Make  $n = 1$  in (2162) and (2163).

**2166** 
$$\int_{nx} x^p (\log x)^m = \int_{nx} e^{(p+n)x} x^m. \quad [\text{Subs. } \log x.]$$

Hence the integral of the logarithmic function may be obtained from that of the equivalent exponential function (2161).

For another method, see (2003-5).

## HYPERBOLIC FUNCTIONS.

**2180** DEFINITIONS.—The hyperbolic cosine, sine, and tangent are written and defined as follows:—

**2181** 
$$\cosh x = \frac{1}{2} (e^x + e^{-x}) = \cos(ix). \quad (768)$$

**2183** 
$$\sinh x = \frac{1}{2} (e^x - e^{-x}) = -i \sin(ix).$$

**2185** 
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = -i \tan(ix). \quad (770)$$

By these equations the following relations are readily obtained.

$$2187 \quad \cosh 0 = 1; \quad \sinh 0 = 0; \quad \cosh \infty = \sinh \infty = \infty.$$

$$2191 \quad \cosh^2 x - \sinh^2 x = 1.$$

$$2192 \quad \sinh (x+y) = \sinh x \cosh y + \cosh x \sinh y.$$

$$2193 \quad \cosh (x+y) = \cosh x \cosh y + \sinh x \sinh y.$$

$$2194 \quad \tanh (x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}.$$

$$2195 \quad \sinh 2x = 2 \sinh x \cosh x.$$

$$2196 \quad \cosh 2x = \cosh^2 x + \sinh^2 x.$$

$$2197 \quad = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x.$$

$$2199 \quad \sinh 3x = 3 \sinh x + 4 \sinh^3 x.$$

$$2200 \quad \cosh 3x = 4 \cosh^3 x - 3 \cosh x.$$

$$2201 \quad \tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}.$$

$$2202 \quad \tanh 3x = \frac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}.$$

$$2203 \quad \sinh \frac{x}{2} = \sqrt{\frac{\cosh x - 1}{2}}; \quad \cosh \frac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}}.$$

$$2205 \quad \tanh \frac{x}{2} = \sqrt{\frac{\cosh x - 1}{\cosh x + 1}} = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1}.$$

$$2208 \quad \cosh x = \frac{1 + \tanh^2 \frac{1}{2}x}{1 - \tanh^2 \frac{1}{2}x}; \quad \sinh x = \frac{2 \tanh \frac{1}{2}x}{1 - \tanh^2 \frac{1}{2}x}.$$

#### INVERSE RELATIONS.

$$2210 \quad \text{Let } u = \cosh x, \therefore x = \cosh^{-1} u = \log (u + \sqrt{u^2 - 1}).$$

$$2211 \quad v = \sinh x, \therefore x = \sinh^{-1} v = \log (v + \sqrt{v^2 + 1}).$$

$$2212 \quad w = \tanh x, \therefore x = \tanh^{-1} w = \frac{1}{2} \log \left( \frac{1+w}{1-w} \right).$$

GEOMETRICAL INTERPRETATION OF  $\tanh S$ .

**2213** *The tangent of the angle which a radius from the centre of a rectangular hyperbola makes with the principal axis, is equal to the hyperbolic tangent of the included area.*

PROOF.—Let  $\theta$  be the angle,  $r$  the radius, and  $S$  the area, in the hyperbola  $x^2 - y^2 = 1$  or  $r^2 = \sec 2\theta$ ; then

$$S = \int_0^\theta \frac{1}{2} \sec 2\theta d\theta = \frac{1}{2} \log \tan \left( \frac{1}{4}\pi + \theta \right). \quad (1942)$$

$$\text{Therefore } e^{2S} = \frac{1 + \tan \theta}{1 - \tan \theta}; \quad \text{therefore } \tan \theta = \frac{e^S - e^{-S}}{e^S + e^{-S}} = \tanh S. \quad (2185)$$

## VALUE OF THE LOGARITHM OF AN IMAGINARY QUANTITY.

$$\mathbf{2214} \quad \log(a+ib) = \frac{1}{2} \log(a^2+b^2) + i \tan^{-1} \frac{b}{a}.$$

$$\text{PROOF.—} \quad \log \frac{a+ib}{\sqrt{(a^2+b^2)}} = \log \sqrt{\frac{1+i\frac{b}{a}}{1-i\frac{b}{a}}} = i \tan^{-1} \frac{b}{a}. \quad \text{By (771).}$$

## DEFINITE INTEGRALS.

## SUMMATION OF SERIES BY DEFINITE INTEGRALS.

$$\mathbf{2230} \quad \int_a^b f(x) dx = [f(a) + f(a+dx) + \dots + f(a+ndx)] dx,$$

where  $n$  increases and  $dx$  diminishes indefinitely, so that  $ndx = b-a$  in the limit.

**2231** Ex. 1.—To find the sum, when  $n$  is infinite, of the series

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} \dots + \frac{1}{n+n}. \quad \text{Put } n = \frac{a}{dx}; \text{ thus,}$$

$$\frac{dx}{a} + \frac{dx}{a+dx} + \frac{dx}{a+2dx} + \dots + \frac{dx}{2a} = \int_a^{2a} \frac{dx}{x} = \log 2.$$

**2232** Ex. 2.—To find the sum, when  $n$  is infinite, of the series

$$\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \dots + \frac{n}{n^2+n^2}. \quad \text{Put } n = \frac{1}{dx}, \text{ then}$$

$$\frac{dx}{1+(dx)^2} + \frac{dx}{1+(2dx)^2} + \dots + \frac{dx}{1+(ndx)^2} = \int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}. \quad (1935)$$

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THEOREMS RESPECTING THE LIMITS OF  
INTEGRATION.

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**2233**  $\int_0^a \phi(x) dx = \int_0^a \phi(a-x) dx.$  [Substitute  $a-x$ .

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**2234**  $\int_0^a \phi(x) dx = 2 \int_0^{\frac{1}{2}a} \phi(x) dx,$

or zero, according as  $\phi(x) = \pm \phi(a-x)$  for all values of  $x$  between 0 and  $a$ .

Ex.—  $\int_0^\pi \sin x dx = 2 \int_0^{\frac{\pi}{2}} \sin x dx. \qquad \int_0^\pi \cos x dx = 0.$

---

If  $\phi(x) = \phi(-x)$ , that is, if  $\phi(x)$  be an even function (1401) for all values of  $x$  between 0 and  $a$ .

**2236**  $\int_{-a}^0 \phi(x) dx = \int_0^a \phi(x) dx = \frac{1}{2} \int_{-a}^a \phi(x) dx.$

Ex.—  $\int_{-\frac{\pi}{2}}^0 \cos x dx = \int_0^{\frac{\pi}{2}} \cos x dx = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx.$

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If  $\phi(x) = -\phi(-x)$ , that is, if  $\phi(x)$  be an odd function for all values of  $x$  between 0 and  $a$ .

**2238**  $\int_{-a}^0 \phi(x) dx = -\int_0^a \phi(x) dx \quad \text{and} \quad \int_{-a}^a \phi(x) dx = 0.$

Ex.—  $\int_{-\frac{\pi}{2}}^0 \sin x dx = -\int_0^{\frac{\pi}{2}} \sin x dx \quad \text{and} \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x dx = 0.$

Given  $a < c < b$ , and that  $x=c$  makes  $\phi(x)$  infinite, the value of  $\int_a^b \phi(x) dx$  may be investigated by putting  $\mu=0$ , after integrating, in the formula

**2240**  $\int_a^b \phi(x) dx = \int_a^{c-\mu} \phi(x) dx + \int_{c+\mu}^b \phi(x) dx.$

If the function  $\phi(x)$  changes sign on becoming infinite, this expression, when  $\mu$  is an indefinitely small quantity, is called the *principal value* of the integral.

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Ex.—  $\int_{-1}^1 \frac{dx}{x^3} = \int_{-1}^{-\mu} \frac{dx}{x^3} + \int_{\mu}^1 \frac{dx}{x^3} = -\frac{1}{2\mu^2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{2\mu^2} = 0,$

which is the *principal value*. If, however,  $\mu$  be made to vanish, the expression takes the indeterminate form  $\infty - \infty$ .

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**2241** Given  $a < c < b$ , the integral  $\int_a^b \frac{\psi(x) dx}{(x-c)^n}$  will always be finite in value while  $n$  is less than unity.

PROOF.—Let  $\mu$  in (2240) be taken so near to  $c$  in value that  $\psi(x)$  shall remain finite and of the same sign for all values of  $x$  comprised between  $c \pm \mu$ . Then the part of the integral in which the fraction becomes infinite, and which is omitted in (2240), will be equal to  $\int_{c-\mu}^{c+\mu} \frac{dx}{(x-c)^n}$ , multiplied by a constant whose value lies between the greatest and least values of  $\psi(x)$  which occur between  $\psi(c-\mu)$  and  $\psi(c+\mu)$ . By integration it appears that the last integral is finite in value when  $n$  is  $< 1$ .

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**2242**  $\int_a^b f(x) dx = (b-a)f\{a+\theta(b-a)\},$

where  $\theta$  lies between 0 and 1 in value.

The equation expresses the fact that the area in (Fig. 1901), bounded by the curve  $y = f(x)$ , the ordinates  $f(a)$ ,  $f(b)$ , and the base  $b-a$  is equal to the rectangle under  $b-a$  and some ordinate lying in value between the greatest and least which occur in passing from  $f(a)$  to  $f(b)$ .

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If  $\psi(x)$  does not change sign while  $x$  varies from  $x=a$  to  $x=b$ ,

**2243**  $\int_a^b f(x) \psi(x) dx = f\{a+\theta(b-a)\} \int_a^b \psi(x) dx.$

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**2244** If  $\phi\left(x, \frac{1}{x}\right)$  is a symmetrical function of  $x$  and  $\frac{1}{x}$ ,

$$\int_0^\infty \phi\left(x, \frac{1}{x}\right) \frac{dx}{x} = 2 \int_0^1 \phi\left(x, \frac{1}{x}\right) \frac{dx}{x}.$$

PROOF.—Separate the integral into two parts by the formula  $\int_0^\infty = \int_0^1 + \int_1^\infty$ , and substitute  $\frac{1}{x}$  in the last integral.

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## METHODS OF EVALUATING DEFINITE INTEGRALS.

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**2245** RULE I.—Substitute a new variable, and adjust the limits accordingly.

For examples, see numbers 2291, 2308, 2342, 2345, 2416, 2425, 2457, 2506, 2605, &c.

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**2246** RULE II.—*Integrate by Parts (1910), so as to introduce a known definite integral.*

*For examples, see numbers 2283, 2290, 2430, 2453, 2465, 2484-5, 2608-13, 2623, 2625, &c.*

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**2247** RULE III.—*Differentiate or integrate with respect to some quantity other than the variable concerned; if a known integral is thus obtained, evaluate it, and then reverse the operation of differentiation or integration before performed with respect to the secondary variable.*

*For examples, see numbers 2346-7, 2364, 2391, 2417, 2421-4, 2426, 2428, 2497-8, 2502-4, 2571, 2575-6, 2591, 2604, 2614, 2617-8, 2632, &c.*

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**2248** RULE IV.—*Substitute imaginary values for constants, and thus transform the expression into one capable of integration.*

*For examples, see numbers 2430, 2494, 2577, 2594, 2598, 2603, 2606, 2615, 2641-2.*

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**2249** RULE V.—*Expand the function, if possible, in a finite or converging series, and integrate the separate terms.*

*For examples, see numbers 2395-7, 2402-3, 2418-9, 2479, 2506, 2571, 2593, 2598, 2614, 2620, 2625, 2629, 2630-2, 2639.*

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**2250** RULE VI.—*Decompose the integral into a number of partial integrals, and change all these by some substitution into integrals having the same limits. By summing the resulting series, a new integral is obtained which may be a known one.*

*For examples, see numbers 2341, 2356-61, 2572, 2638.*

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**2251** RULE VII.—*Separate the function to be integrated into two factors, and replace one of them by its value in the form of a definite integral taken between constant limits with respect to some new variable. The double integral so obtained may frequently be evaluated by changing the order of integration as explained in (2261).*

*For examples, see numbers 2507, 2510, 2573, 2619.*

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**2252** RULE VIII.—*Multiply a known definite integral which is discontinuous between certain values of a constant which it*



contains, by some function of that constant, such that the integral of the product with respect to the constant is known. A new definite integral may thus be obtained.

For examples, see numbers 2518, 2522.

Particular artifices not included in the foregoing rules are employed in 2293, 2305, 2310, 2314-5, 2317, 2367-9, 2404-15, 2422, 2429, 2456, 2495, 2514, 2518, 2585, 2600, 2626, 2635, 2637.

Additional formulæ for integration will be found at 2700, *et seq.*

## DIFFERENTIATION UNDER THE SIGN OF INTEGRATION.

Let  $u = \int_a^b f(x) dx$ , where  $a$ ,  $b$ , and  $f(x)$  are independent of each other; then

$$2253 \quad \frac{du}{db} = f(b) \quad \text{and} \quad \frac{du}{da} = -f(a).$$

PROOF.—Let  $u = \phi(b) - \phi(a)$ .  
Therefore  $u_b = \phi'(b) = f(b)$  and  $u_a = -\phi'(a) = -f(a)$ .

Let  $u = \int_a^b f(x, c) dx$ . Then, when  $a$  and  $b$  are independent of  $c$ ,

$$2255 \quad u_c = \int_a^b \{f(x, c)\}_c dx \quad \text{and} \quad u_{xc} = \int_a^b \{f(x, c)\}_{xc} dx.$$

PROOF.— $\frac{du}{dc} = \left\{ \int_a^b f(x, c+h) dx - \int_a^b f(x, c) dx \right\} \div h$   
 $= \int_a^b \frac{f(x, c+h) - f(x, c)}{h} dx$  (since  $h$  is constant relatively to  $x$ )  $= \int_a^b \frac{df(x, c)}{dc} dx$ .

But if  $a$  and  $b$  also are functions of  $c$ ,

$$2257 \quad \frac{du}{dc} = \int_a^b \left\{ \frac{df(x, c)}{dc} \right\} dx + f(b, c) \frac{db}{dc} - f(a, c) \frac{da}{dc}.$$

PROOF.—The complete derivative of  $u$  with respect to  $c$  will now be  $u_c + u_b b_c + u_a a_c$ . But  $u_b = f(b, c)$  and  $u_a = -f(a, c)$ , by (2253-4).

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 INTEGRATION BY DIFFERENTIATING UNDER THE SIGN OF INTEGRATION.

$$2258 \quad \text{Ex. 1.} \quad \int x^n e^{ax} dx = \int (e^{ax})_{na} dx = d_{na} \int e^{ax} dx \quad (2256) \\ = d_{na} (e^{ax} a^{-1}) = e^{ax} (x + d_a)^n a^{-1},$$

by (1464),  $a$  and  $x$  being transposed.

$$2259 \quad \text{Ex. 2.} \quad \int x^n e^{ax} \sin bx dx = d_{na} \int e^{ax} \sin bx dx.$$

The last integral is given in (1999), putting  $n = 1$ .

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$$2260 \quad \text{Ex. 3.} \quad$$

$$\int x a^{x-1} dx = \int (a^x)_a dx = \left( \int a^x dx \right)_a = \left( \frac{a^x}{\log a} \right)_a = \frac{a^{x-1}}{\log a} \left( x - \frac{1}{\log a} \right).$$


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## INTEGRATION UNDER THE SIGN OF INTEGRATION.

When the limits are constant,

$$2261 \quad \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y) dx dy = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y) dy dx.$$

That is, the order of integration may be changed.

But an exception to this rule occurs when, at any stage of the integration, an infinite value is produced. The double integrals above will not then have the same value.

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 APPROXIMATE INTEGRATION.

## BERNOULLI'S SERIES.

$$2262 \quad \int_0^a f(x) dx = af(a) - \frac{a^2}{1.2} f'(a) + \frac{a^3}{1.2.3} f''(a) - \dots \\ + \frac{(-1)^{n-1} a^n}{1.2 \dots n} f^{n-1}(a) + \frac{(-1)^n}{1.2 \dots n} \int_0^a x^n f^n(x) dx.$$

PROOF.—Integrate successively by Parts,  $\int dx$ ,  $\int x dx$ , &c. Or change  $f'(x)$  into  $f(x)$  in (1510).

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$$2263$$

$$\int_a^b f(x) dx = (b-a)f(a) + \frac{(b-a)^2}{1.2} f'(a) + \frac{(b-a)^3}{1.2.3} f''(a) + \&c.$$

PROOF.—Put  $f(a)$  for  $\phi'(a)$  in the expansion of the right side of equation (1902), by Taylor's theorem (1500); viz.,

$$\int_a^b f(x) dx = \phi(b) - \phi(a) = (b-a) \phi'(a) + \frac{(b-a)^2}{1.2} \phi''(a) + \&c.$$

The following is a nearer approximation :—

Let  $(b-a) = nh$ , where  $n$  is an integer; then

$$\begin{aligned} 2264 \quad \int_a^b f(x) dx &= h \left\{ \frac{1}{2} f(b) + \frac{1}{2} f(a) + f(a+h) + \dots + f(b-h) \right\} \\ &\quad - \frac{h^2 B_2}{[2]} \{f'(b) - f'(a)\} + \frac{h^4 B_4}{[4]} \{f'''(b) - f'''(a)\} \\ &\quad - \frac{h^6 B_6}{[6]} \{f^{(5)}(b) - f^{(5)}(a)\} + \&c. \end{aligned}$$

PROOF.—Expand  $(e^{nhx} - 1) \div (e^{hx} - 1)$  by ordinary division, and also by (1539), and operate upon  $f(x)$  with each result; thus, after multiplying by  $h$ , we obtain, by (1520),

$$\begin{aligned} h \{f(x) + f(x+h) + f(x+2h) + \dots + f(x+n-1 h)\} \\ = \{f(x+nh) - f(x)\} \left( d_x - \frac{h}{2} + \frac{h^2 B_2}{1.2} d_x - \frac{h^4 B_4}{1.2.3.4} d_x + \dots \right), \end{aligned}$$

which expression, by changing  $x$  into  $a$  and  $x+nh$  into  $b$ , is equivalent to the above, since  $d_x \{f(x+nh) - f(x)\} = \int_x^{x+nh} f(x) dx$ .

2265

$$\int_a^{a+h} f(x) dx = \frac{2hf(a-h)}{1.2} + \frac{3^2 h^2 f'(a-2h)}{1.2.3} + \frac{4^3 h^3 f''(a-3h)}{1.2.3.4} + \&c.$$

PROOF.—Assume  $x = ce^{hx}$ . Then  $x$  is equal to the coefficient of  $\frac{1}{x}$  in the expansion of  $-\log \left(1 - \frac{ce^{hx}}{x}\right)$ . Thus

$$x = c + \frac{2hc^2}{1.2} + \frac{3^2 h^2 c^3}{1.2.3} + \frac{4^3 h^3 c^4}{1.2.3.4} + \dots$$

Substitute  $d_x$  for  $x$ , and therefore  $d_x e^{-hx}$  for  $c$  in this equation, and operate with it upon  $\int \phi(x+h) dx$ , employing (1520). Finally, write  $f(x)$  for  $\phi'(x)$ , and  $a$  for  $x$ .

A more general result, obtained in the same way, is

$$\begin{aligned} 2266 \quad \int_a^{a+nh} f(x) dx &= nhf(a-h) + n(n+2) \frac{h^2}{[2]} f'(a-2h) \\ &\quad + (n+3)^2 \frac{h^3}{[3]} f''(a-3h) + \&c. \end{aligned}$$

THE INTEGRALS  $B(l, m)$  AND  $\Gamma(n)$ .EULER'S FIRST INTEGRAL  $B(l, m)$ .

The three principal forms are—

$$2280 \quad \text{I.} \quad B(l, m) = \int_0^1 x^{l-1} (1-x)^{m-1} dx = B(m, l). \quad [\text{By (2233)}]$$

$$2281 \quad \text{II.} \quad B(l, m) = \int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}} dx. \quad [\text{By substituting } \frac{x}{1-x} \text{ in I.}]$$

$$2282 \quad \text{III.} \quad B(l, m) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{l+m}} dx. \quad [\text{By substituting } \frac{1-x}{x} \text{ in I.}]$$

When  $l$  and  $m$  are positive, and  $l$  is an integer,

$$2283 \quad B(l, m) = \frac{1^{(l-1)}}{m^{(l)}}.$$

If  $m$  be the integer, interchange  $l$  and  $m$ . If both  $l$  and  $m$  are integers, the forms are convertible.

PROOF.—Integrate (2280) by parts, thus,

$$\int_0^1 x^{l-1} (1-x)^{m-1} dx = \frac{l-1}{m} \int_0^1 x^{l-2} (1-x)^m dx.$$

Repeat this step successively.

EULER'S SECOND INTEGRAL  $\Gamma(n)$ .

$n$  being a real and positive quantity,

$$2284 \quad \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx = \int_0^1 \left( \log \frac{1}{x} \right)^{n-1} dx.$$

The second form being obtained by substituting  $e^{-x}$  in the first.

$$2286 \quad \Gamma(1) = 1, \quad \Gamma(2) = 1.$$

$$2288 \quad \Gamma(n+1) = n\Gamma(n) = n(n-1) \dots (n-r) \Gamma(n-r).$$

$$2290 \quad \Gamma(n+1) = \underline{n}, \text{ when } n \text{ is an integer.}$$

$$\text{PROOF.—By Parts, } \int_0^\infty e^{-x} x^{n-1} dx = \left[ \frac{x^n}{ne^x} \right]_0^\infty + \frac{1}{n} \int_0^\infty e^{-x} x^n dx.$$

The fraction becomes zero at each limit, as appears by (1580), differentiating the numerator and denominator, each  $r$  times, and taking  $r > n$ .

$$2291 \quad \int_0^{\infty} e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n} = \int_0^1 x^{k-1} \left( \log \frac{1}{x} \right)^{n-1} dx.$$

PROOF.—Substitute  $kx$  in the first integral, and so reduce it to the form (2284). In the second integral, substitute  $-\log x$ , reducing it to the former. When  $n$  is an integer, (2291) may be obtained by differentiating  $n-1$  times for  $k$  the equation  $\int_0^{\infty} e^{-kx} dx = \frac{1}{k}$ .

When  $\mu$  is an indefinitely great integer,

$$2293 \quad \Gamma(n) = \frac{1.2.3 \dots \mu}{n(n+1) \dots (n+\mu)} \mu^n.$$

PROOF.— $\log \frac{1}{x} = \lim. \mu (1 - x^{\frac{1}{\mu}})$  (1583). Give it this value in (2285), and then substitute  $y = x^{\frac{1}{\mu}}$ ; thus, in the limit,

$\Gamma(n) = \mu^{n-1} \int_0^1 (1 - x^{\frac{1}{\mu}})^{n-1} dx = \mu^n \int_0^1 y^{\mu-1} (1-y)^{n-1} dy$ . Then, by (2283), changing  $\mu$  finally into  $\mu+1$  in the fraction.

$\log \Gamma(1+n)$  IN A CONVERGING SERIES.

2294 Let  $n$  be  $< 1$ ,  $\mu$  an indefinitely great integer, and  $S_r = 1 + \frac{1}{2^r} + \frac{1}{3^r} + \dots \frac{1}{\mu^r}$ , then

$$\log \Gamma(1+n).$$

$$2295 = (\log \mu - S_1)n + \frac{1}{2}S_2n^2 - \frac{1}{3}S_3n^3 + \frac{1}{4}S_4n^4 - \frac{1}{5}S_5n^5 + \&c.$$

$$2296 = \frac{1}{2} \log \frac{n\pi}{\sin n\pi} + (\log \mu - S_1)n - \frac{1}{3}S_3n^3 - \frac{1}{5}S_5n^5 - \&c.$$

$$2297 = \frac{1}{2} \log \frac{n\pi(1-n)}{\sin n\pi(1+n)} + n(1 + \log \mu - S_1) + \frac{n^3}{3}(1 - S_3) + \frac{n^5}{5}(1 - S_5) + \&c.$$

$$2298 = \frac{1}{2} \log \frac{n\pi}{\sin n\pi} - \frac{1}{2} \log \frac{1+n}{1-n} + .4227843n - .0673530n^3 - .0073855n^5 - .0011927n^7 - .0002231n^9 - \&c.$$

$$\text{PROOF.—By (2293), } \Gamma(1+n) = \frac{\mu^n 1.2.3 \dots \mu}{(n+1)(n+2) \dots (n+\mu)},$$

since  $\frac{\mu}{n+\mu+1} = 1$ , when  $\mu = \infty$ . Whence

$$\log \Gamma(1+n) = n \log \mu - l(1+n) - l\left(1 + \frac{n}{2}\right) - l\left(1 + \frac{n}{3}\right) - \dots - l\left(1 + \frac{n}{\mu}\right).$$

Developing the logarithms by (155), the series (2295) is obtained. The next series is deduced from this by substituting

$$S_1 n^3 + \frac{1}{2} S_2 n^4 + \frac{1}{3} S_3 n^5 + \frac{1}{4} S_4 n^6 + \&c. = \log n\pi - \log \sin n\pi,$$

a result obtained from (815) by putting  $\theta = n\pi$  and expanding the logarithms by (156).

The series (2297) is deduced from the preceding by adding the expression

$$0 = -\frac{1}{2} \log \frac{1+n}{1-n} + n + \frac{n^3}{3} + \frac{n^5}{5} + \&c., \text{ from (157).}$$

**2305**

$$B(l, m) = \frac{\Gamma(l) \Gamma(m)}{\Gamma(l+m)}.$$

PROOF.—Perform the integrations in the double integral

$$\int_0^\infty \int_0^\infty e^{-x(y+1)} x^{l+m-1} y^{l-1} dx dy,$$

first for  $x$ , by formula (2291), and then for  $y$ , by (2281), and the result is  $B(l, m) \Gamma(l+m)$ . Again perform the integration, first for  $y$ , by (2291), and the result is  $\Gamma(l) \Gamma(m)$ , by (2284).

NOTE.—The double integral may be written by the following rule:—

Write  $xy$  for  $x$  in  $\Gamma(l)$ , and multiply by the factors of  $\Gamma(m+1)$ . We

thus obtain 
$$\int_0^\infty \int_0^\infty e^{-xy} (xy)^{l-1} \times e^{-x} x^m dx dy,$$

which is equivalent to the integral in question.

**2306**

$$B(l, m) B(l+m, n) = B(m, n) B(m+n, l) \\ = B(n, l) B(n+l, m)$$

**2307**

$$= \frac{\Gamma(l) \Gamma(m) \Gamma(n)}{\Gamma(l+m+n)}. \quad [\text{By (2305).}]$$

**2308**

$$\int_0^a x^{l-1} (a-x)^{m-1} dx = a^{l+m-1} B(l, m). \quad [\text{Substitute } \frac{x}{a}.$$

If  $p$  and  $q$  are positive integers,  $p < q$ , and if  $m = \frac{2p+1}{2q}$ .

**2309**

$$\int_0^\infty \frac{x^{2p}}{1+x^{2q}} dx = \frac{\pi}{2q \sin m\pi}.$$

**2310**

$$\int_0^\infty \frac{x^{2p}}{1-x^{2q}} dx = \frac{\pi}{2q \tan m\pi}.$$

PROOF.—(i.) In (2023) put  $l = 2p+1$ ,  $n = 2q$ , and take the value of the integral between the limits  $\pm\infty$ . The first term becomes  $\log 1 = 0$ ; the second gives the series

$$\frac{\pi}{q} \left\{ \sin \frac{l\pi}{2q} + \sin \frac{3l\pi}{2q} + \dots + \sin \frac{(2q-1)l\pi}{2q} \right\} = \frac{\pi}{q \sin m\pi},$$

by (800). The integral required is one-half of this result, by (2237).

(ii.) (2310) is deduced in a similar manner from (2021).



$$2311 \quad \int_0^\infty \frac{x^{m-1}}{1+x} dx = \frac{\pi}{\sin m\pi}, \quad \int_0^\infty \frac{x^{m-1}}{1-x} dx = \frac{\pi}{\tan m\pi},$$

where  $m$  has any value between 0 and 1.

PROOF.—By substituting  $x^{2q}$  in (2309-10). Also, since  $m = \frac{2p+1}{2q}$ , by taking the integers  $p$  and  $q$  large enough, the fraction may, in the limit, be made equal to any quantity whatever lying between 0 and 1 in value.

$$2313 \quad \Gamma(m) \Gamma(1-m) = \frac{\pi}{\sin m\pi}, \quad m \text{ being } < 1.$$

PROOF.—Put  $l+m=1$  in the two values of  $B(l, m)$  (2282) and (2305); thus,

$$\Gamma(m) \Gamma(1-m) = \int_0^1 \frac{x^{m-1}}{1+x} dx = \frac{\pi}{\sin m\pi}, \quad \text{by (2311).}$$

$$2314 \quad \text{COR.—} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

The following is an independent proof:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x} x^{-\frac{1}{2}} dx = 2 \int_0^\infty e^{-y^2} dy = 2 \int_0^\infty e^{-z^2} dz.$$

Now form the product of the last two integrals, and change the variables to  $r, \theta$  by the equations

$$\left. \begin{array}{l} y = r \sin \theta \\ z = r \cos \theta \end{array} \right\}, \text{ from which, by (1609), } dy dz = \frac{d(yz)}{d(r\theta)} dr d\theta = r dr d\theta. \text{ Hence}$$

$$\{\Gamma\left(\frac{1}{2}\right)\}^2 = 4 \int_0^\infty \int_0^\infty e^{-(y^2+z^2)} dy dz = 4 \int_0^\infty \int_0^{\frac{1}{2}\pi} e^{-r^2} r dr d\theta = \pi;$$

the limits for  $r$  and  $\theta$  being obtained from

$$r^2 = y^2 + z^2, \quad \tan \theta = \frac{y}{z}.$$

$$2315 \quad \Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{2}{n}\right) \Gamma\left(\frac{3}{n}\right) \dots \Gamma\left(\frac{n-1}{n}\right) = \sqrt{\frac{(2\pi)^{n-1}}{n}}.$$

PROOF.—Multiply the left side by the same factors in reversed order, and apply (2313) thus

$$\begin{aligned} & \Gamma\left(\frac{1}{n}\right) \Gamma\left(1-\frac{1}{n}\right) \Gamma\left(\frac{2}{n}\right) \Gamma\left(1-\frac{2}{n}\right) \dots \Gamma\left(\frac{n-1}{n}\right) \Gamma\left(1-\frac{n-1}{n}\right) \\ &= \frac{\pi^{n-1}}{\sin \frac{\pi}{n} \sin \frac{2\pi}{n} \dots \sin \frac{(n-1)\pi}{n}} = \frac{2^{n-1} \pi^{n-1}}{n}, \quad \text{by (814).} \end{aligned}$$

$$2316 \quad \frac{n^{nx}}{n\Gamma(nx)} \Gamma(x) \Gamma\left(x+\frac{1}{n}\right) \dots \Gamma\left(x+\frac{n-1}{n}\right) = \sqrt{\frac{(2\pi)^{n-1}}{n}}.$$

PROOF.—Call the expression on the left  $\phi(x)$ . Change  $x$  to  $x+r$ , where  $r$  is any integer, and change each Gamma function by the formula  $\Gamma(x+r) = x^{(r)} \Gamma(x)$  (2288). The result after reduction is  $\phi(x)$ . Hence  $\phi(x) = \phi(x+r)$ , however great  $r$  may be. Therefore  $\phi(x)$  is independent of  $x$ . But, when  $x = \frac{1}{n}$ ,  $\phi(x)$  takes the value in question by (2315). Therefore  $\phi(x)$  always has that value.

The formula may also be obtained by means of (2294).

### NUMERICAL CALCULATION OF $\Gamma(x)$ .

**2317** All values of  $\Gamma(x)$  may be found in terms of values lying between  $\Gamma(0)$  and  $\Gamma(\frac{1}{2})$ .

When  $x$  is  $> 1$ , formula (2289) reduces  $\Gamma(x)$  to the value in which  $x$  is  $< 1$ ; and when  $x$  lies between 1 and  $\frac{1}{2}$ , formula (2313) reduces the function to the value in which  $x$  lies between 0 and  $\frac{1}{2}$ .

Values of  $\Gamma(x)$ , when  $x$  lies between 0 and 1, can also be made to depend upon values in which  $x$  lies between  $\frac{1}{6}$  and  $\frac{1}{3}$ , by the formulæ,

$$\mathbf{2318} \quad \Gamma(x) = 2^{1-2x} \sqrt{\pi} \frac{\Gamma(2x)}{\Gamma(\frac{1}{2}+x)}, \quad \Gamma(x) = \frac{\sqrt{\pi}}{2^{1-x} \cos \frac{\pi x}{2}} \frac{\Gamma(\frac{x}{2})}{\Gamma(\frac{1-x}{2})}.$$

PROOF.—To obtain (2318), make  $n = 2$  in (2316). To obtain (2319), put  $m = \frac{1}{2}(1+x)$  in (2313), and change  $x$  into  $\frac{1}{2}x$  in (2318), and then eliminate  $\Gamma(\frac{1+x}{2})$ .

### Methods of employing the formulæ—

**2320** (i.) When  $x$  lies between  $\frac{2}{3}$  and 1, reduce  $\Gamma(x)$  to  $\Gamma(1-x)$ , by (2313).

**2321** (ii.) When  $x$  lies between  $\frac{1}{3}$  and  $\frac{2}{3}$ , reduce by (2319), the limits on the right of which will then be  $\frac{1}{6}$  and  $\frac{1}{3}$ .

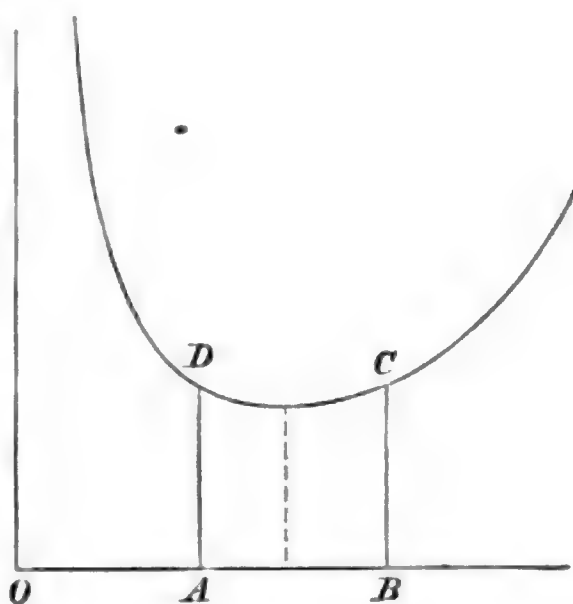
**2322** (iii.) When  $x$  lies between 0 and  $\frac{1}{6}$ , reduce by (2318);  $\Gamma(\frac{1}{2}+x)$  will then involve the limits  $\frac{1}{2}$  and  $\frac{2}{3}$ , and will be reducible by case (ii.)

If  $2x$  is  $< \frac{1}{6}$ , reduce  $\Gamma(2x)$  by (2318), writing  $2x$  for  $x$ . If this gives  $4x < \frac{1}{6}$ , reduce again by the same formula, writing  $4x$  for  $x$ , and so on.

**2323** The figure exhibits the curve whose equation in rectangular coordinates is  $y = \Gamma(x)$ . Let the unit abscissa be  $OA = AB = 1$ . Then the ordinates  $AD, BC$  are also each  $= 1$  by (2286-7).

The minimum value of  $\Gamma(x)$  is approximately 0.8556032, corresponding to  $x = 1.4616321$ .

The values of  $\log \Gamma(x)$  in the table at page 30 correspond to ordinates taken between  $AD = \Gamma(1)$  and  $BC = \Gamma(2)$ .



## INTEGRATION OF ALGEBRAIC FORMS.

$$\mathbf{2341} \quad \int_0^1 \frac{x^{l-1} + x^{m-1}}{(1+x)^{l+m}} dx = B(l, m).$$

PROOF.—Add together (2281) and (2282). Separate the resulting integral into  $\int_0^1 + \int_1^\infty$ , and substitute  $\frac{1}{x}$  in the last part.

$$\mathbf{2342} \quad \int_0^1 \frac{x^{l-1}(1-x)^{m-1}}{(x+a)^{l+m}} dx = \frac{B(l, m)}{a^m(1+a)^l}. \quad \left[ \text{Substitute } \frac{x+ax}{x+a} \right].$$

$$\mathbf{2343} \quad \int_0^1 x^{l-1}(1-x^n)^{m-1} dx = \frac{1}{n} B\left(\frac{l}{n}, m\right). \quad \left[ \text{Substitute } x^n \right].$$

$$\mathbf{2344} \quad \int_0^1 \frac{x^{l-1}(1-x)^{m-1}}{\{ax+b(1-x)\}^{l+m}} dx = \frac{1}{a^l b^m} B(l, m). \quad \left[ \text{Subs. } \frac{ax}{(a-b)x+b} \right].$$

The integral is also equivalent to

$$2 \int_0^{\frac{\pi}{2}} \frac{\sin^{2l-1} \theta \cos^{2m-1} \theta}{(a \sin^2 \theta + b \cos^2 \theta)^{l+m}} d\theta, \quad \text{and similarly in other cases.}$$

$$\mathbf{2345} \quad \int_0^\infty \frac{x^{n-1} dx}{(a+bx)^{n+1}} = \frac{1}{nab^n}. \quad \left[ \text{Substitute } \frac{bx}{a+bx} \right].$$

$$2346 \quad \int_0^\infty \frac{x^{n-1} dx}{(a+bx)^{m+n}} = \frac{1^{(m-1)}}{n^{(m)}} \frac{1}{a^m b^n}.$$

PROOF.—Differentiate (2345)  $m-1$  times for  $a$ . (2255)

$$2347 \quad \int_0^\infty \frac{x^{m+n-1}}{(1+x)^{n+1}} dx = C(m, n) \frac{\pi}{\sin m\pi}, \quad m < 1.$$

PROOF.—Substitute  $x = ay$  in (2311), and then differentiate  $n$  times for  $a$ .

$$2348 \quad \int_0^\infty \frac{x^{m-1} dx}{1+x^n} = \frac{\pi}{n} \operatorname{cosec} \frac{m\pi}{n}, \quad \int_0^\infty \frac{x^{m-1} dx}{1-x^n} = \frac{\pi}{n} \cot \frac{m\pi}{n},$$

where  $m$  and  $n$  are any positive quantities, and  $m$  is  $< n$ .

PROOF.—Change  $m$  into  $\frac{m}{n}$  in (2311-2), and then substitute  $x^{\frac{1}{n}}$ .

When  $n$  is positive and greater than unity,

$$2350 \quad \int_0^\infty \frac{dx}{1+x^n} = \frac{\pi}{n} \operatorname{cosec} \frac{\pi}{n}, \quad \int_0^\infty \frac{dx}{1-x^n} = \frac{\pi}{n} \cot \frac{\pi}{n}.$$

PROOF.—Substitute  $x^m$  in (2311-2) and change  $m$  into  $\frac{1}{n}$ .

$$2352 \quad \int_0^1 \frac{dx}{\sqrt[n]{1-x^n}} = \frac{\pi}{n} \operatorname{cosec} \frac{\pi}{n}, \quad \int_0^1 \frac{dx}{\sqrt[n]{1+x^n}} = \frac{\pi}{n} \cot \frac{\pi}{n}.$$

PROOF.—Substitute  $\frac{x}{\sqrt[n]{1+x^n}}$  in (2350), and  $\frac{x}{\sqrt[n]{1-x^n}}$  in (2351).

When  $m$  lies between 0 and 1,

$$2354 \quad \int_0^\infty \frac{x^m dx}{1+x^2} = -\frac{\pi}{2} \sec \frac{m\pi}{2}, \quad \int_0^\infty \frac{x^m dx}{1-x^2} = -\frac{\pi}{2} \tan \frac{m\pi}{2}.$$

PROOF.—Make  $n = 2$  and write  $m+1$  for  $m$  in (2348-9).

$$2356 \quad \int_0^1 \frac{x^{m-1} + x^{-m}}{1+x} dx = \frac{\pi}{\sin m\pi}, \quad \int_0^1 \frac{x^{m-1} - x^{-m}}{1-x} dx = \frac{\pi}{\tan m\pi},$$

where  $m$  lies between 0 and 1.

PROOF.—Separate (2311-2) each into two integrals by the formula  $\int_0^\infty = \int_0^1 + \int_1^\infty$ , and substitute  $x^{-1}$  in the last integral.

Otherwise, in (2601) substitute  $e^{-\pi x}$ , and change  $a$  into  $\pi a - \frac{1}{2}\pi$ .

$$2358 \quad \int_0^1 \frac{x^m + x^{-m}}{1+x^2} dx = \frac{\pi}{2} \sec \frac{m\pi}{2}. \quad \int_0^1 \frac{x^m - x^{-m}}{1-x^2} dx = \frac{\pi}{2} \tan \frac{m\pi}{2}.$$

PROOF.—From (2354–5) by the method of (2356).

$$2360 \quad \int_0^1 \frac{x^{m-1} + x^{n-m-1}}{1+x^n} dx = \frac{\pi}{n} \operatorname{cosec} \frac{m\pi}{n}.$$

$$2361 \quad \int_0^1 \frac{x^{m-1} - x^{n-m-1}}{1-x^n} dx = \frac{\pi}{n} \cot \frac{m\pi}{n}.$$

PROOF.—In the same way, from (2348–9).

$$2362 \quad \int_0^1 \frac{x^{n-m-1} - x^{n+m-1}}{1+x^{2n}} dx = \frac{\pi}{2n} \sec \frac{m\pi}{2n}.$$

$$2363 \quad \int_0^1 \frac{x^{n-m-1} - x^{n+m-1}}{1-x^{2n}} dx = \frac{\pi}{2n} \tan \frac{m\pi}{2n}.$$

PROOF.—In (2601) substitute  $e^{-\frac{\pi x}{2n}}$  and put  $a = \frac{m\pi}{2n}$ . In (2595) substitute  $e^{-\frac{\pi x}{n}}$  and put  $a = \frac{m\pi}{n}$ .

$$2364 \quad \int_0^\infty \frac{dx}{(x^2+a^2)^n} = \frac{1_2^{(n-1)}}{2_2^{(n-1)}} \frac{\pi}{2a^{2n-1}}, \quad n \text{ being an integer.}$$

PROOF.—By successive reduction by (2062), or by differentiating  $\int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$ ,  $n-1$  times with respect to  $a^2$ . (2255)

$$2365 \quad \int_0^1 \frac{x^{m-1} dx}{(1+bx)(1-x)^m} = \frac{\pi \operatorname{cosec} m\pi}{(1+b)^m}. \quad [\text{Subs. } \frac{1+bx}{1-x}. \quad (2311)]$$

$$2367 \quad \int_0^1 \left( \frac{x^{a-1}}{1-x} - \frac{nx^{na-1}}{1-x^n} \right) dx = \log n.$$

PROOF.—The value when  $a = 1$  is  $\log n$ . The difference, when the value is  $a$ , is  $\int_0^1 \frac{x^{a-1}-1}{1-x} dx - n \int_0^1 \frac{x^{na-1}-x^{n-1}}{1-x^n} dx$ , which, by substituting  $x^n$  in the second integral, is seen to be zero.

$F(x)$  being any integral polynomial,

$$2368 \quad \int_{-1}^{+1} \frac{F(x) dx}{\sqrt{1-x^2}} = A\pi, \quad \text{where } A \text{ is equal to the constant term in the product of } F(x) \text{ and the expansion of } \left(1 - \frac{1}{x^2}\right)^{-\frac{1}{2}}.$$

PROOF.—By successive reduction by (2053), we know that

$$\int \frac{F(x) dx}{\sqrt{1-x^2}} = \phi(x) \sqrt{1-x^2} + A \int \frac{dx}{\sqrt{1-x^2}} \dots\dots\dots (1),$$

where  $\phi(x)$  is some integral polynomial and  $A$  is a constant. Therefore the integral in question  $= A\pi$ . To determine  $A$  write the last equation thus,

$$\int \frac{F(x)}{ix} \left(1 - \frac{1}{x^2}\right)^{-\frac{1}{2}} dx = ix \phi(x) \left(1 - \frac{1}{x^2}\right)^{-\frac{1}{2}} + A \int \frac{1}{ix} \left(1 - \frac{1}{x^2}\right)^{-\frac{1}{2}} dx.$$

Expand each binomial; perform the integrations and equate the coefficients of the two logarithmic terms in the result.

$F(x)$  being an integral polynomial of a degree less than  $n$ ,

$$2369 \quad \int_a^b \frac{F(x)}{(x-c)^n} dx = \frac{1}{n-1} \frac{d^{n-1}}{dc^{n-1}} \left\{ F(c) \log \frac{b-c}{a-c} \right\}.$$

$$\text{PROOF.}— \int_a^b \frac{F(x)}{(x-c)^n} dx = \frac{1}{n-1} \int_a^b \left\{ \frac{d^{n-1}}{dc^{n-1}} \left( \frac{F(x)}{x-c} \right) \right\} dx. \quad (2255)$$

But  $\frac{F(x)}{x-c} = f(x, c) + \frac{F(c)}{x-c}$ , where  $f$  is of a dimension lower than  $n-1$  (421), and therefore  $d_{(n-1)c} f(x, c) = 0$ . Hence the integral on the right

$$= \frac{d^{n-1}}{dc^{n-1}} \int_a^b \frac{F(c)}{x-c} dx = \frac{d^{n-1}}{dc^{n-1}} \left\{ F(c) \log \frac{b-c}{b+c} \right\}.$$

## INTEGRATION OF LOGARITHMIC AND EXPONENTIAL FORMS.

$$2391 \quad \int_0^1 x^p \log x dx = \frac{-1}{(p+1)^2}. \quad \int_0^1 \frac{x^p dx}{\log x} = \log(p+1).$$

PROOF.—These are cases of (2292). Otherwise; to obtain the first integral differentiate, and to obtain the second integrate, the equation  $\int_0^1 x^p dx = \frac{1}{p+1}$  with respect to  $p$  (2255 and 2261).

$$2393 \quad \int_0^1 x^p (\log x)^n dx = (-1)^n \frac{\Gamma(n+1)}{(p+1)^{n+1}}.$$

PROOF.—See (2292). Otherwise, when  $n$  is either a positive or negative integer, the value may be obtained, as in (2391), by performing the differentiation or integration there described,  $n$  times successively, and employing formulæ (2166), and (2163) in the case of integration.

$$2394 \quad \int_0^1 \frac{x^p - x^q}{\log x} dx = \log \frac{p+1}{q+1}. \quad [\text{By (2392).}]$$



$$2395 \quad \int_0^1 \frac{(\log x)^{2n-1}}{1-x} dx = -\frac{2^{2n-2}}{n} B_{2n} \pi^{2n} = -\int_0^\infty \frac{x^{2n-1}}{e^x-1} dx.$$

$$2397 \quad \int_0^1 \frac{(\log x)^{2n-1}}{1+x} dx = -\frac{2^{2n-1}-1}{2n} B_{2n} \pi^{2n} = -\int_0^\infty \frac{x^{2n-1}}{e^x+1} dx.$$

PROOF.—Expand by dividing by  $1 \mp x$ , and integrate by (2393); thus

$$\int_0^1 \frac{(\log x)^{2n-1}}{1-x} dx = -\frac{2n-1}{1} \left( 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \&c. \right),$$

$$\int_0^1 \frac{(\log x)^{2n-1}}{1+x} dx = -\frac{2n-1}{1} \left( 1 - \frac{1}{2^{2n}} + \frac{1}{3^{2n}} - \frac{1}{4^{2n}} + \&c. \right).$$

The first series is summed in (1545). The difference of the two series multiplied by  $2^{2n-1}$  is equal to the first; this gives the value of the second series.

**2399**

$$\int_0^1 \frac{\log x}{1-x} dx = \int_0^1 \frac{\log(1-x)}{x} dx = -1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \&c. = -\frac{\pi^2}{6}.$$

$$2400 \quad \int_0^1 \frac{\log x}{1+x} dx = -1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \&c. = -\frac{\pi^2}{12}.$$

PROOF.—As in (2395–7), making  $n = 1$ .

The series (2399) may also be summed by equating the coefficients of  $\theta^3$  in (764) and (815).

$$2401 \quad \int_0^1 \frac{\log x}{1-x^2} dx = -1 - \frac{1}{3^2} - \frac{1}{5^2} - \frac{1}{7^2} - \&c. = -\frac{\pi^2}{8}.$$

PROOF.—The integral is half the sum of those in (2399, 2400).

$$2402 \quad \int_0^1 \frac{\log(1+x)}{x} dx = \frac{\pi^2}{12}. \quad \int_0^1 \frac{1}{x} \log \frac{1+x}{1-x} dx = \frac{\pi^2}{4}.$$

PROOF.—Expand the logarithms by (155) and (157) and integrate the terms. The series in (2400–1) are reproduced.

$$2404 \quad \text{Let} \quad \int_0^a \frac{\log(1-x)}{x} dx = \phi(a), \quad a \text{ being} < 1.$$

Substitute  $1-x = y$ ; therefore, writing  $l$  for  $\log$ ,

$$\phi(a) = \int_{1-a}^1 \frac{ly dy}{1-y} = \int_0^1 \frac{ly dy}{1-y} - \int_0^{1-a} \frac{ly dy}{1-y}.$$

The second integral by (2399), and the third by Parts, make the right side

$$= -\frac{\pi^2}{6} + la l(1-a) - \int_0^{1-a} \frac{l(1-y)}{y} dy. \quad \text{Therefore}$$

$$2405 \quad \phi(a) + \phi(1-a) = \log a \log(1-a) - \frac{1}{6}\pi^2.$$

$$\text{If } a = \frac{1}{2}, \quad \int_0^1 \frac{\log(1-x)}{x} dx = \frac{1}{2}(\log 2)^2 - \frac{\pi^2}{12}.$$

$$\text{Again, } \phi(x) = \int_0^x \frac{l(1-x)}{x} dx, \quad \therefore \text{ (by 2253) } \phi'(x) = \frac{l(1-x)}{x} \dots (i.);$$

$$d_x \phi\left(\frac{-x}{1-x}\right) = \frac{-1}{(1-x)^2} \phi'\left(\frac{-x}{1-x}\right) = \frac{1}{x(1-x)} l \frac{1}{1-x} \text{ by (i.),}$$

$$\therefore \phi\left(\frac{-x}{1-x}\right) = \int_0^x l \frac{1}{1-x} \frac{dx}{x(x-1)} = \int_0^x l \frac{1}{1-x} \frac{dx}{1-x} + \int_0^x l \frac{1}{1-x} \frac{dx}{x} \\ = \frac{1}{2} \{l(1-x)\}^2 - \phi(x).$$

Put  $\frac{x}{1+x}$  for  $x$ ; then

$$2406 \quad \phi(-x) + \phi\left(\frac{x}{1+x}\right) = \frac{1}{2} \{\log(1+x)\}^2.$$

$$\text{Also, } \phi(x) = - \int_0^x \left(1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \&c.\right) \\ = - \left(x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \&c.\right).$$

$$\text{Hence } \phi(x) + \phi(-x) = - \frac{1}{2} \left(x^2 + \frac{x^4}{2^2} + \frac{x^6}{3^2} + \&c.\right) = \frac{1}{2} \phi(x^2).$$

Eliminate  $\phi(-x)$  by (2406); thus

$$2407 \quad \phi\left(\frac{x}{x+1}\right) + \frac{1}{2} \phi(x^2) - \phi(x) = \frac{1}{2} \{\log(1+x)\}^2.$$

$$\text{Let } \frac{x}{x+1} = x^2, \text{ and therefore } x = -\frac{1}{2} + \frac{1}{2} \sqrt{5} \equiv \beta \text{ say,}$$

$$\therefore \text{ by (2407), } \frac{3}{2} \phi(\beta^2) - \phi(\beta) = \frac{1}{2} \{l(1+\beta)\}^2,$$

$$\text{or } \frac{3}{2} \phi(1-\beta) - \phi(\beta) = \frac{1}{2} (l\beta)^2; \quad [\because \beta^2 = 1-\beta \text{ and } 1+\beta = \frac{1}{\beta};$$

$$\text{and by (2405) } \phi(1-\beta) + \phi(\beta) = 2(l\beta)^2 - \frac{1}{6}\pi^2,$$

$$\therefore \phi(\beta) = (l\beta)^2 - \frac{1}{10}\pi^2 \text{ and } \phi(1-\beta) = (l\beta)^2 - \frac{1}{15}\pi^2, \text{ that is,}$$

$$2408 \quad \int_0^{\frac{\sqrt{5}-1}{2}} \frac{\log(1-x)}{x} dx = \left(\log \frac{\sqrt{5}-1}{2}\right)^2 - \frac{\pi^2}{10}.$$

$$2409 \quad \int_0^{\frac{3-\sqrt{5}}{2}} \frac{\log(1-x)}{x} dx = \left(\log \frac{\sqrt{5}-1}{2}\right)^2 - \frac{\pi^2}{15}.$$

2410 Let  $a$  be  $> 1$ , then  $\phi(a)$  contains imaginary elements, but its value is determinate. We have

$$\phi(a) = \int_0^1 \frac{l(1-x)}{x} dx + \int_1^a \frac{l(1-x)}{x} dx = -\frac{\pi^2}{6} + \int_1^a \frac{\pi i + l(x-1)}{x},$$

3 B

the integration by 2399, and  $l(-1) = \pi i$  by (2214). The last integral  

$$= \pi i a + \int_1^a \left\{ lx + l \left( 1 - \frac{1}{x} \right) \right\} \frac{dx}{x} = \pi i a + \frac{1}{2} (la^2) + \int_1^a l \left( 1 - \frac{1}{x} \right) \frac{dx}{x}.$$

Substitute  $\frac{1}{x} = y$  in the last integral, and it becomes

$$\int_{\frac{1}{a}}^1 \frac{l(1-y)}{y} dy = \int_0^1 \frac{l(1-y)}{y} dy - \int_0^{\frac{1}{a}} \frac{l(1-y)}{y} dy = -\frac{\pi^2}{6} - \phi\left(\frac{1}{a}\right).$$

Hence, when  $a$  is  $> 1$ ,

$$2411 \quad \phi(a) + \phi\left(\frac{1}{a}\right) = -\frac{\pi^2}{3} + \pi i \log a + \frac{1}{2} (\log a)^2.$$

If  $a = 2$ , this result becomes, by employing (2405),

$$2412 \quad \int_0^2 \frac{\log(1-x)}{x} dx = -\frac{\pi^2}{4} + \pi i \log 2.$$

$$2413 \quad \text{Let} \quad \psi(x) = \int_0^x \frac{1}{2x} \log \frac{1+x}{1-x} dx.$$

$$\text{Therefore} \quad \psi'(x) = \frac{1}{2x} l\left(\frac{1+x}{1-x}\right), \quad d_x \psi\left(\frac{1-x}{1+x}\right) = \frac{lx}{1-x^2},$$

$$\text{therefore} \quad \psi(x) + \psi\left(\frac{1-x}{1+x}\right) = \int_0^x \left( l \frac{1+x}{1-x} \frac{1}{2x} + \frac{lx}{1-x^2} \right) dx, \text{ therefore}$$

$$2414 \quad \psi(x) + \psi\left(\frac{1-x}{1+x}\right) = \frac{1}{2} \log x \log \frac{1+x}{1-x}.$$

The constant vanishes, by (2403) and (2401), putting  $x = 1$ .

Let  $x = \frac{1-x}{1+x}$ , and therefore  $x = \sqrt{2}-1$ ; then, by (2414),

$$2415 \quad \int_0^{\sqrt{2}-1} \log \frac{1+x}{1-x} \frac{dx}{x} = -\frac{1}{2} \{ \log(\sqrt{2}-1) \}^2.$$

$$2416 \quad \int_0^1 \frac{\log(1+x)}{1+x^2} dx = \frac{\pi}{8} \log 2.$$

PROOF.—Substitute  $\phi = \tan^{-1}x$ ; then, by (2233),

$$\int_0^{\frac{\pi}{4}} l(1+\tan \phi) d\phi = \int_0^{\frac{\pi}{4}} l \left\{ 1 + \tan \left( \frac{\pi}{4} - \phi \right) \right\} d\phi = \int_0^{\frac{\pi}{4}} l \frac{2}{1+\tan \phi} d\phi.$$

2417 By differentiating or integrating the equations (2341) to (2363) with respect to the index  $m$ , the integrals of func-

tions involving  $\log x$  are produced; thus, from (2356), by integrating for  $m$  between the limits  $\frac{1}{2}$  and  $m$ , we have

$$2418 \quad \int_0^1 \frac{x^{m-1} - x^{-m}}{(1+x) \log x} dx = \log \tan \frac{m\pi}{2}. \quad (1943)$$

Otherwise, this result may be arrived at by forming the expansion of the fraction in powers of  $x$ , and integrating the terms by (2392); the reduction is then effected by 815-6.

In a similar manner, we obtain the more general formula

$$2419 \quad \int_0^1 \frac{(x^{m-1} - x^{n-1}) dx}{(1+x^n) \log x} = \log \frac{m}{n} \frac{n+p}{m+p} \frac{m+2p}{n+2p} \frac{n+3p}{m+3p} \dots$$

$$2420 \quad \int_0^1 \frac{x^{m-1} - x^{n-m-1}}{(1+x^n) \log x} dx = \log \tan \frac{m\pi}{2n}.$$

PROOF.—Integrate (2360) for  $m$  from  $2n$  to  $m$ .

$$2421 \quad \int_0^1 \frac{x^p - x^r + x^q (r-p) \log x}{(\log x)^2} dx$$

$$= (p+1) \log(p+1) - (r+1) \log(r+1) + (r-p) \{1 + \log(q+1)\}$$

PROOF.—Integrate (2394) for  $p$  between the limits  $r$  and  $p$ .

$$2422 \quad \int_0^1 \frac{(q-r) x^p + (r-p) x^q + (p-q) x^r}{(\log x)^2} dx$$

$$= \log \{ (p+1)^{(p+1)(q-r)} (q+1)^{(q+1)(r-p)} (r+1)^{(r+1)(p-q)} \}.$$

PROOF.—Write (2421) symmetrically for  $r, p$ ;  $p, q$  and  $q, r$ . Multiply the three equations, respectively, by  $q, r, p$ , and add, reducing the result by (2394).

$$2423 \quad \int_0^\infty \frac{\log(1+a^2 x^2)}{b^2 + x^2} dx = \frac{\pi}{b} \log(1+ab).$$

PROOF.—Differentiate for  $a$ , and resolve into two fractions. Effect the integration for  $x$ , and integrate finally with respect to  $a$ .

2424

$$\int_0^\infty \log\left(1 + \frac{x^2}{a^2}\right) \log\left(1 + \frac{b^2}{x^2}\right) \frac{dx}{x^2} = 2\pi \left(\frac{1}{a} + \frac{1}{b}\right) \log\left(1 + \frac{b}{a}\right) - \frac{2\pi}{a}.$$

PROOF.—In (2423) put  $a = 1$ , and substitute  $\frac{1}{x} = y$ ; multiply up by  $b$ , and integrate for  $b$  between limits 0 and  $\frac{b}{a}$ , and in the result substitute  $by$ .

$$2425 \quad \int_0^x e^{-kx^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{k}}. \quad [\text{Substitute } kx^2.]$$

$$2426 \quad \int_0^x e^{-kx^2} x^{2n} dx = \frac{1.3 \dots (2n-1)}{2^{n+1}} k^{-\frac{2n+1}{2}} \sqrt{\pi}.$$

PROOF.—Substitute  $kx^2$ . Otherwise, differentiate the preceding equation  $n$  times for  $k$ .

$$2427 \quad \int_0^x \frac{e^{-bx} - e^{-ax}}{x} dx = \log a - \log b.$$

$$2428 \quad \int_0^x \left( \frac{e^{-bx} - e^{-cx}}{x^2} - \frac{(c-b)e^{-ax}}{x} \right) dx = c - b + \log \frac{b^b a^c}{a^b c^c}.$$

PROOF.—Making  $n = 1$  in (2291),  $\int_0^x e^{-ax} dx = \frac{1}{a}$ . Integrate this for  $a$  between limits  $b$  and  $a$  to obtain (2427); and integrate that equation for  $b$  between limits  $b$  and  $c$  to obtain (2428).

$$2429 \quad \int_0^x \left( \frac{e^{-bx} - e^{-ax}}{x^2} - \frac{(a-b)e^{-ax}}{x} \right) dx = a - b + b \log \frac{b}{a}.$$

PROOF.—Make  $c = a$  in (2428).

Otherwise.—Integrating the first term by Parts, the whole reduces to

$$\left[ \frac{e^{-ax} - e^{-bx}}{x} \right]_0^x + b \int_0^x \frac{e^{-ax} - e^{-bx}}{x} dx.$$

The indeterminate fraction is evaluated by (1580) and the integral by (2427).

$$2430 \quad \int_0^x \left\{ \frac{e^{-bx} - e^{-ax}}{x^3} - \frac{(a-b)e^{-ax}}{x^2} - \frac{(a-b)^2 e^{-ax}}{2x} \right\} dx \\ = \frac{1}{4} \{ a^2 + 3b^2 - 4ab + 2b^2 (\log a - \log b) \}.$$

PROOF.—By two successive integrations by Parts,  $\int x^{-3} dx$ , &c.,

$$\int_{\epsilon}^x \frac{e^{-bx} - e^{-ax}}{x^3} dx = \frac{e^{-b\epsilon} - e^{-a\epsilon}}{2\epsilon^2} + \frac{ae^{-a\epsilon} - be^{-b\epsilon}}{2\epsilon} + \frac{1}{2} \int_{\epsilon}^x \frac{b^2 e^{-bx} - a^2 e^{-ax}}{x} dx.$$

Also 
$$\int_{\epsilon}^x \frac{e^{-ax}}{x^2} dx = \frac{e^{-a\epsilon}}{\epsilon} - \int_{\epsilon}^x \frac{ae^{-ax}}{x} dx.$$

Substitute these values, and make  $\epsilon = 0$ . The vanishing fractions are found by (1580), and the one resulting integral is that in (2427).

In a similar manner the value of the subjoined integral may be found.

2431

$$\int_0^{\infty} \left\{ \frac{e^{-bx} - e^{-ax}}{x^4} - \frac{(a-b)e^{-ax}}{x^3} - \frac{(a-b)^2 e^{-ax}}{1.2 x^2} - \frac{(a-b)^3 e^{-ax}}{1.2.3 x^3} \right\} dx.$$

## INTEGRATION OF CIRCULAR FORMS.

NOTATION.—Let  $a_b^{(n)}$  signify the continued product of  $n$  factors in arithmetical progression, the first of which is  $a$ , and the common difference of which is  $b$ , so that

$$2451 \quad a_b^{(n)} \equiv a(a+b)(a+2b) \dots \{a+(n-1)b\}.$$

Similarly, let

$$2452 \quad a_{-b}^{(n)} \equiv a(a-b)(a-2b) \dots \{a-(n-1)b\}.$$

These may be read, respectively, “ $a$  to  $n$  factors, difference  $b$ ”; “ $a$  to  $n$  factors, difference minus  $b$ .”

$$2453 \quad \int_0^{\frac{1}{2}\pi} \sin^n x dx = \frac{n-1}{n} \int_0^{\frac{1}{2}\pi} \sin^{n-2} x dx.$$

PROOF.—By (2048); applying Rule VI., we have, by division,

$$\int \sin^n x dx = \int \sin^{n-2} x dx - \int \sin^{n-2} x \cos^2 x dx,$$

and by Parts, 
$$\int \sin^{n-2} x \cos^2 x dx = \frac{\sin^{n-1} x \cos x}{n-1} + \frac{1}{n-1} \int \sin^n x dx.$$

Therefore 
$$\int_0^{\frac{1}{2}\pi} \sin^{n-2} x \cos^2 x dx = \frac{1}{n-1} \int_0^{\frac{1}{2}\pi} \sin^n x dx.$$

The substitution of this value in the first equation produces the formula.

If  $n$  be an integer, with the notation of (2451),

$$2454 \quad \int_0^{\frac{1}{2}\pi} \sin^{2n+1} x dx = \frac{2^{(n)}_2}{3^{(n)}_2} \quad \text{and} \quad \int_0^{\frac{1}{2}\pi} \sin^{2n} x dx = \frac{1^{(n)}_2}{2^{(n)}_2} \frac{\pi}{2}.$$

PROOF.—By repeated application of formula (2453).

Wallis's Formula.—If  $m$  be any positive integer, we have

$$2456 \quad 2m \left\{ \frac{2^{(m-1)}_2}{1^{(m)}_2} \right\}^2 > \frac{\pi}{2} > (2m-1) \left\{ \frac{2^{(m-1)}_2}{1^{(m)}_2} \right\}^2.$$



And since the ratio of these limits to each other constantly approaches unity as  $m$  increases, the value of either of them when  $m$  is infinite is  $\frac{1}{2}\pi$ .

Ex.—With  $m=4$ ,  $\frac{1}{2}\pi$  lies in magnitude between

$$\frac{2^2 \cdot 4^2 \cdot 6^2 \cdot 8}{1 \cdot 3^2 \cdot 5^2 \cdot 7^2} \quad \text{and} \quad \frac{2^3 \cdot 4^3 \cdot 6^3 \cdot 7}{1 \cdot 3^3 \cdot 5^3 \cdot 7^3}.$$

PROOF.—Put  $2m=n$ , then

$$\int_0^{\frac{1}{2}\pi} \sin^{n-1} x dx, \quad \int_0^{\frac{1}{2}\pi} \sin^n x dx, \quad \text{and} \quad \frac{n-1}{n} \int_0^{\frac{1}{2}\pi} \sin^{n-1} x dx$$

are in descending order of magnitude; the first and second because  $\sin x$  is  $< 1$ ; the second and third by (2453); then substitute the factorial values by (2454–5).

$$2457 \quad \int_0^{\frac{1}{2}\pi} \tan^{2m-1} \phi d\phi = \frac{\pi}{2 \sin m\pi}. \quad [\text{Subs. } x = \tan^2 \phi \text{ in (2311)}].$$

$$2458 \quad \int_0^\pi \sin^n x dx = 2 \int_0^{\frac{1}{2}\pi} \sin^n x dx. \quad [\text{By (2234)}].$$

$$2459 \quad \int_0^\pi \sin^n x \cos^p x dx = 2 \int_0^{\frac{1}{2}\pi} \sin^n x \cos^p x dx \quad \text{or zero,} \\ \text{according as } p \text{ is an even or odd integer.} \quad [\text{By (2234)}].$$

$$2461 \quad \int_0^{\frac{1}{2}\pi} \sin^n x \cos^p x dx = \int_0^{\frac{1}{2}\pi} \sin^p x \cos^n x dx. \quad [\text{By (2233)}].$$

$$2462 \quad \int_0^{\frac{1}{2}\pi} \sin^n x \cos^p x dx = \frac{1}{2} B\left(\frac{n+1}{2}, \frac{p+1}{2}\right). \quad [\text{Subs. } \sin^2 x \text{ (2280)}].$$

Let either of the integers  $n$  and  $p$ , in (2461), be odd, and the other either odd or even; thus, let  $n$  be odd and  $= 2m+1$ , then

$$2463 \quad \int_0^{\frac{1}{2}\pi} \sin^{2m+1} x \cos^p x dx = \frac{2^{(m)}_2}{(p+1)^{(m+1)}_2}. \quad (2451)$$

PROOF.—Transposing the indices by (2461), we have, by Parts (2067),

$$\int_0^{\frac{1}{2}\pi} \sin^p x \cos^{2m+1} x dx = \frac{2m}{p+1} \int_0^{\frac{1}{2}\pi} \sin^{p-2} x \cos^{2m-1} x dx.$$

By repeating the reduction, the integral finally arrived at is

$$2464 \quad \int_0^{\frac{1}{2}\pi} \sin^{p-2m} x \cos x dx = \frac{1}{p+2m+1}.$$

If both the indices are even, then

$$2465 \quad \int_0^{\frac{1}{2}\pi} \sin^{2m} x \cos^{2p} x dx = \frac{1_2^{(m)} 1_2^{(p)}}{2_2^{(m+p)}} \frac{p}{2}. \quad (2451)$$

PROOF.—Reduce by Parts as before. The final integral is  $\int_0^{\frac{1}{2}\pi} \sin^{2m+2p} x dx$ , the value of which is given at (2455).

**2466** Should either of the indices be a negative integer, the value of the integral is infinite, as the foregoing reduction shows, for the factor zero will then occur somewhere in the denominator.

$$2467 \quad \int_0^{\pi} \sin nx \sin px dx = \int_0^{\pi} \cos nx \cos px dx = 0,$$

when  $n$  and  $p$  are unequal integers.

$$2469 \quad \int_0^{\pi} \sin nx \cos px dx = \frac{2n}{n^2 - p^2} \text{ or zero,}$$

according as the difference of the integers  $n$  and  $p$  is *odd* or *even*. [By (1973–5).]

$$2470 \quad \int_0^{\pi} \sin^2 nx dx = \int_0^{\pi} \cos^2 nx dx = \frac{1}{2}\pi,$$

when  $n$  is an integer.

PROOF.—Express in terms of  $\cos 2nx$ , and then integrate.

$$2472 \quad \int_0^{\frac{1}{2}\pi} \sin^n x dx = \int_0^{\frac{1}{2}\pi} \cos^n x dx = \int_0^1 (1-x^2)^{\frac{n-1}{2}} dx.$$

The following four integrals (2473–9) all vanish for integral values of  $n$  and  $p$  excepting in the cases here specified.

$$2474 \quad \int_0^{\pi} \sin^p x \sin nx dx = (-1)^{\frac{n-1}{2}} C\left(p, \frac{p-n}{2}\right) \frac{\pi}{2^p},$$

when  $p$  and  $n$  are *both odd*, and  $n$  is not greater than  $p$ .

**2475** But if  $p$  be *even*, and  $n$  *odd*, the value is

$$(-1)^{\frac{p}{2}} \frac{n}{2^{p-2}} \left\{ \frac{1}{n^2 - p^2} - \frac{C(p, 1)}{n^2 - (p-2)^2} + \frac{C(p, 2)}{n^2 - (p-4)^2} - \dots \right. \\ \left. \dots (-1)^{\frac{p}{2}} \frac{C(p, \frac{1}{2}p)}{2n^2} \right\}.$$

$$2476 \quad \int_0^\pi \sin^p x \cos nx dx = (-1)^{\frac{n}{2}} C\left(p, \frac{p-n}{2}\right) \frac{\pi}{2^p},$$

when  $p$  and  $n$  are both even, and  $n$  is not greater than  $p$ .

2477 But if  $p$  be odd and  $n$  even, the value is

$$(-1)^{\frac{p-1}{2}} \frac{1}{2^{p-2}} \left\{ \frac{p}{p^2-n^2} - \frac{C(p, 1)(p-2)}{(p-2)^2-n^2} + \frac{C(p, 2)(p-4)}{(p-4)^2-n^2} - \dots \right. \\ \left. \dots (-1)^{\frac{p-1}{2}} \frac{C\left\{p, \frac{1}{2}(p-1)\right\}}{1-n^2} \right\}.$$

$$2478 \quad \int_0^\pi \cos^p x \cos nx dx = C\left(p, \frac{p-n}{2}\right) \frac{\pi}{2^p},$$

when  $p$  and  $n$  are either both odd or both even, and  $n$  is not greater than  $p$ .

2479  $\int_0^\pi \cos^p x \sin nx dx$ , when  $p \sim n$  is odd, takes the value

$$\frac{n}{2^{p-2}} \left\{ \frac{1}{n^2-p^2} + \frac{C(p, 1)}{n^2-(p-2)^2} + \frac{C(p, 2)}{n^2-(p-4)^2} + \&c. \right\},$$

the last term within the brackets being

$$\frac{C\left(p, \frac{p-1}{2}\right)}{n^2-1} \quad \text{when } p \text{ is odd and } n \text{ even, or} \quad \frac{C\left(p, \frac{p}{2}\right)}{n^2} \quad \text{when } p \text{ is even and } n \text{ odd.}$$

PROOF.—(For 2474 to 2479.)—Expand by (772-4), and apply (2467-2470) to the separate terms.

COROLLARIES.— $n$  being any integer,

$$2480 \quad \int_0^\pi \cos^n x \cos nx dx = \frac{\pi}{2^n}, \quad \int_0^{1\pi} \cos^n x \cos nx dx = \frac{\pi}{2^{n+1}}.$$

$$2482 \quad \int_0^\pi \sin^{2n} x \cos 2nx dx = (-1)^n \frac{\pi}{2^{2n}},$$

$$\int_0^\pi \sin^{2n+1} x \sin (2n+1)x dx = (-1)^n \frac{\pi}{2^{2n+1}}.$$

$$2484 \quad \int_0^{1\pi} \cos^p x \cos nx dx = \frac{p(p-1)}{p^2-n^2} \int_0^{1\pi} \cos^{p-2} x \cos nx dx.$$

2485

$$\int_0^{1\pi} \cos^p x \sin nx dx = \frac{p(p-1)}{p^2-n^2} \int_0^{1\pi} \cos^{p-2} x \sin nx dx - \frac{n}{p^2-n^2}.$$

PROOF.—(For either formula) By Parts,  $\int \cos x dx$ ; and the new integral of highest dimensions in  $\cos x$ , by Parts,  $\int \cos^{p-1} x \sin x dx$ .

$$2486 \quad \int_0^{1\pi} \cos^{n-2} x \cos nx dx = 0. \quad \int_0^{1\pi} \cos^{n-2} x \sin nx dx = \frac{1}{n-1}.$$

PROOF.—Make  $p=n$  in (2484-5).

When  $k$  is a positive integer,

$$2488 \quad \int_0^{1\pi} \cos^{n-2k} x \cos nx dx = 0,$$

$$2489 \quad \int_0^{1\pi} \cos^{n+2k} x \cos nx dx = \frac{(n+2k)_{-1}^{(k)}}{1^{(k)}} \frac{\pi}{2^{n+2k+1}}.$$

PROOF.—The first, by putting  $p=n-2, n-4, \dots, n-2k$  successively in (2484) and employing (2486). The second, by putting  $p=n+2, n+4, \dots, n+2k$  successively and employing (2481).

When  $k$  is *not* an integer,

$$2490 \quad \int_0^{1\pi} \cos^{n-2k} x \cos nx dx = 2^{2-n+1} \sin k\pi B(n-2k+1, k).$$

PROOF.—In (2706) take  $f(a) = a^{n-2k}$ , and transform by (766). The coefficient of  $i$  vanishes by (2239), and the limits are changed by (2237).

$$2491 \quad \int_0^{1\pi} \cos^n x \sin nx dx = \frac{1}{2^{n+1}} \left( 2 + \frac{2^2}{2} + \frac{2^3}{3} + \dots + \frac{2^n}{n} \right).$$

PROOF.—By successive reduction by (1970), making  $m=n$ , and the integral definite.

2492 When  $p$  and  $n$  are integers, one odd and the other even,

$$\int_0^{1\pi} \cos^p x \cos nx dx = \frac{(-1)^{\frac{1}{2}(p+n+1)} 1^{(p)}}{(n-p)_{\frac{1}{2}}^{(p+1)}} \begin{cases} +, & \text{with } p \text{ odd,} \\ -, & \text{with } p \text{ even.} \end{cases}$$

PROOF.—Reduce successively by (2484). The final integral, according as  $p$  is odd or even, will be

$$\int_0^{1\pi} \cos x \cos nx dx = \frac{\cos \frac{1}{2}n\pi}{1-n^2} = \frac{(-1)^{\frac{1}{2}n}}{1-n^2} \quad \text{or} \quad \int_0^{1\pi} \cos nx dx = \frac{\sin \frac{1}{2}n\pi}{n} = \frac{(-1)^{\frac{1}{2}(n-1)}}{n}.$$

3 C

$$2493 \quad \int_0^\pi \cos^p x \cos nx dx = \frac{p^{(n)}_{-1}}{1^{(n)}_2} \int_0^\pi \sin^{2n} x \cos^{p-n} x dx,$$

where  $n$  and  $p$  are any integers whatever such that  $p-n$  is  $> 0$ .

PROOF.—When  $p-n$  is odd, each integral vanishes, by (2478) and (2459).

When  $p-n$  is even, let it  $= 2k$ ; then, by (2488),

$$\begin{aligned} \int_0^\pi \cos^{n+2k} x \cos nx dx &= \frac{(n+2k)^{(k)}_{-1}}{1^{(k)}_2} \frac{\pi}{2^{n+2k-1}} = \frac{(n+2k)^{(n)}_{-1}}{1^{(n)}_2} \frac{1^{(n)}_2 1^{(k)}_2}{2^{(n+k)}_2} \frac{\pi}{2} \\ &= \frac{(n+2k)^{(n)}_{-1}}{1^{(n)}_2} \int_0^\pi \sin^{2n} x \cos^{2k} x dx, \quad (\text{by 2465}). \end{aligned}$$

But  $n+2k=p$ , and by (2234) the limit may be doubled. Hence the result.

$$2494 \quad \int_0^{\frac{1}{2}\pi} x \cos^{n-2} x \sin nx dx = \frac{\pi}{2^n(n-1)}.$$

PROOF.—In (2707), put  $k=1$  and  $f(x) = x^{n-2}$ . Give  $e^{inx}$  its value from (766). The imaginary term in the result vanishes, and the limits are changed, by (2237). Finally, write  $x$  instead of  $\theta$ .

2495

$$\int_0^\pi f^n(\cos x) \sin^{2n} x dx = 1.3 \dots (2n-1) \int_0^\pi f(\cos x) \cos nx dx.$$

PROOF.—Let  $z = \cos x$ . By (1471), we have

$$d_{(n-1)z}(1-z^2)^{n-\frac{1}{2}} = (-1)^{n-1} 1.3 \dots (2n-1) \frac{\sin nx}{n} \dots \dots \dots (i.)$$

Also, by integrating  $n$  times by Parts,

$$\begin{aligned} \int_{-1}^1 f^n(z) (1-z^2)^{n-\frac{1}{2}} dz &= (-1)^n \int_{-1}^1 f(z) d_{nz} (1-z^2)^{n-\frac{1}{2}} dz \\ &= -1.3 \dots (2n-1) \int_{-1}^1 f(z) dz \left( \frac{\sin nx}{n} \right) dz, \quad \text{by (i.)} \end{aligned}$$

Then substitute  $z = \cos x$ .

Otherwise.—Let  $f(z) = A_0 + A_1 z + A_2 z^2 + \&c. = \Sigma A_p z^p$ ,

$$\therefore f^n(z) = \Sigma p(p-1) \dots (p-n+1) A_p z^{p-n},$$

$$\begin{aligned} \therefore \int_0^\pi f(\cos x) \cos nx dx &= \Sigma A_p \int_0^\pi \cos^p x \cos nx dx \\ &= \frac{1}{1.3 \dots (2n-1)} \int_0^\pi f^n(\cos x) \sin^{2n} x dx, \quad \text{by (2493)}. \end{aligned}$$

$$2496 \quad \int_0^{\frac{1}{2}\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{2ab}. \quad (1982)$$

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$$2497 \quad \int_0^{\frac{1}{2}\pi} \frac{\cos^2 x dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} = \frac{\pi}{4a^3b}. \quad [\text{Differentiate (2496) for } a.]$$


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$$2498 \quad \int_0^{\frac{1}{2}\pi} \frac{\sin^2 x dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} = \frac{\pi}{4ab^3}. \quad [\text{Differentiate (2496) for } b.]$$


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$$2499 \quad \int_0^{\frac{1}{2}\pi} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} = \frac{\pi}{4ab} \left( \frac{1}{a^2} + \frac{1}{b^2} \right).$$

[Add together (2497-8)]

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$$2500 \quad \int_0^{\frac{1}{2}\pi} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^3} = \frac{\pi}{16ab} \left( \frac{3}{a^4} + \frac{2}{a^2b^2} + \frac{3}{b^4} \right).$$


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$$2501 \quad \int_0^{\frac{1}{2}\pi} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} = \frac{\pi}{32ab} \left( \frac{5}{a^6} + \frac{3}{a^4b^2} + \frac{3}{a^2b^4} + \frac{5}{b^6} \right).$$

(2500) and (2501) are obtained by repeating upon (2499) the operations by which that integral was itself obtained from (2496).

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$$2502 \quad \int_0^1 \frac{\tan^{-1} ax}{x \sqrt{(1-x^2)}} dx = \frac{\pi}{2} \log(a + \sqrt{1+a^2}).$$

PROOF.—Denote the integral by  $u$ .

$$\frac{du}{da} = \int_0^1 \frac{dx}{(1+a^2x^2)\sqrt{1-x^2}} = \frac{\pi}{2} \frac{1}{\sqrt{1+a^2}}, \quad [\text{by (2008)}]$$

$$\therefore u = \frac{\pi}{2} \int_0^a \frac{da}{\sqrt{1+a^2}} = \frac{\pi}{2} \log(a + \sqrt{1+a^2}). \quad (1928)$$


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$$2503 \quad \int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)} dx = \frac{\pi}{2} \log(1+a).$$

PROOF.—Differentiate for  $a$ . Integrate for  $x$  by partial fractions, and then integrate for  $a$ .

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$$2504 \quad \int_0^\infty \tan^{-1} \frac{x}{a} \tan^{-1} \frac{x}{b} \frac{dx}{x^2} = \frac{\pi}{2} \log \left\{ \left(1 + \frac{a}{b}\right)^{\frac{1}{a}} \left(1 + \frac{b}{a}\right)^{\frac{1}{b}} \right\}.$$

PROOF.—From (2503) we obtain

$$\int_0^\infty \frac{\tan^{-1} x}{x(a^2+x^2)} dx = \frac{\pi}{2} \frac{\log(1+a)}{a^2}.$$

Integrate for  $a$  between limits  $\frac{a}{b}$  and  $\infty$ , and in the result substitute  $bx$ .



$$2505 \quad \int_0^{\infty} \frac{\tan^{-1} ax - \tan^{-1} bx}{x} dx = \frac{\pi}{2} \log \frac{a}{b}.$$

PROOF.—Applying (2700),  $\phi(0)$  here vanishes. Also, by Parts, we have

$$\int_{\frac{a}{b}}^{\frac{a}{a}} \frac{\tan^{-1}(bx)}{x} dx = \frac{\pi}{2} \int_{\frac{a}{a}}^{\frac{a}{b}} \frac{dx}{x}$$

since  $bx$  is infinite and therefore  $\tan^{-1}(bx) = \frac{\pi}{2}$  in every element of the integral. Hence the required value is

$$\frac{\pi}{2} \int_{\frac{a}{b}}^{\frac{a}{a}} \frac{dx}{x} = \frac{\pi}{2} \log \frac{a}{b}.$$

$$2506 \quad \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi^2}{4}.$$

PROOF.—(i.) Substitute  $\pi - x$ , and the integral is reproduced, and is thus shown to be

$$= \frac{\pi}{2} \int_0^{\pi} \frac{\sin y}{1 + \cos^2 y} dy = -\frac{\pi}{2} (\tan^{-1} \cos \pi - \tan^{-1} \cos 0) = \frac{\pi^2}{4}.$$

(ii.) Otherwise, expand by dividing by the denominator, and integrate each term of the result by Parts. Employing (2478) we obtain the series

$$\pi \left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c. \dots \right) = \frac{\pi^2}{4}. \quad [\text{By (2945)}].$$

$$2507 \quad \int_0^{\infty} \frac{\cos x}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2}} = \int_0^{\infty} \frac{\sin x}{\sqrt{x}} dx.$$

PROOF.—By the method of (2251), putting

$$\frac{1}{\sqrt{x}} = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-xy^2} dy,$$

the integral becomes

$$\frac{2}{\sqrt{\pi}} \int_0^{\infty} \int_0^{\infty} e^{-xy^2} \cos x dx dy = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \int_0^{\infty} e^{-xy^2} \cos x dy dx \quad (2261)$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{y^2 dy}{1+y^4} \quad (2610) = \sqrt{\frac{\pi}{2}}. \quad (2348)$$

The second integral is obtained in a similar manner.

$$2509 \quad \int_0^{\infty} \cos y^2 dy = \frac{\sqrt{\pi}}{2\sqrt{2}} = \int_0^{\infty} \sin y^2 dy.$$

PROOF.—Substitute  $y^2$ , and (2507-8) are produced.

When  $n$  and  $p$  are integers,

$$2510 \quad \int_0^\infty \frac{\sin^n x}{x^p} dx = \frac{1}{1.2 \dots (p-1)} \int_0^\infty \int_0^\infty z^{p-1} e^{-zx} \sin^n x dz dx.$$

The integration for  $x$  in the double integral is given in (2608-9), and the original integral is thus reduced to the integral of a rational fraction.

PROOF.—By the method of (2251), putting

$$\frac{1}{x^p} = \frac{1}{\Gamma(p)} \int_0^\infty e^{-zx} z^{p-1} dz. \quad [\text{By (2291).}]$$

$$2511 \quad \int_0^\infty \frac{\sin^2 x}{x^2} dx = 2 \int_0^\infty \frac{dz}{z^2+4} = \frac{\pi}{2}. \quad [\text{By (2510).}]$$

$$2512 \quad \int_0^\infty \frac{\sin^3 x}{x} dx = 6 \int_0^\infty \frac{dz}{(z^2+1)(z^2+9)} = \frac{\pi}{4}. \quad [\text{By (2510) \& (2081).}]$$

$$2513 \quad \int_0^\infty \frac{\cos qx - \cos px}{x} dx = \log \frac{p}{q}.$$

PROOF.—By (2700). Transforming the numerator by (673), and putting  $\frac{1}{2}(p+q) = a$ ,  $\frac{1}{2}(p-q) = b$ , this becomes

$$2514 \quad \int_0^\infty \frac{\sin ax \sin bx}{x} dx = \frac{1}{2} \log \frac{a+b}{a-b}.$$

$$2515 \quad \int_0^\infty \frac{\cos qx - \cos px}{x^2} dx = \frac{\pi}{2} (p-q).$$

PROOF.—Integrate (2572) for  $r$  between the limits  $p$  and  $q$ .

If  $a$  and  $b$  are positive quantities,

$$2516 \quad \int_0^\infty \frac{\sin ax \cos bx}{x} dx = \frac{\pi}{2} \quad \text{or } 0,$$

according as  $a$  is  $>$  or  $<$   $b$ .

PROOF.—Change by (666), and employ (2572).

$$2518 \quad \int_0^\infty \frac{\sin ax \sin bx}{x^2} dx = \frac{\pi a}{2} \quad \text{or} \quad \frac{\pi b}{2},$$

according as  $a$  or  $b$  is the least of the two numbers.

PROOF.—From (2515), exactly as in (2513).

Otherwise, as an illustration of the method in (2252), as follows. Denoting the integral in (2516) by  $u$ , we have, (i.) when  $b$  is  $> a$ ,

$$\int_0^b u db = \int_0^a \frac{\pi}{2} db + \int_a^b 0 db = \frac{\pi a}{2};$$

that is, 
$$\frac{\pi a}{2} = \int_0^b \int_0^a \frac{\sin ax \cos bx}{x} db dx = \int_0^a \frac{\sin ax \sin bx}{x^2} dx. \quad (2261)$$

(ii.) When  $b$  is  $< a$ , 
$$\int_0^b u db = \int_0^b \frac{\pi}{2} db = \frac{\pi b}{2}.$$

If  $a$  is a positive quantity,

**2520** 
$$\int_0^\infty \frac{\sin^2 x \cos ax}{x^2} dx = \frac{\pi}{4} (2-a) \text{ or } 0,$$

according as  $a$  is or is not less than 2.

PROOF.— $\sin^2 x \cos ax = \frac{1}{2} \sin x \{ \sin (1+a)x + \sin (1-a)x \}$ ; and the result then follows from (2518), the value of the integral being in the two cases  $\frac{\pi}{4} - \frac{\pi}{4} = 0$  and  $\frac{\pi}{4} - \frac{\pi}{4} (a-1) = \frac{\pi}{4} (2-a)$ .

**2522** 
$$\int_0^\infty \frac{\sin^2 x \sin ax}{x^3} dx = \frac{\pi}{2} \text{ or } \frac{a\pi}{2} \left(1 - \frac{a}{4}\right),$$

according as  $a$  is  $>$  or  $<$  2.

PROOF.—Denote the integral in (2520) by  $u$ ; then, when  $a$  is  $> 2$ , the present integral is equal to

$$\int_0^a u da = \int_0^2 \frac{\pi}{4} (2-a) da + \int_2^a 0 da = \frac{\pi}{2}.$$

And, when  $a$  is  $< 2$ , 
$$\int_0^a u da = \int_0^a \frac{\pi}{4} (2-a) da = \frac{\pi a}{2} - \frac{\pi a^2}{8}.$$

## INTEGRATION OF CIRCULAR LOGARITHMIC AND EXPONENTIAL FORMS.

**2571** 
$$\int_0^\infty \frac{e^{-ax} \sin rx}{x} dx = \tan^{-1} \frac{r}{a}.$$

PROOF.—Differentiate for  $r$ , and integrate by (2584).

Otherwise.—Expand  $\sin rx$  by (764), and integrate the terms by (2291). Gregory's series (791) is the result.

$$2572 \quad \int_0^\infty \frac{\sin rx}{x} dx = \frac{\pi}{2}.$$

PROOF.—(i.) By making  $a = 0$  in (2571).

(ii.) *Otherwise.* By the method of (2250). First, observing that the integral is independent of  $r$ , which may be proved by substituting  $rx$ , let  $r = 1$ .

Then 
$$\int_0^\pi \frac{\sin x}{x} dx = \int_0^\pi \frac{\sin x}{x} dx + \int_\pi^{2\pi} \frac{\sin x}{x} dx + \int_{2\pi}^{3\pi} \frac{\sin x}{x} dx + \&c.$$

Now,  $n$  being an integer, the general term is either

$$\int_{(2n-1)\pi}^{2n\pi} \frac{\sin x}{x} dx = \int_0^\pi \frac{-\sin y dy}{(2n-1)\pi + y}, \text{ by substituting } x = (2n-1)\pi + y,$$

or 
$$\int_{(2n-2)\pi}^{(2n-1)\pi} \frac{\sin x}{x} dx = \int_0^\pi \frac{\sin y dy}{(2n-1)\pi - y}, \text{ by substituting } x = (2n-1)\pi - y;$$

$$\begin{aligned} \therefore \int_0^\pi \frac{\sin x}{x} dx &= \int_0^\pi \sin y \left\{ \frac{1}{\pi - y} - \frac{1}{\pi + y} + \frac{1}{3\pi - y} - \frac{1}{3\pi + y} + \frac{1}{5\pi - y} - \&c. \right\} dy \\ &= \frac{1}{2} \int_0^\pi \sin y \tan \frac{y}{2} dy \quad (2913) = \int_0^\pi \sin^2 \frac{y}{2} dy = \frac{\pi}{2}. \end{aligned}$$

$$2573 \quad \int_0^\infty \frac{\cos rx}{1+x^2} dx = \frac{\pi}{2} e^{-r} = \int_0^\infty \frac{r \cos x}{r^2 + x^2} dx.$$

PROOF.—(i.) By (2251), putting  $\frac{1}{1+x^2} = 2 \int_0^\infty e^{-(1+x^2)y^2} dy$  (2291), the integral takes the form

$$\begin{aligned} 2 \int_0^\infty \int_0^\infty \cos rx e^{-(1+x^2)y^2} y dx dy &= 2 \int_0^\infty \int_0^\infty e^{-y^2} y e^{-y^2 x^2} \cos rx dy dx \\ &= \sqrt{\pi} \int_0^\infty e^{-y^2} \frac{r}{y^2} dy \quad (2614) = \frac{\pi e^{-r}}{2} \quad (2604). \end{aligned}$$

(ii.) *Otherwise.* By the method of (2252), putting  $u = \int_0^\infty \frac{\sin ax \cos bx}{x} dx$ , it follows from (2516) that

$$\int_0^\infty u e^{-a} da = \int_0^b 0 e^{-a} da + \int_b^\infty \frac{\pi}{2} e^{-a} da = \frac{\pi}{2} e^{-b}.$$

Therefore 
$$\frac{\pi}{2} e^{-b} = \int_0^\infty \int_0^\infty \frac{\sin ax \cos bx}{x} e^{-a} da dx = \int_0^\infty \frac{\cos bx}{1+x^2} dx, \text{ by (2583).}$$

$$2575 \quad \int_0^\infty \frac{x \sin rx}{1+x^2} dx = \frac{\pi}{2} e^{-r}.$$

$$2576 \quad \int_0^\infty \frac{\sin rx}{x(1+x^2)} dx = \frac{\pi}{2} (1 - e^{-r}).$$

PROOF.—For (2575) differentiate, and for (2576) integrate equation (2573) with respect to  $r$ .

$$2577 \quad \int_0^\infty e^{-ax} x^{n-1} \frac{\sin}{\cos}(bx) dx = \frac{\Gamma(n)}{(a^2+b^2)^{\frac{1}{2}n}} \frac{\sin}{\cos}\left(n \tan^{-1} \frac{b}{a}\right).$$

PROOF.—By (2291),  $\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}.$

Put  $k = a + ib$ , and  $a = r \cos \theta$ ,  $b = r \sin \theta$ ; thus

$$\int_0^\infty e^{-(a+ib)x} x^{n-1} dx = (\cos n\theta - i \sin n\theta) \frac{\Gamma(n)}{r^n},$$

by (757). Substitute on the left side for  $e^{-ibx}$  from (767), and equate real and imaginary parts. Otherwise, as in (2259).

$$2579 \quad \int_0^\infty x^{n-1} \frac{\sin}{\cos}(bx) dx = \frac{\Gamma(n)}{b^n} \frac{\sin}{\cos}\left(\frac{n\pi}{2}\right).$$

PROOF.—Make  $a = 0$  in (2577).

$$2581 \quad \int_0^\infty \frac{\frac{\sin}{\cos}(bx)}{x^m} dx = \frac{b^{m-1} \pi}{\Gamma(m) 2^{\frac{\sin}{\cos}\left(\frac{m\pi}{2}\right)}}.$$

PROOF.—Put  $n = 1 - m$  in (2579), and employ

$$\Gamma n \Gamma(1-n) = \frac{\pi}{\sin n\pi} = \frac{\pi}{\sin m\pi}.$$

$$2583 \quad \int_0^\infty e^{-ax} \sin bx dx = \frac{b}{a^2+b^2}, \quad \int_0^\infty e^{-ax} \cos bx dx = \frac{a}{a^2+b^2}.$$

PROOF.—Make  $n = 1$  in (2577-8).

Otherwise.—Directly from (1999), putting  $n = 1$ , and  $-a$  for  $a$ .

$$2585 \quad \int_0^{\frac{1}{2}\pi} \cos^{p-n-1} \theta \sin^{n-1} \theta \frac{\sin}{\cos}(p\theta) d\theta = \frac{\Gamma(p-n) \Gamma(n)}{\Gamma(p)} \frac{\sin}{\cos}\left(\frac{n\pi}{2}\right),$$

where  $n$  is a positive integer  $> 1$ .

PROOF.—In (2577), put  $\tan^{-1} \frac{b}{a} = \theta$ , thus, writing  $p$  for  $n$ ,

$$\int_0^\infty e^{-ax} x^{p-1} \sin bx dx = \frac{\Gamma(p)}{a^p} \cos^p \theta \sin p\theta.$$

Multiply this equation by  $b^{n-1} db = a^n \tan^{n-1} \theta \sec \theta d\theta$ .

and integrate from  $b = 0$  to  $\infty$ , by (2579). Then the corresponding limits in the integration for  $\theta$  will be 0 and  $\frac{1}{2}\pi$ .

$$2587 \quad \int_0^{1\pi} \sin^{p-2} \theta \frac{\sin}{\cos}(p\theta) d\theta = \frac{-\cos\left(\frac{p\pi}{2}\right)}{+\sin\left(\frac{p\pi}{2}\right)} \frac{1}{p-1}.$$

PROOF.—Put  $n = p-1$  in (2585).

$$2589 \quad \int_0^\infty e^{-x} x^{n-1} \frac{\sin}{\cos} \left\{ (x \tan \theta) \right\} dx = \Gamma(n) \cos^n \theta \frac{\sin}{\cos} \left\{ n\theta \right\}.$$

PROOF.—In (2709), let  $\phi(x) = \cos(x \tan \theta)$ ;

$\therefore$  by (765),  $A_0 = 1$ ,  $A_1 = -\frac{\tan^2 \theta}{1.2}$ ,  $A_2 = \frac{\tan^4 \theta}{1.2.3.4}$ , &c.;  $A_1, A_3$ , &c. vanishing. Therefore

$$1 - \frac{\alpha(\alpha+1)}{1.2} \tan^2 \theta + \frac{\alpha^{(4)}}{1^{(4)}} \tan^4 \theta - \frac{\alpha^{(6)}}{1^{(6)}} \tan^6 \theta + \dots = \frac{\int_0^\infty e^{-x} x^{\alpha-1} \cos(x \tan \theta) dx}{\int_0^\infty e^{-x} x^{\alpha-1} dx}.$$

The series on the left  $= \frac{1}{2}(1+i \tan \theta)^{-\alpha} + \frac{1}{2}(1-i \tan \theta)^{-\alpha}$ , which by the values (770) and (768) reduces to  $\cos \alpha \theta \cos^\alpha \theta$ . Then change  $\alpha$  into  $n$ . Similarly, with *sine* in the place of *cosine*.

$$2591 \quad \int_0^\infty \frac{e^{-ax} - e^{-\beta x}}{x} \sin bx dx = \tan^{-1} \frac{\beta}{b} - \tan^{-1} \frac{a}{b}.$$

PROOF.—Integrate (2583) for  $a$  between  $a = a$  and  $a = \beta$ .

$$2592 \quad \int_0^\infty e^{-kx} \cos ax \sin^n x dx,$$

where  $n$  is any positive integer.

See (2717-20) for the values of this integral.

$$2593 \quad \int_0^\infty \frac{e^{ax} + e^{-ax}}{e^{\pi x} - e^{-\pi x}} \sin mx dx = \frac{1}{2} \frac{e^m - e^{-m}}{e^m + 2 \cos a + e^{-m}},$$

$a$  being  $< \pi$ .

PROOF.—The function expanded by division becomes

$$(e^{ax} + e^{-ax}) \sin mx (e^{-\pi x} + e^{-3\pi x} + e^{-5\pi x} + \&c.)$$

Multiply in and integrate by (2583). The result is

$$\sum \frac{m}{\{(2n-1)\pi - a\}^2 + m^2} + \sum \frac{m}{\{(2n-1)\pi + a\}^2 + m^2}.$$

But this series is also produced by differentiating the logarithm of equation (2953). Hence the result.



$$2594 \quad \int_0^\infty \frac{e^{ax} - e^{-ax}}{e^{\pi x} - e^{-\pi x}} \cos mx \, dx = \frac{\sin a}{e^m + 2 \cos a + e^{-m}}.$$

PROOF.—Change  $m$  into  $i\theta$  in (2593), thus

$$\int_0^\infty \frac{(e^{ax} + e^{-ax})(e^{\theta x} - e^{-\theta x})}{e^{\pi x} - e^{-\pi x}} \, dx = \frac{\sin \theta}{\cos a + \cos \theta}.$$

Now change  $a$  into  $im$  and write  $a$  instead of  $\theta$ .

$$2595 \quad \int_0^\infty \frac{e^{ax} - e^{-ax}}{e^{\pi x} - e^{-\pi x}} \, dx = \frac{\tan \frac{1}{2}a}{2}. \quad \int_0^\infty \frac{\sin mx \, dx}{e^{\pi x} - e^{-\pi x}} = \frac{1}{4} \frac{e^{\frac{1}{2}m} - e^{-\frac{1}{2}m}}{e^{\frac{1}{2}m} + e^{-\frac{1}{2}m}}.$$

PROOF.—Make  $m = 0$  in (2594), and  $a = 0$  in (2593).

$$2597 \quad \int_0^\infty \frac{e^{\pi x} + e^{-\pi x}}{e^{\pi x} - e^{-\pi x}} \sin mx \, dx = \frac{1}{2} \frac{e^{\frac{1}{2}m} + e^{-\frac{1}{2}m}}{e^{\frac{1}{2}m} - e^{-\frac{1}{2}m}}.$$

PROOF.—Make  $a = \pi$  in (2593).

$$2598 \quad \int_0^\infty \frac{x^{2n-1} \, dx}{e^{\pi x} - e^{-\pi x}} = \frac{2^{2n} - 1}{4n} B_{2n}.$$

PROOF.—Expand  $\sin mx$  on the left side of (2596) by (764). The right side is  $= -\frac{1}{4}i \tan(\frac{1}{2}im)$  by (770). Expand this by (2917), and equate the coefficients of the same powers of  $m$ .

$$2599 \quad \int_0^\infty \frac{e^{ax} - e^{-ax}}{e^{\frac{1}{2}\pi x} + e^{-\frac{1}{2}\pi x}} \sin mx \, dx = \frac{2(e^m - e^{-m}) \sin a}{e^{2m} + 2 \cos 2a + e^{-2m}}.$$

$$2600 \quad \int_0^\infty \frac{e^{ax} + e^{-ax}}{e^{\frac{1}{2}\pi x} + e^{-\frac{1}{2}\pi x}} \cos mx \, dx = \frac{2(e^m + e^{-m}) \cos a}{e^{2m} + 2 \cos 2a + e^{-2m}}.$$

PROOF.—To obtain (2599), put  $a + \frac{1}{2}\pi$  and  $a - \frac{1}{2}\pi$  successively for  $a$  in equation (2593), and take the difference of the results. (2600) is obtained in the same way from (2594).

$$2601 \quad \int_0^\infty \frac{e^{ax} + e^{-ax}}{e^{\frac{1}{2}\pi x} + e^{-\frac{1}{2}\pi x}} \, dx = \sec a.$$

PROOF.—Make  $m = 0$  in (2600).

$$2602 \quad \int_0^\infty \sin(cx)^2 \, dx = \int_0^\infty \cos(cx)^2 \, dx = \frac{\sqrt{\pi}}{2c\sqrt{2}}.$$

PROOF.—By (2425)  $\int_0^\infty e^{-a^2 x} dx = \frac{\sqrt{\pi}}{2a}.$

Put  $a = \frac{1+i}{\sqrt{2}} c$ . Substitute on the left from (766), and equate real and imaginary parts.

**2604**  $\int_0^\infty e^{-(x^2 + \frac{a^2}{x^2})} dx = \frac{\sqrt{\pi}}{2} e^{-2a}.$

PROOF.—Denote the integral by  $u$ . Differentiate the equation for  $a$ , and substitute  $\frac{a}{x}$  in the resulting integral to prove that  $\frac{du}{da} = -2u$ , and therefore  $u = Ce^{-2a}$ . When  $a = 0$ , we get

$$\int_0^\infty e^{-x^2} dx = C, \therefore C = \frac{1}{2}\sqrt{\pi} \text{ (2425).}$$

**2605**  $\int_0^\infty e^{-(x^2 + \frac{a^2}{x^2})k} dx = \frac{\sqrt{\pi}}{2\sqrt{k}} e^{-2ak}.$

PROOF.—Substitute  $x\sqrt{k}$ , and integrate by (2604).

**2606**  $\int_0^\infty e^{-(x^2 + \frac{a^2}{x^2})\cos\theta} \frac{\cos\left[\left(x^2 + \frac{a^2}{x^2}\right)\sin\theta\right]}{\sin\left[\left(x^2 + \frac{a^2}{x^2}\right)\sin\theta\right]} dx$   
 $= \frac{\sqrt{\pi}}{2} e^{-2a\cos\theta} \frac{\cos\left(2a\sin\theta + \frac{\theta}{2}\right)}{\sin\left(2a\sin\theta + \frac{\theta}{2}\right)}.$

PROOF.—In (2605) put  $k = \cos\theta + i\sin\theta$ ; substitute from (766), and equate real and imaginary parts.

**2608**  $\int_0^\infty e^{-ax} \sin^{2n+1} x dx = \frac{1.2.3 \dots (2n+1)}{(a^2+1)(a^2+3^2) \dots (a^2+2n+1^2)}.$

**2609**  $\int_0^\infty e^{-ax} \sin^{2n} x dx = \frac{1.2.3 \dots 2n}{a(a^2+2^2)(a^2+4^2) \dots (a^2+2n^2)}.$

**2610**  $\int_0^\infty e^{-ax} \cos^{2n+1} x dx = \frac{a}{a^2 + (2n+1)^2} + \frac{a(2n+1)2n}{(a^2 + 2n+1^2)(a^2 + 2n-1^2)}$   
 $+ \frac{a(2n+1)2n(2n-1)(2n-2)}{(a^2 + 2n+1^2)(a^2 + 2n-1^2)(a^2 + 2n-3^2)} + \dots + \frac{a \cdot 2n+1}{(a^2 + 2n+1^2) \dots (a^2 + 1)}.$

$$2611 \quad \int_0^x e^{-ax} \cos^{2n} x \, dx = \frac{a}{a^2 + (2n)^2} + \frac{a \, 2n \, (2n-1)}{(a^2 + 2n^2)(a^2 + 2n-2^2)} \\ + \frac{a \, 2n \, (2n-1) \, (2n-2) \, (2n-3)}{(a^2 + 2n^2)(a^2 + 2n-2^2)(a^2 + 2n-4^2)} + \dots + \frac{a \, |2n|}{(a^2 + 2n^2) \dots (a^2 + 2^2)}.$$

PROOF OF (2608-11). — Reduce successively by (1999). The integral part after each reduction disappears between the limits in the cases (2608-9), but not in the cases (2610-1). See also (2721).

2612

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{-ax} \cos^{2n+1} x \, dx = \frac{1.2.3 \dots (2n+1)}{(a^2+1)(a^2+3^2) \dots (a^2+2n+1^2)} (e^{\frac{a\pi}{2}} + e^{-\frac{a\pi}{2}})$$

2613

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{-ax} \cos^{2n} x \, dx = \frac{1.2.3 \dots 2n}{a(a^2+2^2)(a^2+4^2) \dots (a^2+2n^2)} (e^{\frac{a\pi}{2}} - e^{-\frac{a\pi}{2}}).$$

PROOF.—By successive reduction by (1999).

2614

$$\int_0^\infty e^{-a^2 x^2} \cos 2bx \, dx = \frac{\sqrt{\pi}}{2a} e^{-\frac{b^2}{a^2}}.$$

PROOF.—Denote the integral by  $u$ , then

$$\frac{du}{db} = - \int_0^\infty e^{-a^2 x^2} 2x \sin 2bx \, dx = - \int_0^\infty \frac{2b}{a^2} e^{-a^2 x^2} \cos 2bx \, dx = - \frac{2bu}{a^2},$$

the second integration being effected by parts,  $\int e^{-a^2 x^2} 2x \, dx$ . Therefore  $\log u = \log C - \frac{b^2}{a^2}$ ; and  $b = 0$  gives  $C = \frac{\sqrt{\pi}}{2a}$  (2425).

Otherwise.—Expand the cosine by (765), and integrate the terms of the product by (2426). Thus the general term is

$$(-1)^n \frac{(2b)^{2n}}{|2n|} \int_0^\infty e^{-a^2 x^2} x^{2n} \, dx = (-1)^n \frac{(2b)^{2n}}{|2n|} \frac{1.3.5 \dots (2n-1)}{2^{n+1} a^{2n+1}} \sqrt{\pi}. \\ = (-1)^n \frac{\sqrt{\pi}}{2a} \left(\frac{b}{a}\right)^{2n} \frac{1}{1.2 \dots n}, \text{ which gives the required result by (150).}$$

2615

$$\int_0^\infty e^{-a^2 x^2} \cosh 2bx = \frac{\sqrt{\pi}}{2a} e^{\left(\frac{b}{a}\right)^2}. \quad (2181)$$

PROOF.—Change  $b$  into  $ib$  in (2614).

$$2617 \quad \int_0^\infty e^{-x} x \sin 2bx dx = \frac{b\sqrt{\pi}}{2} e^{-b^2}.$$

$$2618 \quad \int_0^\infty e^{-x} x^{n+1} \sin (2bx + \tfrac{1}{2}n\pi) dx = \frac{\sqrt{\pi}}{2^{n+1}} \frac{d^n}{db^n} (be^{-b^2}).$$

PROOF.—To obtain (2617), put  $a=1$  in (2614), and differentiate for  $b$ . To obtain (2618), differentiate, in all,  $n+1$  times for  $b$ .

$$2619 \quad \int_0^\infty \frac{\cos x - e^{-ax}}{x} dx = \log a.$$

PROOF.—By (2251) putting  $\frac{1}{x} = \int_0^\infty e^{-xy} dy$  (2291), and changing the order of integration, the integral becomes

$$\begin{aligned} \int_0^\infty \int_0^\infty (\cos x - e^{-ax}) e^{-xy} dy dx &= \int_0^\infty \int_0^\infty (e^{-xy} \cos x - e^{-(a+y)x}) dy dx \\ &= \int_0^\infty \left( \frac{y}{1+y^2} - \frac{1}{a+y} \right) dy \quad (2584, 2291) = \log a. \end{aligned}$$

$$2620 \quad \int_0^\pi \log (1 - 2a \cos x + a^2) dx = 0,$$

when  $a$  is equal to, or less than, unity; but is equal to  $2\pi \log a$ , when  $a$  is greater than unity.

PROOF.—(i.)  $a=1$ . By (2635), since

$$\log 2 (1 - \cos x) = \log 4 + 2 \log \sin \tfrac{1}{2}x.$$

(ii.)  $a < 1$ . By integrating (2922) from 0 to  $\pi$ .

(iii.)  $a > 1$ . As in (2926), integrating from 0 to  $\pi$ .

$$2622 \quad \int_0^\pi \log (1 - n \cos x) dx.$$

When  $n$  is less than unity, the values of this integral depend on those of (2620). See (2933).

$$2623 \quad \int_0^\pi \frac{x \sin x dx}{1 - 2a \cos x + a^2} = \frac{\pi}{a} \log (1+a), \text{ or } \frac{\pi}{a} \log \left(1 + \frac{1}{a}\right),$$

according as  $a$  is less or greater than unity.

PROOF.—Integrate  $\int_0^\pi \log (1 - 2a \cos x + a^2) dx$  by Parts,  $\int dx$ , and apply (2620).

$$2625 \quad \int_0^\pi \cos rx \log (1 - 2a \cos x + a^2) dx = -\frac{\pi a^r}{r}, \text{ or } -\frac{\pi a^{-r}}{r},$$

according as  $a$  is less or greater than unity.

PROOF.—Substitute the value of the logarithm obtained in (2922). The integral of every term of the resulting expansion, excepting the one in which  $u = r$ , vanishes by (2467).

$$2627 \quad \int_0^\pi \frac{\sin x \sin rx \, dx}{1 - 2a \cos x + a^2} = \frac{\pi a^{r-1}}{2}, \quad \text{or} \quad \frac{\pi a^{-(r+1)}}{2},$$

according as  $a$  is less or greater than unity.

PROOF.—Integrate (2625) by Parts,  $\int \cos rx \, dx$ .

$$2629 \quad \int_0^\pi \frac{\cos rx \, dx}{1 - 2a \cos x + a^2} = \frac{\pi a^r}{1 - a^2}, \quad a \text{ being } < 1.$$

PROOF.—The fraction  $= \cos rx (1 + 2a \cos x + 2a^2 \cos 2x + 2a^3 \cos 3x + \dots) \div (1 - a^2)$ , by (2919), and the result follows as in (2625).

$$2630 \quad \int_0^\infty \frac{1}{1+x^2} \cdot \frac{dx}{1 - 2a \cos cx + a^2} = \frac{\pi}{2(1-a^2)} \frac{1+ae^{-c}}{1-ae^{-c}}.$$

PROOF.—Expand the second factor by (2919), and integrate the terms by (2573).

$$2631 \quad \int_0^\infty \frac{\log(1 - 2a \cos cx + a^2) \, dx}{1+x^2} = \pi \log(1 - ae^{-c}).$$

PROOF.—Expand the numerator by (2922), and integrate the terms by (2573).

$$2632 \quad \int_0^\infty \frac{x \sin cx \, dx}{(1+x^2)(1 - 2a \cos cx + a^2)} = \frac{\pi}{2(e^c - a)}.$$

PROOF.—By differentiating (2631) for  $c$ .

Otherwise.—Expand by (2921), and integrate the terms by (2574).

$$2633 \quad \int_0^{\frac{1}{2}\pi} \frac{\log(1 + c \cos x)}{\cos x} \, dx = \frac{1}{2} \left\{ \frac{\pi^2}{4} - (\cos^{-1} c)^2 \right\}.$$

PROOF.—Put  $a = 1$  in (1951), and take the integral between the limits 0 and  $\frac{1}{2}\pi$ , then integrate for  $b$  between limits 0 and  $c$ ; the result is

$$\int_0^{\frac{1}{2}\pi} \frac{\log(1 + c \cos x)}{\cos x} \, dx = 2 \int_0^c \frac{1}{\sqrt{1-b^2}} \tan^{-1} \sqrt{\frac{1-b}{1+b}} \, db,$$

and the integral on the right is found by substituting  $\cos^{-1} b$ .

$$2634 \quad \int_0^\pi \frac{\log(1+c \cos x)}{\cos x} dx = \pi \sin^{-1} c.$$

PROOF.—As in (2633), by taking 0 and  $\pi$  for the limits of  $x$ .

$$2635 \quad \int_0^{\frac{1}{2}\pi} \log \sin x dx = \frac{\pi}{2} \log \frac{1}{2} = \int_0^1 \frac{\log x}{\sqrt{1-x^2}} dx.$$

PROOF.— $\int_0^{\frac{1}{2}\pi} \sin x dx = \int_0^{\frac{1}{2}\pi} \cos x dx$  (2233). Add these integrals and substitute  $2x$ , applying (2234) to the result.

$$2637 \quad \int_0^\pi x \log \sin x dx = \frac{\pi^2}{2} \log \frac{1}{2}.$$

PROOF.— $\int_0^\pi x^2 \log \sin x dx = \int_0^\pi (\pi-x)^2 \log \sin x dx$ , by (2233). Equate the difference of these integrals to zero.

$$2638 \quad \int_0^{n\pi} x \log \sin^2 x dx = -n^2 \pi^2 \log 2, \quad n \text{ being an integer.}$$

PROOF.—Method of (2250),

$$\begin{aligned} \int_0^{n\pi} x \log \sin^2 x dx &= \int_0^\pi x \log \sin^2 x dx + \int_\pi^{2\pi} x \log \sin^2 x dx + \dots + \int_{(n-1)\pi}^{n\pi} x \log \sin^2 x dx \\ &= \int_0^\pi x \log \sin^2 x dx + \int_0^\pi (\pi+y) \log \sin^2 y dy + \dots + \int_0^\pi \{(n-1)\pi+y\} \log \sin^2 y dy. \end{aligned}$$

Each integral reduces by (2635) and (2637); for example,

$$\begin{aligned} \int_0^\pi (\pi+y) \log \sin^2 y dy &= 2 \int_0^\pi (\pi+y) \log \sin y dy = 2\pi \int_0^\pi \log \sin y dy + 2 \int_0^\pi y \log \sin y dy \\ &= -2\pi^2 \log 2 - \pi^2 \log 2 = -3\pi^2 \log 2. \end{aligned}$$

The result is  $-\{1+3+5+\dots+(2n-1)\} \pi^2 \log 2 = -n^2 \pi^2 \log 2$ .

$$2639 \quad \int_0^\infty \frac{\sin mx}{e^{2\pi x}-1} dx = \frac{1}{2} \left( \frac{1}{e^m-1} + \frac{1}{2} - \frac{1}{m} \right).$$

PROOF.—Develop  $\sin mx$  by (764); integrate the terms by (2393), and sum the series by (1539).

$$2640 \quad \int_0^\infty \frac{\sin mx}{e^x+1} dx = \frac{1}{2m} - \frac{\pi}{e^{m\pi}-e^{-m\pi}}.$$



PROOF.—Develope  $\sin mx$  by (764); integrate the terms by (2398). The resulting series is  $= \frac{1}{2m} + \frac{\pi}{2i} \operatorname{cosec} im\pi$ , by (2918), which is equivalent to the above by (769).

$$2641 \quad \int_0^1 \frac{\cos(m \log x) - \cos(n \log x)}{\log x} dx = \frac{1}{2} \log \frac{1+m^2}{1+n^2}.$$

$$2642 \quad \int_0^1 \frac{\sin(m \log x) - \sin(n \log x)}{\log x} dx = \tan^{-1} m - \tan^{-1} n.$$

PROOF.—Put  $p=im$  and  $q=in$  in (2394), and equate corresponding parts. See (2214).

$$2643 \quad \int_0^1 \frac{\sin(n \log x)}{\log x} dx = \tan^{-1} n. \quad \int_0^1 \frac{\operatorname{vers}(n \log x)}{\log x} dx = \frac{1}{2} \log \frac{1}{1+n^2}.$$

PROOF.—Put  $m=0$  in (2641) and (2642).

## MISCELLANEOUS THEOREMS.

### FRULLANI'S FORMULA.

$$2700 \quad \int_0^\infty \frac{\phi(ax) - \phi(bx)}{x} dx = \phi(0) \log \frac{b}{a} + \int_{\frac{h}{a}}^{\frac{h}{b}} \frac{\phi(bx)}{x} dx,$$

$h$  being  $= \infty$ , and the last term generally  $= 0$ .

PROOF.—In the integral  $\int_0^h \frac{\phi(z) - \phi(0)}{z} dz$  substitute  $z=ax$  and  $z=bx$ , and equate the results thus,

$$\int_0^{\frac{h}{a}} \frac{\phi(ax)}{x} dx - \int_0^{\frac{h}{b}} \frac{\phi(bx)}{x} dx = \int_0^{\frac{h}{a}} \frac{\phi(0)}{x} dx - \int_0^{\frac{h}{b}} \frac{\phi(0)}{x} dx,$$

$$\text{or} \quad \int_0^{\frac{h}{a}} \frac{\phi(ax) - \phi(bx)}{x} dx - \int_{\frac{h}{a}}^{\frac{h}{b}} \frac{\phi(bx)}{x} dx = \int_0^{\frac{h}{a}} \frac{\phi(0)}{x} dx - \int_0^{\frac{h}{b}} \frac{\phi(0)}{x} dx = \phi(0) \log \frac{b}{a}.$$

Then make  $h$  infinite. For applications see (2513) and (2505).

$$2701 \quad \int_0^\infty \frac{\phi(ax) - \phi(bx)}{x^2} dx = (a-b) \int_0^\infty \frac{\phi'(x)}{x} dx \\ + \phi'(0) \left( \log \frac{a^a}{b^b} - a + b \right) - (a-b) \frac{\phi(h)}{h} + \int_{\frac{h}{a}}^{\frac{h}{b}} \frac{\phi(bx)}{x^2} dx,$$

with  $h = \infty$ .

$$\text{PROOF.} \quad d_a \left\{ \int_0^{\frac{h}{a}} \frac{\phi(ax)}{x^2} dx \right\} = \int_0^{\frac{h}{a}} \frac{\phi'(ax)}{x} dx - \frac{\phi(h)}{h} \\ (2257) = \int_0^{\frac{h}{a}} \frac{\phi'(x)}{x} dx - \phi'(0) la - \frac{\phi(h)}{h},$$

by making  $b = 1$  in the proof of (2700). Integrate for  $a$  between limits

$$a \text{ and } b, \text{ thus} \quad \int_0^{\frac{h}{a}} \frac{\phi(ax)}{x^2} dx - \int_0^{\frac{h}{b}} \frac{\phi(bx)}{x^2} dx \\ = (a-b) \int_0^{\frac{h}{a}} \frac{\phi'(x)}{x} dx + \phi'(0) \{a la - b lb - a + b\} - \frac{(a-b)\phi(h)}{h},$$

$$\text{and the left is} \quad = \int_0^{\frac{h}{a}} \frac{\phi(ax) - \phi(bx)}{x^2} dx - \int_{\frac{h}{a}}^{\frac{h}{b}} \frac{\phi(bx)}{x^2} dx.$$

POISSON'S FORMULÆ.

$$2702 \quad \int_0^\pi \frac{f(a+e^{ix}) + f(a+e^{-ix})}{1-2c \cos x + c^2} dx = \frac{2\pi}{1-c^2} f(a+c),$$

$c$  being  $< 1$ .

PROOF.—By Taylor's theorem (1500), and by (2919), the fraction is equal to the product of the two expansions

$$2 \left\{ f(a) + f'(a) \cos x + \frac{1}{1 \cdot 2} f''(a) \cos 2x + \frac{1}{1 \cdot 2 \cdot 3} f'''(a) \cos 3x + \dots \right\}$$

and  $\{1 + 2c \cos x + 2c^2 \cos 2x + 2c^3 \cos 3x + \dots\}$

divided by  $(1-c^2)$ . By (2468) the integral of every term of the product vanishes, except when it is of the form  $2 \int_0^\pi \cos^2 nx$ , and this is  $= \pi$ , by (2471). Hence the result.

2703

$$\int_0^\pi \frac{f(a+e^{ix}) + f(a+e^{-ix})}{1-2c \cos x + c^2} (1-c \cos x) dx = \pi \{f(a+c) + f(a)\}.$$

2704

$$\int_0^\pi \frac{f(a+e^{ix}) - f(a+e^{-ix})}{1-2c \cos x + c^2} \sin x dx = \frac{i\pi}{c} \{f(a+c) - f(a)\}.$$

PROOF.—As in (2702), adding unity to each side of (2919), and employing (2921, 2467, 2470).

## ABEL'S FORMULA.

Given that  $F(x+a)$  can be expanded in powers of  $e^{-a}$ , then

$$2705 \quad \int_0^\infty \frac{F(x+iat) + F(x-iat)}{1+t^2} dt = \pi F(x+a).$$

PROOF.—Assume  $F(x+a) = A + A_1 e^{-a} + A_2 e^{-2a} + A_3 e^{-3a} + \&c.$ ,

$$\therefore F(x+iat) + F(x-iat) = 2A + 2A_1 \cos at + 2A_2 \cos 2at + \&c.$$

Substitute and integrate by (1935) and (2573).

$$\text{Ex.—Let } F(x) = \frac{1}{x}, \text{ then } \int_0^\infty \frac{dt}{(1+t^2)(x^2+a^2t^2)} = \frac{\pi}{2x(x+a)}.$$

## KUMMER'S FORMULA.

$$2706 \quad \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(2x \cos \theta e^{i\theta}) e^{2ik\theta} d\theta = \sin k\pi \int_0^1 (1-z)^{k-1} f(xz) dz.$$

PROOF.—If  $h = xe^{2i\theta}$ , then  $x+h = 2x \cos \theta e^{i\theta}$  by (766). Substitute these values in the expansion of  $f(x+h)$  by (1500); multiply by  $e^{2ik\theta}$  and integrate; thus, after reducing by (769),

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(2x \cos \theta e^{i\theta}) e^{2ik\theta} d\theta = \sin k\pi \left\{ \frac{f(x)}{k} - \frac{xf'(x)}{k+1} + \frac{x^2 f''(x)}{1 \cdot 2 \cdot (k+2)} - \&c. \right\}$$

Again, putting  $h = -x\phi$  in (1500), multiplying by  $\phi^{k-1} d\phi$ , and integrating, we have  $\int_0^1 \phi^{k-1} f(x-x\phi) d\phi =$  the foregoing series within the brackets. Equating the two values and changing  $\phi$  into  $1-z$ , the formula is obtained.

For an application see (2490).

2707 When  $k$  is an integer,

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(2x \cos \theta e^{i\theta}) e^{2ik\theta} \theta d\theta = \frac{\pi \cos k\pi}{2i} \int_0^1 (1-z)^{k-1} f(xz) dz.$$

PROOF.—Divide equation (2706) by  $\sin k\pi$ , and evaluate the indeterminate fraction by (1580), differentiating with respect to  $k$ .

For applications see (2490), (2494).

2708 If  $X$  be a function of  $x$  so chosen that

$$\int_a^b Xf(x, k) dx = C_k \int_a^b Xf(x, 0) dx \dots\dots\dots (i.),$$

and if the series

$$A_0 f(x, 0) + A_1 f(x, 1) + A_2 f(x, 2) + \&c. \dots = \phi(x) \dots (ii.),$$

where  $\phi$  is a known function, then

$$A_0C_0 + A_1C_1 + A_2C_2 + \&c. \dots = \frac{\int_a^b X\phi(x) dx}{\int_a^b Xf(x, 0) dx} \dots\dots (iii.)$$

PROOF.—Multiply (ii.) by  $X$ , and integrate from  $a$  to  $b$ , employing (i.)

**2709** If the sum of the series

$$A_0 + A_1x + A_2x^2 + A_3x^3 + \&c. \dots = \phi(x)$$

be known, then

$$A_0 + A_1a + A_2a(a+1) + A_3a(a+1)(a+2) + \&c. \dots \\ = \frac{\int_0^\infty e^{-x}x^{a-1}\phi(x) dx}{\int_0^\infty e^{-x}x^{a-1} dx}.$$

PROOF.—In (2708) let  $X = e^{-x}x^{a-1}$  and  $f(x, k) = x^k$ . Then since, by Parts, we have  $\int_0^\infty e^{-x}x^{a+k-1} dx = a(a+1)\dots(a+k-1)\int_0^\infty e^{-x}x^{a-1} dx$ , it follows that  $C_k = a(a+1)\dots(a+k-1)$ . Hence, conditions (i.) and (ii.) being fulfilled, result (iii.) is established.  
For an application see (2589).

THEOREM.—Let  $f(x+iy) = P+iQ\dots\dots\dots (i.)$

**2710** Then  $\int_a^b \int_a^\beta \frac{dQ}{dy} dx dy = \int_a^\beta \int_a^b \frac{dP}{dx} dy dx \dots\dots\dots (ii.)$

**2711**  $\int_a^b \int_a^\beta \frac{dP}{dy} dx dy = -\int_a^\beta \int_a^b \frac{dQ}{dx} dy dx \dots\dots\dots (iii.)$

PROOF.—Differentiating (i.) independently for  $x$  and  $y$ ,  
 $f'(x+iy) = P_x + iQ_x$ ,  $if'(x+iy) = P_y + iQ_y$ ,  
 $\therefore P_x + iQ_x = Q_y - iP_y$ ,  $\therefore P_x = Q_y$  and  $Q_x = -P_y$ .  
Hence by (2261) the equalities (ii.) and (iii.) are obtained.

Ex.—Let  $f(x+iy) = e^{-(x+iy)^2} = e^{-x^2-y^2}(\cos 2xy - i \sin 2xy)$ .  
Here  $P = e^{-x^2-y^2} \cos 2xy$ ,  $Q = -e^{-x^2-y^2} \sin 2xy$ , therefore, by (iii.),  
 $\int_a^b e^{-x^2} (e^{\beta^2} \cos 2\beta x - e^{a^2} \cos 2ax) dx = \int_a^\beta e^{y^2} (e^{-b^2} \sin 2by - e^{-a^2} \sin 2ay) dy$ .  
Put  $a = a = 0$ ,  $b = \infty$ ; therefore  
 $\int_0^\infty e^{-x^2} (e^{\beta^2} \cos 2\beta x - 1) dx = 0$ ,  $\therefore e^{\beta^2} \int_0^\infty e^{-x^2} \cos 2\beta x dx = \int_0^\infty e^{-x^2} dx$ .

## CAUCHY'S FORMULA.

**2712** Let  $\int_0^{\infty} x^{2n} F(x^2) dx = A_{2n}$ ,  $n$  being an integer, then

$$\begin{aligned} & \int_0^{\infty} x^{2n} F\left\{\left(x - \frac{1}{x}\right)^2\right\} dx \\ &= A_0 + \frac{n(n+1)}{1 \cdot 2} A_2 + \frac{(n-1)^{(4)}}{1^{(4)}} A_4 + \frac{(n-2)^{(6)}}{1^{(6)}} A_6 + \&c. \end{aligned}$$

PROOF.—In the integral  $\int_{-\infty}^{\infty} z^{2n} F(z^2) dz = 2A_{2n}$ , substitute  $z = x - \frac{1}{x}$ , and it becomes  $\int_0^{\infty} \left(x - \frac{1}{x}\right)^{2n} \left(x + \frac{1}{x}\right) F\left\{\left(x - \frac{1}{x}\right)^2\right\} \frac{dx}{x} = 2A_{2n} \dots\dots (i.)$

Let the integral sought be denoted by  $C_{2n}$ , then

$$\int_0^{\infty} \frac{1}{x^{2n+1}} F\left\{\left(x - \frac{1}{x}\right)^2\right\} \frac{dx}{x} = \int_0^{\infty} x^{2n+1} F\left\{\left(x - \frac{1}{x}\right)^2\right\} \frac{dx}{x} = C_{2n}.$$

This is proved by substituting  $\frac{1}{x}$  in the first integral. Therefore by addition

$$\int_0^{\infty} \left(x^{2n+1} + \frac{1}{x^{2n+1}}\right) F\left\{\left(x - \frac{1}{x}\right)^2\right\} \frac{dx}{x} = 2C_{2n} \dots\dots\dots (ii.)$$

Now, in the expansion of  $\cos(2n+1)\theta$  (776), put  $2\cos\theta = x + \frac{1}{x}$  and  $2i\sin\theta = x - \frac{1}{x}$ , where  $x = e^{i\theta}$  by (768-9), and multiply the equation by  $F\left\{\left(x - \frac{1}{x}\right)^2\right\} \frac{dx}{x}$ , and integrate from  $x=0$  to  $x=\infty$ . Then, by (i.) and (ii.), the required result is obtained.

**2713** Ex.—Let  $F(x) = e^{-ax}$ , then

$$A_{2n} = \int_0^{\infty} x^{2n} e^{-ax^2} dx = \frac{1 \cdot 3 \dots (2n-1)}{2^{n+1} a^{n+\frac{1}{2}}} \sqrt{\pi} \quad \text{and} \quad A_0 = \frac{\sqrt{\pi}}{2\sqrt{a}}.$$

Therefore

$$\begin{aligned} & \int_0^{\infty} x^{2n} e^{-a\left(x^2 - \frac{1}{x^2}\right)} dx \\ &= \frac{e^{-2a} \sqrt{\pi}}{2\sqrt{a}} \left\{ 1 + \frac{n(n+1)}{4a} + \frac{(n-1)^{(4)}}{1^{(2)} 4^2 a^2} + \frac{(n-2)^{(6)}}{1^{(3)} 4^3 a^3} + \frac{(n-3)^{(8)}}{1^{(4)} 4^4 a^4} + \&c. \right\}. \end{aligned}$$

## FINITE VARIATION OF A PARAMETER.

**2714** Theorem (2255) may be extended to the case of a finite change in the value of a quantity under the sign of integration.

Let  $a$  be independent of  $a$  and  $b$ , and let  $\Delta$  be the difference caused by an increase of unity in the value of  $a$ , then

$$\int_a^b \Delta \phi(x, a) dx = \Delta \int_a^b \phi(x, a) dx.$$

**2715** Ex. 1.  $\int_0^\infty e^{-ax} dx = \frac{1}{a}$ ,  $\therefore \int_0^\infty \Delta e^{-ax} dx = \Delta \frac{1}{a}$ , that is

$$\int_0^\infty e^{-ax} (e^{-x} - 1) dx = -\frac{1}{a(a+1)}.$$

Also, by repeating the operation,

$$\int_0^\infty \Delta^n e^{-ax} dx = \Delta^n \frac{1}{a}, \text{ that is}$$

**2716**  $\int_0^\infty e^{-ax} (e^{-x} - 1)^n dx = \frac{(-1)^n [n]}{a(a+1) \dots (a+n)}.$

**2717** Ex. 2.—In (2583-4) put  $k$  for  $a$  and  $(2a-m)$  for  $b$ , then

$$\int_0^\infty e^{-kx} \Delta \sin (2a-m)x dx = \Delta \frac{2a-m}{k^2 + (2a-m)^2} \dots\dots\dots (i.),$$

$$\int_0^\infty e^{-kx} \Delta \cos (2a-m)x dx = \Delta \frac{k}{k^2 + (2a-m)^2} \dots\dots\dots (ii.).$$

In (ii.) let  $m = 2p$ , an even integer, then

$$\begin{aligned} \Delta^{2p} \cos (2a-2p)x &= \cos (2a+2p)x - 2p \cos (2a+2p-2)x + \dots \\ &\quad \dots + \cos (2a-2p)x \\ &= \cos 2ax [\cos 2px - 2p \cos (2p-2)x + O(2p, 2) \cos (2p-4)x - \dots \\ &\quad \dots + \cos 2px] \\ &\quad - \sin 2ax [\sin 2px - 2p \sin (2p-2)x + \dots \\ &\quad \dots - \sin 2px]. \end{aligned}$$

The coefficient of  $\cos 2ax$ , in which equidistant terms are equal, is  $= (-1)^p 2^{2p} \sin^{2p} x$  (773); while the coefficient of  $\sin 2ax$  vanishes because the equidistant terms destroy each other. Therefore

$$\Delta^{2p} \cos (2a-2p)x = (-1)^p 2^{2p} \cos 2ax \sin^{2p} x.$$

Hence (ii.) becomes

**2718**  $\int_0^\infty e^{-kx} \cos 2ax \sin^{2p} x dx = \frac{(-1)^p}{2^{2p}} \Delta^{2p} \frac{k}{k^2 + (2a-2p)^2}.$

**2719** Again, in (i.) let  $m = 2p+1$ , an odd integer, then

$$\begin{aligned} \Delta^{2p+1} \sin (2a-2p-1)x &= \sin (2a+2p+1)x - (2p+1) \sin (2a+2p-1)x \\ &\quad + C(2p+1, 2) \sin (2a+2p-3)x - \dots - \sin (2a-2p-1)x \\ &= \sin 2ax [\cos (2p+1)x - (2p+1) \cos (2p-1)x + \dots - \cos (2p+1)x] \\ &\quad + \cos 2ax [\sin (2p+1)x - (2p+1) \sin (2p-1)x + \dots + \sin (2p+1)x]. \end{aligned}$$

The coefficient of  $\sin 2ax$  vanishes as before, while that of  $\cos 2ax$  is

$$= (-1)^p 2^{2p+1} \sin^{2p+1} x \text{ (774).}$$



Therefore equation (i.) becomes

**2720**

$$\int_0^\infty e^{-kx} \cos 2ax \sin^{2p+1} x dx = \frac{(-1)^p}{2^{2p+1}} \Delta^{2p+1} \frac{2a-2p-1}{k^2 + (2a-2p-1)^2}.$$

To compute the right member of equation (2718), we have

$$\Delta^{2p} \frac{k}{k^2 + (2a-2p)^2} = k \left[ \frac{1}{k^2 + (2a+2p)^2} - \frac{2p}{k^2 + (2a+2p-2)^2} + \frac{C(2p, 2)}{k^2 + (2a+2p-4)^2} - \dots + \frac{1}{k^2 + (2a-2p)^2} \right].$$

Let  $a=0$ , then the equidistant terms are equal, and we obtain in this case

$$\mathbf{2721} \quad \Delta^{2p} \frac{k}{k^2 + (2a-2p)^2} = \frac{(-1)^p 1 \cdot 2 \dots 2p \cdot 2^{2p}}{k (k^2 + 4)(k^2 + 16) \dots \{k^2 + (2p)^2\}}.$$

Thus formula (2609) is proved.

Similarly, by making  $a=0$  in (2720) after expansion, formula (2608) is obtained.

Let  $p$  be any integer, and let  $q$  and  $a$  be arbitrary, but  $q < 2p$  in (2722), and  $< 2p+1$  in (2723).

**2722**

$$\int_0^\infty \frac{\cos 2ax \sin^{2p} x}{x^{q+1}} dx = \frac{(-1)^p}{2^{2p} \Gamma(q+1)} \int_0^\infty \Delta^{2p} \frac{z^{q+1}}{z^2 + (2a-2p)^2} dz.$$

**2723**

$$\begin{aligned} \int_0^\infty \frac{\cos 2ax \sin^{2p+1} x}{x^{q+1}} dx \\ = \frac{(-1)^p}{2^{2p+1} \Gamma(q+1)} \int_0^\infty \Delta^{2p+1} \frac{(2a-2p-1) z^q}{z^2 + (2a-2p-1)^2} dz, \end{aligned}$$

where  $\Delta$  has the signification in (2714).

PROOF.—Employing the method of (2510), replace

$$\frac{1}{x^{q+1}} \text{ by } \frac{1}{\Gamma(q+1)} \int_0^\infty e^{-xz} z^q dz,$$

$q$  being integral or fractional; therefore

$$\int_0^\infty \frac{\cos 2ax \sin^{2p} x}{x^{q+1}} dx = \frac{1}{\Gamma(q+1)} \int_0^\infty \int_0^\infty \cos 2ax \sin^{2p} x e^{-xz} z^q dz dx,$$

by changing the order of integration. Substitute the value in (2718) for the integral containing  $x$ , writing the factor  $z^q$  under the operator  $\Delta$ , since it is independent of  $a$ .

Similarly, with  $2p+1$  in the place of  $p$ , we substitute from (2720).

It may be shown that, whenever  $a > p$ , formula (2722) reduces to

**2724**

$$\int_0^\infty \frac{\cos 2ax \sin^{2p} x}{x^{q+1}} dx = \frac{(-1)^{p+1} \pi}{2^{2p+1} \Gamma(q+1) \sin \frac{q\pi}{2}} \Delta^{2p} (2a-2p)^q.$$

For a complete investigation, see Cauchy's "Mémoire de l'Ecole Polytechnique," tome xvii.

**2725**

Ex.—Let  $a = 2$ ,  $p = 1$ ,  $q = \frac{1}{2}$ ,

$$\int_0^\infty \frac{\cos 4x \sin^2 x}{x^{\frac{3}{2}}} dx = \frac{\pi}{8\Gamma(\frac{3}{2}) \sin \frac{\pi}{6}} \Delta^2 (2a-2)^{\frac{1}{2}},$$

and  $\Delta^2 (2a-2)^{\frac{1}{2}} = (2a+2)^{\frac{1}{2}} - 2(2a)^{\frac{1}{2}} + (2a-2)^{\frac{1}{2}} = 6^{\frac{1}{2}} - 2 \cdot 4^{\frac{1}{2}} + 2^{\frac{1}{2}}.$

## FOURIER'S FORMULA.

**2726**

$$\int_0^h \frac{\sin ax}{\sin x} \phi(x) dx = \frac{\pi}{2} \phi(0),$$

when  $a = \infty$  and  $h$  is not greater than  $\frac{1}{2}\pi$ .

PROOF.—(i.) Let  $\phi(x)$  be a continuous, finite, positive quantity, decreasing in value as  $x$  increases from zero to  $h$ .

$$\int_0^h \frac{\sin ax}{\sin x} \phi(x) dx = \int_0^{\frac{\pi}{a}} + \int_{\frac{\pi}{a}}^{\frac{2\pi}{a}} + \int_{\frac{2\pi}{a}}^{\frac{3\pi}{a}} + \dots + \int_{\frac{(r-1)\pi}{a}}^{\frac{r\pi}{a}} + \int_{\frac{r\pi}{a}}^h \dots\dots\dots (i.),$$

$\frac{r\pi}{a}$  being the greatest multiple of  $\frac{\pi}{a}$  contained in  $h$ . The terms are alternately positive and negative, as appears from the sign of  $\sin ax$ . The following investigation shows that the terms decrease in value. Take two consecutive terms

$$\int_{\frac{n\pi}{a}}^{\frac{(n+1)\pi}{a}} \frac{\sin ax}{\sin x} \phi(x) dx, \quad \int_{\frac{(n+1)\pi}{a}}^{\frac{(n+2)\pi}{a}} \frac{\sin ax}{\sin x} \phi(x) dx.$$

Substituting  $x - \frac{\pi}{a}$  in the second integral, it becomes

$$- \int_{\frac{n\pi}{a}}^{\frac{(n+1)\pi}{a}} \frac{\sin ax}{\sin \left(x + \frac{\pi}{a}\right)} \phi \left(x + \frac{\pi}{a}\right) dx,$$

and since  $\phi$  decreases as  $x$  increases, an element of this integral is less than the corresponding element of the first integral.

Now, by substituting  $ax = y$ , we have

$$\int_{\frac{\pi\pi}{a}}^{\frac{(n+1)\pi}{a}} \frac{\sin ax}{\sin x} \phi(x) dx = \int_{n\pi}^{(n+1)\pi} \frac{\sin y}{a \sin \frac{y}{a}} \phi\left(\frac{y}{a}\right) dy = \phi(0) \int_{n\pi}^{(n+1)\pi} \frac{\sin y}{y} dy \dots (ii.),$$

when  $a$  is infinite, because then  $\phi\left(\frac{y}{a}\right) = \phi(0)$  and  $a \sin \frac{y}{a} = y$ .

Hence the sum of  $n$  terms of (i.) may be replaced by  $\phi(0) \int_0^{\pi} \frac{\sin y}{y} dy$ , which, when  $n$  is infinite, takes the value  $\phi(0) \frac{1}{2}\pi$  by (2572); while the sum of the remaining terms vanishes, because (the signs alternating) that sum is less than the  $n+1^{\text{th}}$  term, which itself vanishes when  $n$  is infinite.

(ii.) If  $\phi(x)$ , while always *decreasing*, becomes negative, let  $C$  be a constant such that  $C + \phi(x)$  remains always positive while  $x$  varies from 0 to  $h$ . The theorem is true for  $C + \phi(x)$ , and also for a function constant and equal to  $C$ , and it is therefore true for the decreasing function  $\phi$  whatever its sign.

If  $\phi(x)$  is a function always *increasing* in value,  $-\phi(x)$  is a decreasing function. The theorem applies to the last function, and therefore also to  $\phi(x)$ .

**2727** COR.—Hence the same integral taken between any two limits lying between zero and  $\frac{1}{2}\pi$ , vanishes when  $a$  is infinite.

$$\mathbf{2728} \quad \int_0^h \frac{\sin ax}{\sin x} \phi(x) dx$$

$$= \pi \left\{ \frac{1}{2}\phi(0) + \phi(\pi) + \phi(2\pi) + \dots + \phi(n-1)\pi + \phi(n\pi) \right\},$$

when  $a$  is an indefinitely great *odd integer*, and  $n\pi$  is the greatest multiple of  $\pi$  less than  $h$ . But when  $a$  is an indefinitely great *even integer*, the second and alternate terms of the series have the minus sign.

$$\text{PROOF.} \quad \int_0^h \frac{\sin ax}{\sin x} \phi(x) dx = \int_0^{n\pi} \frac{\sin ax}{\sin x} \phi(x) dx + \int_{n\pi}^h \frac{\sin ax}{\sin x} \phi(x) dx \dots (i.),$$

decompose the second integral into  $2n$  others with the limits 0 to  $\frac{1}{2}\pi$ ,  $\frac{1}{2}\pi$  to  $\pi$ ,  $\pi$  to  $\frac{3}{2}\pi$ , ...  $(2n-1)\frac{1}{2}\pi$  to  $n\pi$ ; and in these integrals put successively  $x = y$ ,  $\pi - y$ ,  $\pi + y$ ,  $2\pi - y$ ,  $2\pi + y$ , ...  $n\pi - y$ . The new limits will be 0 to  $\frac{1}{2}\pi$ ,  $\frac{1}{2}\pi$  to 0 alternately, with the even terms negative, so that, by changing the signs of the even terms, the limits for each will be 0 to  $\frac{1}{2}\pi$ . Also, if  $a$  is an odd integer,  $\frac{\sin ax}{\sin x}$  is changed into  $\frac{\sin ay}{\sin y}$  by each substitution, so that (i.) becomes

$$\int_0^{\frac{1}{2}\pi} \frac{\sin ay}{\sin y} \{ \phi(y) + \phi(\pi - y) + \phi(\pi + y) + \dots + \phi(n\pi - y) \} dy + \int_{n\pi}^h \frac{\sin ax}{\sin x} \phi(x) dx \dots (iii.)$$

But, when  $a$  is even, the substitution of  $n\pi \mp y$  for  $x$  makes  $\frac{\sin ay}{\sin y}$  minus

whenever  $r$  is odd. The limit of the first part of (iii.) is

$$\frac{\pi}{2} \{ \phi(0) + 2\phi(\pi) + 2\phi(2\pi) + \dots + 2\phi(n-1)\pi + \phi(n\pi) \}, \text{ by (2726).}$$

In the last part of (iii.) put  $x = n\pi + y$ , and the integral becomes

$$\int_0^{h-n\pi} \frac{\sin ay}{\sin y} \phi(n\pi + y) dy = \frac{\pi}{2} \phi(n\pi), \text{ if } h-n\pi \text{ is } \neq \frac{\pi}{2}, \text{ by (2725).}$$

If  $h-n\pi$  lies between  $\frac{1}{2}\pi$  and  $\pi$ , decompose the integral into two others; the one with limits 0 to  $\frac{1}{2}\pi$  will converge towards  $\frac{1}{2}\pi\phi(n\pi)$ , while the other with limits  $\frac{1}{2}\pi$  to  $h-n\pi$  becomes, by putting  $y = \pi - z$ ,

$$\int_{(n+1)\pi-h}^{\frac{1}{2}\pi} \frac{\sin az}{\sin z} \phi[(n+1)\pi - z] dz = 0,$$

the limit by (2727). Hence the last term of (iii.) is  $\frac{1}{2}\pi\phi(n\pi)$ . Substituting these values, (2728) is obtained.

**2729** Ex.—By (2614),  $\int_0^\infty e^{-a^2x^2} \cos 2bx dx = \frac{\sqrt{\pi}}{2a} e^{-\frac{b^2}{a^2}}.$

Put  $b = 0, 1, 2 \dots n$  successively, and add, after multiplying the first equation by  $\frac{1}{2}$ , thus

$$\begin{aligned} \int_0^\infty e^{-a^2x^2} \left( \frac{1}{2} + \cos 2x + \cos 4x + \dots + \cos 2nx \right) dx \\ = \frac{\sqrt{\pi}}{2a} \left\{ \frac{1}{2} + e^{-\frac{1}{a^2}} + e^{-\frac{4}{a^2}} + \dots + e^{-\frac{n^2}{a^2}} \right\}. \end{aligned}$$

The left side  $= \frac{1}{2} \int_0^\infty e^{-a^2x^2} \frac{\sin(2n+1)x}{\sin x} dx$ , by (801),

and, if  $n = \infty$ , becomes

$$\frac{\pi}{2} \left\{ \frac{1}{2} + e^{-\pi^2a^2} + e^{-4\pi^2a^2} + e^{-9\pi^2a^2} + \dots \right\}, \text{ by (2728);}$$

$$\therefore \pi \left\{ \frac{1}{2} + e^{-\pi^2a^2} + e^{-4\pi^2a^2} + e^{-9\pi^2a^2} + \dots \right\} = \frac{\sqrt{\pi}}{a} \left\{ \frac{1}{2} + e^{-\frac{1}{a^2}} + e^{-\frac{4}{a^2}} + e^{-\frac{9}{a^2}} + \dots \right\}.$$

Put  $\pi a = a$  and  $\frac{1}{a} = \beta$ ; therefore

$$\begin{aligned} \mathbf{2730} \quad \sqrt{a} \left\{ \frac{1}{2} + e^{-a^2} + e^{-4a^2} + e^{-9a^2} + \dots \right\} \\ = \sqrt{\beta} \left\{ \frac{1}{2} + e^{-\beta^2} + e^{-4\beta^2} + e^{-9\beta^2} + \dots \right\}, \end{aligned}$$

with the condition  $a\beta = \pi$ .

**2731**  $\int_0^h \frac{\sin ax}{x} \phi(x) dx = \frac{\pi}{2} \phi(0),$

when  $a$  is an infinite integer.

PROOF.—The integral may be put in the form

$$\int_0^h \frac{\sin ax}{\sin x} \Phi(x) dx, \text{ where } \Phi(x) = \frac{\sin x}{x} \phi(x),$$

therefore, by (2726), when  $h$  is  $\geq \frac{1}{2}\pi$ , and by (2728), if  $h$  is  $> \frac{1}{2}\pi$ , the value is  $\frac{1}{2}\pi\Phi(0)$ , since in (2728)  $\Phi(\pi)$ ,  $\Phi(2\pi)$ , &c. all vanish. But  $\Phi(0) = \phi(0)$ . Hence the theorem is proved.

When  $a$  and  $\beta$  are both positive,

$$2732 \quad \int_a^\beta \frac{\sin ax}{x} \phi(x) dx = 0 = \int_{-a}^{-\beta} \frac{\sin ax}{x} \phi(x) dx.$$

$$2733 \quad \int_{-a}^\beta \frac{\sin ax}{x} \phi(x) dx = \pi\phi(0).$$

PROOF.—(i.)  $\int_a^\beta = \int_0^\beta - \int_0^a = \frac{\pi}{2}\phi(0) - \frac{\pi}{2}\phi(0)$ , by (2729).

(ii.)  $\int_{-a}^\beta = \int_{-a}^0 + \int_0^\beta = \frac{\pi}{2}\phi(0) + \frac{\pi}{2}\phi(0)$ ,

by substituting  $-x$  in the second integral.

$$2734 \quad \int_0^a \int_0^h \phi(x) \cos ux du dx = \frac{\pi}{2} \phi(0), \quad \text{when } a = \infty.$$

PROOF.  $\frac{\sin ax}{x} = \int_0^a \cos ux du$ . Substitute this in (2731).

When  $a$  and  $\beta$  are positive, the limit when  $a$  is infinite of

$$2735 \quad \int_0^a \int_a^\beta \phi(x) \cos tu \cos ux du dx,$$

or of  $\int_0^a \int_a^\beta \phi(x) \sin tu \sin ux du dx,$

is  $\frac{1}{2}\pi\phi(t)$ , if  $t$  lies between  $a$  and  $\beta$ ,  $\frac{1}{4}\pi\phi(t)$  if  $t = a$ , and zero for any other value of  $t$ .

PROOF.—When  $a = \infty$  we have, by (668), and integrating with respect to  $u$ ,

$$\begin{aligned} \int_a^\beta \int_0^a \phi(x) \cos ux \cos tu dx du &= \frac{1}{2} \int_a^\beta \frac{\sin a(x-t)}{x-t} \phi(x) dx + \frac{1}{2} \int_a^\beta \frac{\sin a(x+t)}{x+t} \phi(x) dx \\ &= \frac{1}{2} \int_{a-t}^{\beta-t} \frac{\sin az}{z} \phi(z+t) dz + \frac{1}{2} \int_{a+t}^{\beta+t} \frac{\sin az}{z} \phi(z-t) dz \dots\dots\dots(i.), \end{aligned}$$

by substituting  $z = x-t$  and  $z = x+t$  in the two integrals respectively.

When  $a$  is infinite, the limit of each integral is known.

When  $a$  and  $\beta$  are positive and  $t$  lies between them in value, the limit of (i.) is  $\frac{1}{2}\pi\phi(t)$ , by (2732-3)..... (ii.)

When  $a$  and  $\beta$  are positive and  $t$  does not lie between them, the value is zero, by (2732)..... (iii.)

If  $\alpha = t$  in (i.), the first integral becomes  $= \frac{1}{2}\pi\phi(t)$  by (2731), and the second vanishes as before; so that the value, in this case, is  $\frac{1}{2}\pi\phi(t)$ ... (iv.)

The same demonstration applies in the case of (2736), transforming by (669) instead of (668).

Hence, by (ii.), if  $t$  be always positive,

$$\begin{aligned} 2737 \quad \int_0^\infty \int_0^\infty \phi(x) \cos tu \cos ux \, du \, dx &= \frac{\pi}{2} \phi(t) \\ &= \int_0^\infty \int_0^\infty \phi(x) \sin tu \sin ux \, du \, dx. \end{aligned}$$

$$\begin{aligned} 2739 \quad \text{Ex.—Let} \quad \phi(x) &= e^{-ax}, \\ \therefore \int_0^\infty \int_0^\infty e^{-ax} \cos tu \cos ux \, du \, dx &= \frac{\pi}{2} e^{-at}. \end{aligned}$$

$$\text{Therefore, by (2584),} \quad \int_0^\infty \frac{a \cos tu}{a^2 + u^2} \, du = \frac{\pi}{2} e^{-at},$$

which is equivalent to (2574), with  $t = 1$ .

The expressions in (2737–8) being even functions of  $u$ , we have, supposing  $t$  to be always positive,

$$\int_{-\infty}^\infty \int_0^\infty \phi(x) \cos tu \cos ux \, du \, dx = \pi \phi(t) = \int_{-\infty}^\infty \int_0^\infty \phi(x) \sin tu \sin ux \, du \, dx \dots (i.)$$

Replacing  $\phi(x)$  by  $\phi(-x)$ , and afterwards substituting  $-x$ , these equations become

$$\begin{aligned} \int_{-\infty}^\infty \int_{-\infty}^0 \phi(x) \cos tu \cos ux \, du \, dx &= \pi \phi(-t) \\ &= - \int_{-\infty}^\infty \int_{-\infty}^0 \phi(x) \sin tu \sin ux \, du \, dx \dots \dots \dots (ii.) \end{aligned}$$

From (i) and (ii.), by addition and subtraction, we get

$$2740 \quad \int_{-\infty}^\infty \int_{-\infty}^\infty \phi(x) \cos tu \cos ux \, du \, dx = \pi [\phi(t) + \phi(-t)],$$

$$2741 \quad \int_{-\infty}^\infty \int_{-\infty}^\infty \phi(x) \sin tu \sin ux \, du \, dx = \pi [\phi(t) - \phi(-t)].$$

Whence, by addition,

$$2742 \quad \int_{-\infty}^\infty \int_{-\infty}^\infty \phi(x) \cos u(t-x) \, du \, dx = 2\pi\phi(t),$$

the original formula of Fourier's.



### THE FUNCTION $\psi(x)$ .

The function  $d_x \log \Gamma(x)$  is denominated  $\psi(x)$ .

$$2743 \quad \psi(x) = \log \mu - \frac{1}{x} - \frac{1}{x+1} - \frac{1}{x+2} - \dots - \frac{1}{x+\mu},$$

when  $\mu$  is an indefinitely great integer.

PROOF.—By differentiating the logarithm of (2293).

$$2744 \quad \text{COR. } \psi(1) = \log \mu - 1 - \frac{1}{2} - \frac{1}{3} - \dots - \frac{1}{\mu+1},$$

when  $\mu = \infty$ ,

$$= -0.577215,664901,532860,60 \dots \text{ (Euler).}$$

All other values of  $\psi(x)$ , when  $x$  is a commensurable quantity, may be made to depend upon the value of  $\psi(1)$ .

When  $x$  is less than 1,

$$2745 \quad \psi(1-x) - \psi(x) = \pi \cot \pi x.$$

PROOF.—Differentiate the logarithm of the equation

$$\Gamma(x) \Gamma(1-x) = \pi \div \sin \pi x \quad (2313).$$

$$2746 \quad \psi(x) + \psi\left(x + \frac{1}{n}\right) + \psi\left(x + \frac{2}{n}\right) + \dots + \psi\left(x + \frac{n-1}{n}\right) \\ = n\psi(nx) - n \log n.$$

PROOF.—Differentiate the logarithm of equation (2316).

2747 To compute the value of  $\psi\left(\frac{p}{q}\right)$  when  $\frac{p}{q}$  is a proper fraction.

Find  $\psi\left(\frac{p}{q}\right)$  from the two equations

$$2748 \quad \psi\left(1 - \frac{p}{q}\right) - \psi\left(\frac{p}{q}\right) = \pi \cot \frac{p}{q} \pi, \quad (2745)$$

2749

$$\psi\left(1 - \frac{p}{q}\right) + \psi\left(\frac{p}{q}\right) = 2 \left\{ \psi(1) - \log q + \cos \frac{2p\pi}{q} \log \left(2 \text{vers} \frac{2\pi}{q}\right) \right. \\ \left. + \cos \frac{4p\pi}{q} \log \left(2 \text{vers} \frac{4\pi}{q}\right) + \cos \frac{6p\pi}{q} \log \left(2 \text{vers} \frac{6\pi}{q}\right) + \&c. \right\}.$$



Let  $\frac{2\pi}{q} = \omega$ . Then, by giving to  $\phi$  in equation (v.) its different values  $\omega, 2\omega, 3\omega \dots (q-1)\omega$ , we obtain  $q-1$  linear equations in the unknown quantities  $\psi\left(\frac{1}{q}\right), \psi\left(\frac{2}{q}\right) \dots \psi\left(\frac{q-1}{q}\right)$ . To solve these equations for  $\psi\left(\frac{p}{q}\right)$   $p$  being an integer less than  $q$ , multiply them respectively by

$$\cos p\omega, \cos 2p\omega \dots \cos (q-1)p\omega,$$

and join to their sum equation (2746), after putting  $x = \frac{1}{q}$  and  $n = q$ .

The coefficient of  $\psi\left(\frac{k}{q}\right)$  in the result,  $k$  being any integer less than  $q$ , is

$$\cos p\omega \cos k\omega + \cos 2p\omega \cos 2k\omega + \dots + \cos (q-1)p\omega \cos (q-1)k\omega + 1.$$

By expanding each term by (668), we see by (iv.) that this coefficient vanishes excepting for the values  $k = q-p$  and  $k = p$ , in each of which cases it becomes  $= \frac{1}{2}q$ . Hence, dividing by  $\frac{1}{2}q$ , we obtain

$$\psi\left(\frac{q-p}{q}\right) + \psi\left(\frac{p}{q}\right) = 2\psi(1) - 2\log 2 + \cos p\omega \log(2 - 2\cos \omega)$$

$$+ \cos 2p\omega \log(2 - 2\cos 2\omega) + \dots + \cos (q-1)p\omega \log\{2 - 2\cos(q-1)\omega\}.$$

The last term  $= \cos p\omega \log(2 - 2\cos \omega) =$  the third term; the last but one  $= \cos 2p\omega \log(2 - 2\cos 2\omega) =$  the second term, and so on, forming pairs of equal terms. But, if  $q$  be even, there is the odd term

$$\cos \frac{1}{2}qp\omega \log(2 - 2\cos \frac{1}{2}q\omega) = \pm 2 \log 2,$$

according as  $p$  is even or odd.

EXAMPLES.—By (2748–9) we obtain

$$2750 \quad \psi\left(\frac{1}{2}\right) = \psi(1) - 3 \log 2 + \frac{\pi}{2}, \quad \psi\left(\frac{1}{4}\right) = \psi(1) - 3 \log 2 - \frac{\pi}{2},$$

$$2752 \quad \psi\left(\frac{2}{3}\right) = \psi(1) - \frac{1}{2} \log 3 + \frac{\pi}{2\sqrt{3}}, \quad \psi\left(\frac{1}{3}\right) = \psi(1) - \frac{1}{2} \log 3 - \frac{\pi}{2\sqrt{3}},$$

$$2754 \quad \psi\left(\frac{1}{2}\right) = \psi(1) - 2 \log(2).$$

#### DEVELOPMENTS OF $\psi(a+x)$ .

When  $x$  is any integer,

$$2755 \quad \psi(a+x) = \psi(a) + \frac{1}{a} + \frac{1}{a+1} + \frac{1}{a+2} + \dots + \frac{1}{a+x-1}.$$

PROOF.—By (2289), putting  $n = a+x-1$  and  $r = x-1$ ,

$$\Gamma(a+x) = (a+x-1)(a+x-2) \dots (a+2)(a+1)a \Gamma(a).$$

Differentiate the logarithm of this equation with respect to  $a$ .

$$2756 \quad \psi(a+x) = \psi(a) + \frac{x}{a} - \frac{x(x-1)}{2a(a+1)} + \frac{x(x-1)(x-2)}{3a(a+1)(a+2)} \\ - \frac{x(x-1)(x-2)(x-3)}{4a(a+1)(a+2)(a+3)} + \&c.$$

If  $x$  be a positive integer, the number of terms in this series is finite, and the value of  $\psi(a+x)$  can be found from that of  $\psi(a)$ .

Hence, by this or the preceding formula, in conjunction with (2747), the value of  $\psi(N)$ , when  $N$  is any commensurable quantity, may be found in terms of  $\psi(1)$ .

PROOF.—Let  $\psi(a+x) = A + Bx + Cx(x-1) + Dx(x-1)(x-2) + \&c.$

Change  $x$  into  $x+1$ ; then,

$$\Delta\psi(a+x) = \psi(a+x+1) - \psi(a+x) = d_x \{ \log \Gamma(a+x+1) - \log \Gamma(a+x) \} \\ = d_x \log(a+x) \quad (2288) = \frac{1}{a+x},$$

$\Delta x = 1$ ,  $\Delta x(x-1) = 2x$ ,  $\Delta x(x-1)(x-2) = 3x(x-1)$ , &c. Therefore

$$\frac{1}{a+x} = B + 2Cx + 3Dx(x-1) + 4Ex(x-1)(x-2) +,$$

$$\Delta \frac{1}{a+x} = 2C + 2 \cdot 3Dx + 3 \cdot 4Ex(x-1) +,$$

$$\Delta^2 \frac{1}{a+x} = 2 \cdot 3D + 2 \cdot 3 \cdot 4 \cdot Ex +,$$

$$\Delta^3 \frac{1}{a+x} = 2 \cdot 3 \cdot 4E +.$$

Put  $x=0$  in each equation to determine the coefficients  $A, B, C, D$ , &c.;

thus  $A = \psi(a)$ ,  $B = \frac{1}{a}$ ,  $2C = \Delta \frac{1}{a} = \frac{1}{a+1} - \frac{1}{a} = -\frac{1}{a(a+1)}$ ,

$$2 \cdot 3D = \Delta^2 \frac{1}{a} = \Delta \frac{-1}{a(a+1)} = \frac{2}{a(a+1)(a+2)},$$

$$2 \cdot 3 \cdot 4E = \Delta^3 \frac{1}{a} = \Delta \frac{2}{a(a+1)(a+2)} = -\frac{2 \cdot 3}{a(a+1)(a+2)(a+3)}, \text{ and so on.}$$

#### SUMMATION OF SERIES BY THE FUNCTION $\psi(x)$ .

**2757** *Formula I.*  $\frac{a}{b} + \frac{a}{b+c} + \frac{a}{b+2c} + \dots + \frac{a}{b+nc}$

$$= \frac{a}{b} - \frac{a}{c} \psi\left(\frac{b}{c} + 1\right) + \frac{a}{c} \psi\left(\frac{b}{c} + n + 1\right).$$

PROOF.—Let  $S_n$  denote the  $n$  terms of the series to be summed. We have

$$S_{n+1} - S_n = \frac{a}{c} \div \left(\frac{b}{c} + n + 1\right) = \frac{a}{c} \left[ \psi\left(\frac{b}{c} + n + 2\right) - \psi\left(\frac{b}{c} + n + 1\right) \right] \quad (2288)$$

or  $S_{n+1} - \frac{a}{c} \psi\left(\frac{b}{c} + n + 2\right) = S_n - \frac{a}{c} \psi\left(\frac{b}{c} + n + 1\right).$

Hence the difference is independent of  $n$ , and therefore

$$S_n - \frac{a}{c} \psi\left(\frac{b}{c} + n + 1\right) = S_0 - \frac{a}{c} \psi\left(\frac{b}{c} + 1\right) = \frac{a}{b} - \frac{a}{c} \psi\left(\frac{b}{c} + 1\right).$$

**2758** Ex.  $1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n+1} = 1 - \frac{1}{2}\psi\left(\frac{3}{2}\right) + \frac{1}{2}\psi\left(n + \frac{3}{2}\right).$

**2759** Formula II.  $\frac{a}{b} - \frac{a}{b+c} + \frac{a}{b+2c} - \dots - \frac{a}{b+(2n+1)c}$   
 $= \frac{ac}{b(b+c)} + \frac{a}{2c} \left\{ \psi\left(\frac{b+3c}{2c}\right) - \psi\left(\frac{b+2c}{2c}\right) + \psi\left(\frac{b+2c}{2c} + n\right) - \psi\left(\frac{b+3c}{2c} + n\right) \right\}.$

PROOF.—The series is equivalent to

$$\frac{a}{b} + \frac{a}{b+2c} + \frac{a}{b+4c} + \dots + \frac{a}{b+2nc} - \left\{ \frac{a}{b+c} + \frac{a}{b+c+2c} + \dots + \frac{a}{b+c+2nc} \right\},$$

and the result follows by Formula I.

**2760** Formula III.—

$$\frac{1}{b} - \frac{1}{b+c} + \frac{1}{b+2c} - \dots \text{ in inf.}$$

$$= \frac{c}{b(b+c)} + \frac{1}{2c} \left\{ \psi\left(\frac{b+3c}{2c}\right) - \psi\left(\frac{b+2c}{2c}\right) \right\}.$$

PROOF.—Make  $n = \infty$  in Formula II. The last two terms become equal.

**2761** Ex. 1.—In (2760) let  $b = c = 1$ , then

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \&c. = \frac{1}{2} + \frac{1}{2} \left\{ \psi(2) - \psi\left(\frac{3}{2}\right) \right\} = \log 2.$$

For  $\psi(2) = 1 + \psi(1)$ , by (2755);  $\psi\left(\frac{3}{2}\right) = 2 + \psi(1) - 2 \log 2$ , by (2754-5).

**2762** Ex. 2.—In (2760) let  $b = 1$ ,  $c = 2$ , then

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c. = \frac{2}{3} + \frac{1}{4} \psi\left(\frac{5}{4}\right) - \frac{1}{4} \psi\left(\frac{3}{4}\right) = \frac{\pi}{4}.$$

**2763**  $\psi(1+a) = \int_0^1 \frac{x^a - 1}{x-1} dx + \psi(1).$

PROOF.  $\psi(1+a) = \psi(1) + a - \frac{a(a-1)}{2 \cdot 1 \cdot 2} + \frac{a(a-1)(a-2)}{3 \cdot 1 \cdot 2 \cdot 3} - \&c.$  [by (2756)]

But  $\frac{1-(1-x)^a}{x} = a - \frac{a(a-1)}{1 \cdot 2} x + \frac{a(a-1)(a-2)}{1 \cdot 2 \cdot 3} x^2 - \&c.,$

therefore  $\psi(1+a) = \int_0^1 \frac{1-(1-x)^a}{x} dx + \psi(1).$

Substitute  $1-x$  in the integral.

**2764**  $\psi(1+a) - \psi(1+b) = \int_0^1 \frac{x^a - x^b}{x-1} dx. \quad [\text{By (2763)}]$

Ex.—Put  $b = -a$ ; then

$$\psi(1+a) - \psi(1-a) = \frac{1}{a} + \psi(a) - \psi(1-a) \quad (2756) = \frac{1}{a} - \pi \cot \pi a \quad (2745).$$

**2765** Therefore 
$$\int_0^1 \frac{x^a - x^{-a}}{x-1} dx = \frac{1}{a} - \pi \cot \pi a.$$

$\psi(x)$  AS A DEFINITE INTEGRAL INDEPENDENT OF  $\psi(1)$ .

**2766** 
$$\psi(x) = -\int_0^1 \left( \frac{1}{\log u} + \frac{u^{x-1}}{1-u} \right) du.$$

PROOF.  $\psi(x) = \log \mu - \frac{1}{x} - \frac{1}{x+1} - \dots - \frac{1}{x+\mu-1}$  with  $\mu = \infty$  (2743).

But 
$$\frac{1}{x} + \frac{1}{x+1} + \dots + \frac{1}{x+\mu-1} = \int_0^1 \frac{z^{x-1} - z^{x+\mu-1}}{1-z} dz,$$

by actual division and integration.

Also 
$$\log \mu = \int_0^1 \left( \frac{1}{1-z} - \frac{\mu z^{\mu-1}}{1-z^\mu} \right) dz \quad (2367).$$

$$\therefore \psi(x) = \int_0^1 \frac{1-z^{x-1}}{1-z} dz + \int_0^1 \left\{ \frac{z^{x+\mu-1}}{1-z} - \frac{\mu z^{\mu-1}}{1-z^\mu} \right\} dz, \quad \mu = \infty \dots (i.)$$

Put  $z = y^\mu$  in the first integral, therefore

$$\int_0^1 \mu \frac{1-y^{\mu x-\mu}}{1-y^\mu} y^{\mu-1} dy = \mu \int_0^1 \frac{y^{\mu-1} - y^{\mu x-1}}{1-y^\mu} dy.$$

Replace  $y$  by  $z$ , and suppress the term common with the second integral

of (i.), and we get 
$$\psi(x) = \int_0^1 \left\{ \frac{z^{x+\mu-1}}{1-z} - \frac{\mu z^{\mu x-1}}{1-z^\mu} \right\} dz.$$

Put  $z^\mu = u$ , and this becomes

$$\psi(x) = \int_0^1 \left\{ \frac{u^{\frac{x}{\mu}}}{\mu(1-u^{\frac{1}{\mu}})} - \frac{u^{x-1}}{1-u} \right\} du.$$

But when  $\mu = \infty$  the product  $\mu(1-u^{\frac{1}{\mu}})$  has  $-\log u$  for its limit (1584); and  $u^{\frac{x}{\mu}} = 1$ . Hence the result.

**2767** 
$$\psi(x) = \int_0^\infty \left\{ e^{-a} - \frac{1}{(1+a)^x} \right\} \frac{da}{a}.$$

PROOF.  $\Gamma(x) = \int_0^\infty e^{-z} z^{x-1} dz$ ;  $d_x \Gamma(x) = \int_0^\infty e^{-z} z^{x-1} \log z dz.$

But, by (2427), 
$$\log z = \int_0^\infty \frac{e^{-a} - e^{-az}}{a} da,$$



$$\begin{aligned}
\therefore d_x \Gamma(x) &= \int_0^x \int_0^\infty e^{-z} z^{x-1} \frac{e^{-a} - e^{-ax}}{a} dz da \\
&= \int_0^\infty \left[ e^{-a} \int_0^x e^{-z} z^{x-1} dz - \int_0^x e^{-(1+a)z} z^{x-1} dz \right] \frac{da}{a} \\
&= \Gamma(x) \int_0^\infty \left[ e^{-a} - \frac{1}{(1+a)^x} \right] \frac{da}{a} \quad (2291),
\end{aligned}$$

which establishes the formula since

$$d_x \Gamma(x) \div \Gamma(x) = d_x \log \Gamma(x) = \psi(x).$$

$$2768 \quad \log \Gamma(x) = \int_0^\infty \left[ (x-1) e^{-\xi} - \frac{e^{-\xi} - e^{-\xi x}}{1 - e^{-\xi}} \right] \frac{d\xi}{\xi}$$

$$2769 \quad = \int_0^1 \left[ \frac{1 - z^{x-1}}{1 - z} - x + 1 \right] \frac{dz}{z}.$$

$$2770 \quad \psi(x) = \int_0^\infty \left[ \frac{e^{-\xi}}{\xi} - \frac{e^{-\xi x}}{1 - e^{-\xi}} \right] d\xi.$$

PROOF.—Integrate (2767) for  $x$  between the limits 1 and  $x$ , observing that  $\log \Gamma(1) = 0$ ; thus

$$\log \Gamma(x) = \int_0^\infty \left\{ (x-1) e^{-a} - \frac{(1+a)^{-1} - (1+a)^{-x}}{\log(1+a)} \right\} \frac{da}{a}.$$

Subtract from this the equation obtained from it by making  $x = 2$ , and multiplying the result by  $x - 1$ . We thus obtain

$$\log \Gamma(x) = \int_0^\infty \left[ (x-1)(1+a)^{-x} - \frac{(1+a)^{-1} - (1+a)^{-x}}{a} \right] \frac{da}{\log(1+a)}.$$

Substitute  $\xi = \log(1+a)$ , and (2768) is the result. To obtain (2769), substitute  $z = (1+a)^{-1}$ . Lastly, (2770) is the result of differentiating (2768) for  $x$ .

## NUMERICAL CALCULATION OF $\log \Gamma(x)$ .

**2771** The second member of (2768) can be divided into two parts, one of which appears under a finite form, and the other vanishes with  $x$ . If we put

$$P = \left( x - 1 - \frac{1}{1 - e^{-\xi}} \right) \frac{e^{-\xi}}{\xi}, \text{ and } Q = \frac{1}{\xi(1 - e^{-\xi})},$$

then 
$$\log \Gamma(x) = \int_0^\infty (P + Q e^{-\xi x}) d\xi \dots\dots\dots (i.)$$

If  $Q$  be developed in ascending powers of  $\xi$ , the terms which contain negative indices are  $\frac{1}{\xi^2} + \frac{1}{2\xi} = R$  say.

$$\begin{aligned} \text{Put } F(x) &= \int_0^\infty (P + Re^{-tx}) d\xi \\ &= \int_0^\infty \left[ \left( (x-1) - \frac{1}{1-e^{-\xi}} \right) e^{-\xi} + \left( \frac{1}{\xi} + \frac{1}{2} \right) e^{-\xi x} \right] \frac{d\xi}{\xi} \dots\dots\dots(\text{ii.}), \end{aligned}$$

$$\text{and } \varpi(x) = \int_0^\infty (Q-R) e^{-\xi x} d\xi = \int_0^\infty \left( \frac{1}{1-e^{-\xi}} - \frac{1}{\xi} - \frac{1}{2} \right) e^{-\xi x} \frac{d\xi}{\xi} \dots (\text{iii.})$$

$$\text{Then, by (i.), } \log \Gamma(x) = F(x) + \varpi(x) \dots\dots\dots(\text{iv.})$$

$F(x)$  can now be calculated in a finite form, and  $\varpi(x)$  will have zero for its limit as  $x$  increases.

First, to show that  $F(\frac{1}{2})$  and  $\varpi(\frac{1}{2})$  can be exactly calculated.

$$\varpi\left(\frac{1}{2}\right) = \int_0^\infty \left( \frac{1}{1-e^{-\xi}} - \frac{1}{\xi} - \frac{1}{2} \right) e^{-\frac{1}{2}\xi} \frac{d\xi}{\xi},$$

and, by substituting  $\frac{1}{2}\xi$ ,

$$\varpi\left(\frac{1}{2}\right) = \int_0^\infty \left( \frac{1}{1-e^{-2\xi}} - \frac{1}{2\xi} - \frac{1}{2} \right) e^{-\xi} \frac{d\xi}{\xi} \dots\dots\dots(\text{v.})$$

Again, putting  $x=1$  in (iii.), we have

$$\varpi(1) = \int_0^\infty \left( \frac{1}{1-e^{-\xi}} - \frac{1}{\xi} - \frac{1}{2} \right) e^{-\xi} \frac{d\xi}{\xi} \dots\dots\dots(\text{vi.});$$

and, by substituting  $\frac{1}{2}\xi$ ,

$$\varpi(1) = \int_0^\infty \left( \frac{1}{1-e^{-2\xi}} - \frac{1}{2\xi} - \frac{1}{2} \right) e^{-2\xi} \frac{d\xi}{\xi} \dots\dots\dots(\text{vii.})$$

The difference of (vi.) and (vii.) gives

$$0 = \int_0^\infty \left( \frac{1}{1-e^{-2\xi}} - \frac{2-e^{-\xi}}{2\xi} - \frac{1-e^{-\xi}}{2} \right) e^{-\xi} \frac{d\xi}{\xi} \dots\dots\dots(\text{viii.}),$$

since

$$\frac{e^{-\xi}}{1-e^{-\xi}} - \frac{e^{-2\xi}}{1-e^{-2\xi}} = \frac{e^{-\xi}}{1-e^{-2\xi}}.$$

Subtract (viii.) from (v.), thus

$$\varpi\left(\frac{1}{2}\right) = \frac{1}{2} \int_0^\infty \left( \frac{e^{-\xi} - e^{-2\xi}}{\xi^2} - \frac{e^{-2\xi}}{\xi} \right) d\xi = \frac{1}{2} - \frac{\log 2}{2} \quad (2429).$$

$$\text{Also, by (iv.), } F\left(\frac{1}{2}\right) + \varpi\left(\frac{1}{2}\right) = l\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}l\pi, \therefore F\left(\frac{1}{2}\right) = \frac{1}{2}\log(2\pi) - \frac{1}{2} \dots(\text{ix.})$$

$F(x)$  may now be found by calculating  $F(x) - F(\frac{1}{2})$  as follows:—

$$\begin{aligned} \text{By (ii.), } F(x) - F\left(\frac{1}{2}\right) &= \int_0^\infty \left[ \left( x - \frac{1}{2} \right) e^{-\xi} + \left( \frac{1}{\xi} + \frac{1}{2} \right) (e^{-\xi x} - e^{-\frac{1}{2}\xi}) \right] \frac{d\xi}{\xi} \\ &= \int_0^\infty \frac{e^{-\xi x} - e^{-\frac{1}{2}\xi} + \xi \left( x - \frac{1}{2} \right) e^{-\xi}}{\xi^2} d\xi + \frac{1}{2} \int_0^\infty \frac{e^{-\xi x} - e^{-\frac{1}{2}\xi}}{\xi} d\xi \\ &= \frac{1}{2} - x + \left( x - \frac{1}{2} \right) \log x \quad (2427-8), \end{aligned}$$

$$\therefore F(x) = \frac{1}{2}\log(2\pi) + \left( x - \frac{1}{2} \right) \log x - x, \text{ by (ix.);}$$

$$\therefore \text{ by (iv.) } \log \Gamma(x) = \frac{1}{2}\log(2\pi) + \left( x - \frac{1}{2} \right) \log x - x + \varpi(x) \dots\dots\dots(\text{x.});$$

**2772**

$$\therefore \Gamma(x) = e^{-x} x^{x-\frac{1}{2}} \sqrt{(2\pi)} e^{\varpi(x)} \dots\dots\dots(\text{xi.})$$

When  $x$  is very large,  $e^{\varpi(x)}$  differs but little from unity. For  $\varpi(x)$  diminishes without limit as  $x$  increases, by the value (iii.)

Replacing  $\varpi(x)$  in (x.) by its value (iii.), and observing that

$$\log \Gamma(x+1) = \log x + \log \Gamma(x),$$

$$\text{we get } \log \Gamma(x+1) = \frac{1}{2} \log(2\pi) + \left(x + \frac{1}{2}\right) \log x - x + \int_0^{\infty} \left( \frac{1}{1-e^{-\xi}} - \frac{1}{\xi} - \frac{1}{2} \right) e^{-\xi x} \frac{d\xi}{\xi} \dots\dots\dots \text{(xii.)}$$

Now, by (1539),

$$\left( \frac{1}{1-e^{-\xi}} - \frac{1}{\xi} - \frac{1}{2} \right) \frac{1}{\xi} = \frac{B_1}{1 \cdot 2} - \frac{B_3 \xi^2}{1 \cdot 2 \cdot 3 \cdot 4} + \dots \pm \frac{B_{2n} \xi^{2n-2}}{1 \dots 2n} \mp \frac{\theta B_{2n+2} \xi^{2n}}{1 \dots 2n+2},$$

where  $\theta$  is  $< 1$ . Also

$$\int_0^{\infty} e^{-\xi x} \xi^{2\mu} d\xi = \frac{1 \dots 2\mu}{x^{2\mu+1}}, \quad \int_0^{\infty} \theta e^{-\xi x} \xi^{2n} d\xi = \theta_1 \frac{1 \dots 2n}{x^{2n+1}}.$$

So that equation (xii.) produces

$$\begin{aligned} \text{2773 } \log \Gamma(x+1) &= \frac{\log(2\pi)}{2} + \left(x + \frac{1}{2}\right) \log x - x \\ &+ \frac{B_2}{1 \cdot 2x} - \frac{B_4}{3 \cdot 4x^3} + \dots \mp \frac{\theta B_{2n+2}}{(2n+1)(2n+2) x^{2n+1}}. \end{aligned}$$

This series is divergent, the terms increasing indefinitely. The complementary term, which increases with  $n$  and is very great when  $n$  is very great, is, however, *very small for considerable values of  $n$* . For instance, when  $x=10$ , the values obtained for  $\log \Gamma(11)$ , by taking 3, 4, 5, or 6 terms of the series, are respectively,

$$16.090820096, \quad 16.104415343, \quad 16.104412565, \quad 16.104112563.$$

## CHANGE OF THE VARIABLES IN A DEFINITE MULTIPLE INTEGRAL.

**2774** Let  $x, y, z$  be connected with  $\xi, \eta, \zeta$  by three equations  $u=0, v=0, w=0$ .

Then, when the limits of the integral containing the new variables can be assigned independently, we have

$$\begin{aligned} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} F(x, y, z) dx dy dz \\ = - \int_{\xi_1}^{\xi_2} \int_{\eta_1}^{\eta_2} \int_{\zeta_1}^{\zeta_2} \Phi(\xi, \eta, \zeta) \frac{\frac{d(urw)}{d(\xi\eta\zeta)}}{\frac{d(uvw)}{d(xyz)}} d\xi d\eta d\zeta, \end{aligned}$$

where  $\Phi$  is what  $F$  becomes when the values of  $x, y, z$ , in terms of  $\xi, \eta, \zeta$ , obtained by solving the equations  $u, v, w$ , are substituted.

PROOF. 
$$\iiint F(x, y, z) \, dx \, dy \, dz = \iiint \Phi(\xi, \eta, \zeta) \frac{dx}{d\xi} \frac{dy}{d\eta} \frac{dz}{d\zeta} \, d\xi \, d\eta \, d\zeta.$$

To find  $x_\xi$ , consider  $\eta$  and  $\zeta$  constant, and differentiate the three equations  $u, v, w$  for  $\xi$ , as in (1723). To find  $y_\eta$ , consider  $\zeta$  and  $x$  constant, and differentiate for  $\eta$ . To find  $z_\zeta$ , consider  $x$  and  $y$  constant, and differentiate for  $\zeta$ . We thus obtain

$$\frac{dx}{d\xi} \frac{dy}{d\eta} \frac{dz}{d\zeta} = - \frac{\frac{d(uvw)}{d(\xi yz)} \frac{d(uvw)}{d(\eta z\xi)} \frac{d(uvw)}{d(\zeta \xi \eta)}}{\frac{d(uvw)}{d(xyz)} \frac{d(uvw)}{d(yz\xi)} \frac{d(uvw)}{d(z\xi\eta)}} = - \frac{\frac{d(uvw)}{d(\xi \eta \zeta)}}{\frac{d(uvw)}{d(xyz)}},$$

observing that two interchanges of columns in a determinant do not alter its value or sign (559).

Similarly in the case of any number of independent variables.

When, however, the limits in the transformed integral have to be discovered from the given equations, the process is not so simple.

In the first place, we shall show how to change the *order* of integration merely.

**2775** Taking a double integral in its most general form, we shall have

$$\int_a^b \int_{\psi(x)}^{\phi(x)} F(x, y) \, dx \, dy = \Sigma \int_a^\beta \int_{\Psi(y)}^{\Phi(y)} F(x, y) \, dy \, dx \dots\dots\dots (i.)$$

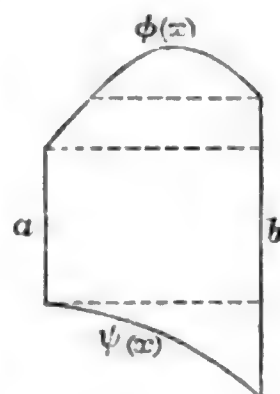
The right member will generally consist of more than one integral, and  $\Sigma$  denotes their sum. The limits of the integration for  $x$  may be, one or both, constants, or, one or both, functions of  $y$ .  $\Psi$  is the inverse of the function  $\psi$ , and is obtained by solving the equation  $y = \psi(x)$ , so that  $x = \Psi(y)$ . Similarly with regard to  $\phi$  and  $\Phi$ .

An examination of the solid figure described in (1907), whose volume this integral represents, will make the matter clearer. The integration, the order of which has to be changed, extends over an area which is the projection of the solid upon the plane of  $xy$ , and which is bounded by the two straight lines  $x = a, x = b$ , and the two curves  $y = \psi(x), y = \phi(x)$ .

The summation of the elements  $PQqp$  extends from  $a$  to  $b$ , and includes in the one integral on the left of equation (i.) the whole of the solid in question.

But, on the right, the different integrals represent the summation of elements like  $PQqp$ , but all parallel to  $OX$ , between planes  $y = \alpha, y = \beta$ , &c. drawn through points where the limits of  $x$  change their character on account of the boundaries  $y = \psi(x), y = \phi(x)$  not being straight lines parallel to  $OX$ .

**2776** EXAMPLE.—Let the figure represent the projected area on the  $xy$  plane, bounded by the curves  $y = \psi(x)$ ,  $y = \phi(x)$ , and the straight lines  $x = a$ ,  $x = b$ . Let  $y = \phi(x)$  have a maximum value when  $x = c$ . The values of  $y$  at this point will be  $\phi(c)$ , and at the points where the straight lines meet the curves the values will be  $\phi(a)$ ,  $\phi(b)$ ,  $\psi(a)$ ,  $\psi(b)$ .



According to the drawing, the right member of equation (i.) will now stand as follows,  $U$  being written for  $F(x, y)$ ,

$$\int_{\psi(b)}^{\psi(a)} \int_{\psi(y)}^b U dy dx + \int_{\psi(a)}^{\phi(a)} \int_a^b U dy dx + \int_{\phi(a)}^{\phi(b)} \int_{\phi(y)}^b U dy dx + \int_{\phi(b)}^{\phi(c)} \int_{\phi_1(y)}^{\phi_2(y)} U dy dx.$$

The four integrals represent the four areas into which the whole is divided by the dotted lines drawn parallel to the  $X$  axis. In the last integral,  $\phi_1(y)$  and  $\phi_2(y)$  are the two values of  $x$  corresponding to one of  $y$  in that part of the curve  $y = \phi(x)$  which is cut twice by any  $x$  coordinate.

**2777** To change the order of integration in a triple integral, from  $z, y, x$  to  $y, x, z$ , we shall have an equation of the form

$$\int_{x_1}^{x_2} \int_{\psi_1(x)}^{\psi_2(x)} \int_{\phi_1(x,y)}^{\phi_2(x,y)} F(x, y, z) dz dy dx = \Sigma \int_{z_1}^{z_2} \int_{\Psi_1(z)}^{\Psi_2(z)} \int_{\Phi_1(z,x)}^{\Phi_2(z,x)} F(x, y, z) dz dx dy \quad \dots\dots\dots (ii.)$$

Here the most general form for the integrals whose sum is indicated by  $\Sigma$  is that in which the limits of  $y$  are functions of  $z$  and  $x$ , the limits of  $x$  functions of  $z$ , and the limits of  $z$  constant. Referring to the figure in (1906), the total value of the integral is equivalent to the following. Every element  $dx dy dz$  of the solid described in (1907) is multiplied by  $F(xyz)$ , a function of the coordinates of the element, and the sum of the products is taken.

This process is indicated by one triple integral on the left of equation (ii.); the limits of the integration for  $z$  being two unrestricted curved surfaces  $z = \phi_1(x, y)$ ,  $z = \phi_2(x, y)$ ; the limits for  $y$ , two cylindrical surfaces  $y = \psi_1(x)$ ,  $y = \psi_2(x)$ ; and the limits for  $x$ , two planes  $x = x_1$ ,  $x = x_2$ .

But, with the changed order of integration, several integrals may be required. The most general form which any of them can take is that shown on the right of equation (ii.) Solving the equation  $z = \phi_1(x, y)$ , let  $y_1 = \Phi_1(z, x)$ ,  $y_2 = \Phi_2(z, x)$  be two resulting values of  $y$ ; then the integration for  $y$  may be effected between these limits over all parts of the solid where the surface  $z = \phi_1(x, y)$  is cut twice by the same  $y$  coordinate.

The next integration is with respect to  $x$ , and is limited by the cylindrical surface, whose generating lines, parallel to  $OY$ , touch the surface  $z = \phi_1(x, y)$ . At the points of contact,  $x$  will have a maximum or minimum value for each value of  $z$ ; therefore  $d_x \phi_1(x, y) = 0$ . Eliminating  $y$  between this equation and that of the surface, we get  $x = \Psi_1(z)$ ,  $x = \Psi_2(z)$  for the limits of  $x$ .

Lastly, the result of the previous summations is integrated for  $z$  between two parallel planes  $z = z_1$ ,  $z = z_2$ , drawn so as to include all that portion of the solid over which the limits for  $x$  and  $y$ , already determined, remain the same.

The remaining integrations will take place between  $z = z_1$  and similar successive parallel planes; and, according to the portion of the solid which any two of these planes intercept, the limits of  $x$  for that integral will be one or other of the bounding surfaces, curved or plane, the limits of  $y$ , one or other of the curved surfaces.

The general problem to change the variables in a multiple integral, and determine the limits from the given equations, may now be solved.

**2778** First, in the case of a double integral,

$$\int_{x_1}^{x_2} \int_{\mu_1(x)}^{\mu_2(x)} F(x, y) dx dy \dots\dots\dots (iii).,$$

to change from  $x, y$  to  $\xi, \eta$ , having given the equations  $u = 0, v = 0$ , involving the four variables.

To change  $y$  for  $\eta$ , eliminate  $x$  between these equations; thus  $y = f(x, \eta)$  and  $dy = f_\eta(x, \eta) d\eta$ . Substituting these values, we shall have

$$F(x, y) dy = F\{x, f(x, \eta)\} f_\eta(x, \eta) d\eta = F_1(x, \eta) d\eta.$$

Also, if  $\eta_1$  corresponds to  $y_1$ , the equations  $y_1 = \mu_1(x)$  and  $y_1 = f(x, \eta_1)$  will give  $\eta_1 = \psi_1(x)$ . Similarly  $\eta_2 = \psi_2(x)$ .

Hence the integral (iii.) may now be written

$$\int_{x_1}^{x_2} \int_{\psi_1(x)}^{\psi_2(x)} F_1(x, \eta) dx d\eta = \Sigma \int_{\eta_1}^{\eta_2} \int_{\Psi_1(\eta)}^{\Psi_2(\eta)} F_1(x, \eta) d\eta dx \dots\dots\dots (iv.),$$

the form on the right being obtained by changing the order of integration, as explained in (2775).

Next, to change  $x$  for  $\xi$ , eliminate  $y$  between the equations  $u = 0, v = 0$ ; thus,  $x = g(\xi, \eta)$  and  $dx = g_\xi(\xi, \eta) d\xi$ . Substituting as before, we shall have

$$F_1(x, \eta) dx = F_2(\xi, \eta) d\xi.$$

Also,  $\xi_1$  corresponding to  $x_1$ , the equations  $x_1 = \Psi_1(\eta)$  and  $x_1 = g(\xi_1, \eta)$  produce  $\xi_1 = m_1(\eta)$ , and in the same way  $\xi_2 = m_2(\eta)$ . Hence, finally,

$$\int_{x_1}^{x_2} \int_{\mu_1(x)}^{\mu_2(x)} F(x, y) dx dy = \Sigma \int_{\eta_1}^{\eta_2} \int_{m_1(\eta)}^{m_2(\eta)} F_2(\xi, \eta) d\eta d\xi \dots\dots\dots (v.)$$

In the last transformation from  $x$  to  $\xi$ , the most general form of the integrals which may be included under  $\Sigma$  has been chosen. When any of the limits of  $x$  are constants, the process is simplified.

**2779** Again, to change the variables from  $x, y, z$  to  $\xi, \eta, \zeta$ , in the triple integral,

$$\int_{x_1}^{x_2} \int_{\psi_1(x)}^{\psi_2(x)} \int_{\chi_1(x, y)}^{\chi_2(x, y)} F(x, y, z) dx dy dz \dots\dots\dots (vi.),$$

having given the equations  $u = 0, v = 0, w = 0$  between the six variables  $x, y, z, \xi, \eta, \zeta$ .



First, to change from  $z$  to  $\zeta$ , eliminate  $\xi$  and  $\eta$  between the three equations, and let the resulting equation be  $z = f(x, y, \zeta)$ . From this  $dz = f_\zeta(x, y, \zeta) d\zeta$ ; therefore

$$F(x, y, z) dz = F\{x, y, f(x, y, \zeta)\} f_\zeta(x, y, \zeta) d\zeta = F_1(x, y, \zeta) d\zeta.$$

Also, if  $\zeta_1$  corresponds to the limit  $z_1$ , the equations  $z_1 = \chi_1(x, y)$  and  $z_1 = f(x, y, \zeta_1)$  give  $\zeta_1 = \phi_1(x, y)$ . Similarly  $\zeta_2 = \phi_2(x, y)$ .

The integral (vi.) may therefore be written

$$\int_{x_1}^{x_2} \int_{\psi_1(x)}^{\psi_2(x)} \int_{\phi_1(x, y)}^{\phi_2(x, y)} F_1(x, y, \zeta) dx dy d\zeta = \Sigma \int_{\zeta_1}^{\zeta_2} \int_{\Psi_1(\zeta)}^{\Psi_2(\zeta)} \int_{\Phi_1(\zeta, x)}^{\Phi_2(\zeta, x)} F_1(x, y, \zeta) d\zeta dx dy \dots\dots\dots \text{(vii.)}$$

the last form being the result of changing the order of integration, as explained in (2777). We have now to change from  $y$  to  $\eta$ ; we therefore eliminate  $z$  and  $\xi$  from the equations  $u, v, w$ , obtaining an equation of the form  $y = f(\zeta, x, \eta)$ , and proceed exactly as before. The result, as respects the general form of integral in (vii.), will be

$$\int_{\zeta_1}^{\zeta_2} \int_{\Psi_1(\zeta)}^{\Psi_2(\zeta)} \int_{\lambda_1(\zeta, x)}^{\lambda_2(\zeta, x)} F_1(x, \eta, \zeta) d\zeta dx d\eta \dots\dots\dots \text{(viii.)}$$

The order of  $x$  and  $\eta$  has now to be changed by (2775). Since  $\zeta$  is a constant with respect to integrations for  $x$  and  $\eta$ ,  $\Psi_1(\zeta), \Psi_2(\zeta)$  will also be constants, while  $\lambda_1(\zeta, x), \lambda_2(\zeta, x)$  will be functions of the single variable  $x$ .

Suppose  $\eta = \lambda_1(\zeta, x)$  gives  $x = \Lambda_1(\zeta, \eta)$ . Similarly,  $x = \Lambda_2(\zeta, \eta)$  may be the other limit.

At the point where  $x = \Psi_1(\zeta)$  and  $\eta = \lambda_1(\zeta, x)$ , we shall obtain by eliminating  $x$ , say,  $\eta = \mu_1(\zeta)$ . Similarly, from  $x = \Psi_2(\zeta)$  and  $\eta = \lambda_2(\zeta, x)$  suppose, we get  $\eta = \mu_2(\zeta)$  for the next limit; then a general form for the transformed integral will be

$$\int_{\zeta_1}^{\zeta_2} \int_{\mu_1(\zeta)}^{\mu_2(\zeta)} \int_{\Lambda_1(\zeta, \eta)}^{\Lambda_2(\zeta, \eta)} F_1(x, \eta, \zeta) d\zeta d\eta dx \dots\dots\dots \text{(ix.)}$$

It now remains to change from the variable  $x$  to  $\xi$ . Eliminating  $y$  and  $z$  between the equations  $u, v, w$ , we have a result of the form  $x = f(\xi, \eta, \zeta)$ . Substituting for  $x$  and  $dx$  as before, we arrive finally at the form

$$\int_{\zeta_1}^{\zeta_2} \int_{\mu_1(\zeta)}^{\mu_2(\zeta)} \int_{\pi_1(\zeta, \eta)}^{\pi_2(\zeta, \eta)} F_1(\xi, \eta, \zeta) d\zeta d\eta d\xi \dots\dots\dots \text{(x.)}$$

It should be noticed that the limits  $x = \Lambda_1(\zeta, \eta), x = \Lambda_2(\zeta, \eta)$ , in (ix.), are not necessarily different curves. They may, in some of the partial integrals, be different portions of the same curve. This was exemplified in the last integral of (2776).

## MULTIPLE INTEGRALS.

The following theorems, (2825) to (2830), which are given for three variables only, hold good for any number.

Let  $x, y, z$  be quantities which can take any positive values

subject to the condition that their sum is not greater than unity; then

$$2825 \quad \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^{l-1} y^{m-1} z^{n-1} dx dy dz = \frac{\Gamma(l) \Gamma(m) \Gamma(n)}{\Gamma(l+m+n+1)}.$$

Here  $x+y+z=1$  is the limiting equation.

PROOF.—Integrate for  $z$ ; then for  $y$  by (2308); finally for  $x$  by (2280), and change to the gamma function by (2305).

$$2826 \quad \iiint \xi^{l-1} \eta^{m-1} \zeta^{n-1} d\xi d\eta d\zeta = \frac{\alpha^l \beta^m \gamma^n}{pqr} \frac{\Gamma\left(\frac{l}{p}\right) \Gamma\left(\frac{m}{q}\right) \Gamma\left(\frac{n}{r}\right)}{\Gamma\left(\frac{l}{p} + \frac{m}{q} + \frac{n}{r} + 1\right)}$$

when  $\left(\frac{\xi}{\alpha}\right)^p + \left(\frac{\eta}{\beta}\right)^q + \left(\frac{\zeta}{\gamma}\right)^r = 1$  is the limiting equation.

PROOF.—Substitute  $x = \left(\frac{\xi}{\alpha}\right)^p$ ,  $y = \left(\frac{\eta}{\beta}\right)^q$ ,  $z = \left(\frac{\zeta}{\gamma}\right)^r$ , and apply (2825).

2827 When the limiting equation is simply  $\xi + \eta + \zeta = h$ , the value of the last integral becomes

$$h^{l+m+n} \frac{\Gamma(l) \Gamma(m) \Gamma(n)}{\Gamma(l+m+n+1)}.$$

2828 The value of the same integral, taken between the limits  $h$  and  $h+dh$  of the sum of the variables, is

$$h^{l+m+n-1} \frac{\Gamma(l) \Gamma(m) \Gamma(n)}{\Gamma(l+m+n)} dh.$$

PROOF.—Let  $u$  be the value in (2827); then, by Taylor's theorem, the value required is

$$\frac{du}{dh} dh = (l+m+n) h^{l+m+n-1} \frac{\Gamma(l) \Gamma(m) \Gamma(n)}{\Gamma(l+m+n+1)} dh,$$

which reduces to the above, by (2288).

$$2829 \quad \iiint x^{l-1} y^{m-1} z^{n-1} f(x+y+z) dx dy dz \\ = \frac{\Gamma(l) \Gamma(m) \Gamma(n)}{\Gamma(l+m+n)} \int_0^c f(h) h^{l+m+n-1} dh,$$

3 H

if  $x+y+z=h$  and  $h$  varies from 0 to  $c$ . In other words, the variables must take all positive values allowed by the condition that their sum is not greater than  $c$ .

PROOF.—For each value of  $h$  the integration with respect to  $x, y, z$  gives, by (2828),

$$f(h) h^{l+m+n-1} \frac{\Gamma(l) \Gamma(m) \Gamma(n)}{\Gamma(l+m+n)} dh,$$

the variations of  $x, y, z$  not affecting  $h$ . This expression has then to be integrated as a function of  $h$  from 0 to  $c$ .

$$\begin{aligned} 2830 \quad & \iiint \xi^{l-1} \eta^{m-1} \zeta^{n-1} f \left\{ \left( \frac{\xi}{a} \right)^p + \left( \frac{\eta}{\beta} \right)^q + \left( \frac{\zeta}{\gamma} \right)^r \right\} d\xi d\eta d\zeta \\ &= \frac{a^l \beta^m \gamma^n}{pqr} \frac{\Gamma\left(\frac{l}{p}\right) \Gamma\left(\frac{m}{q}\right) \Gamma\left(\frac{n}{r}\right)}{\Gamma\left(\frac{l}{p} + \frac{m}{q} + \frac{n}{r}\right)} \int_0^c f(h) h^{\frac{l}{p} + \frac{m}{q} + \frac{n}{r} - 1} dh, \end{aligned}$$

with the limiting equation  $\left(\frac{\xi}{a}\right)^p + \left(\frac{\eta}{\beta}\right)^q + \left(\frac{\zeta}{\gamma}\right)^r = c$ .

PROOF.—From (2829) by substituting  $x = \left(\frac{\xi}{a}\right)^p$ , &c.

**2831** If  $x, y, z$  be  $n$  variables, taking all positive values subject to the restriction  $x^2 + y^2 + z^2 + \dots \leq 1$ ; then

$$\iiint \frac{dx dy dz \dots}{\sqrt{(1-x^2-y^2-z^2-\dots)}} = \frac{\pi^{\frac{1}{2}(n+1)}}{2^n \Gamma\left\{\frac{1}{2}(n+1)\right\}}.$$

But, if negative values of the variables are permitted, omit the factor  $2^n$  in the denominator.

PROOF.—In (2830) put  $l=m=\dots=1$ ;  $a=\beta=\dots=1$ ;  $p=q=\dots=2$ ;  $f(h) = \frac{1}{\sqrt{(1-h)}}$ ;  $c=1$ ; and the expression on the right becomes

$$\frac{\{\Gamma(\frac{1}{2})\}^n}{2^n \Gamma(\frac{1}{2}n)} \int_0^1 \frac{h^{\frac{1}{2}n-1}}{(1-h)^{\frac{1}{2}}} dh.$$

The integral is  $= B\left(\frac{1}{2}n, \frac{1}{2}\right)$  (2280)  $= \frac{\Gamma(\frac{1}{2}n) \Gamma(\frac{1}{2})}{\Gamma\left\{\frac{1}{2}(n+1)\right\}}$  (2305). Hence the result.

But if negative values of the variables are allowed under the same restriction,  $x^2 + y^2 + z^2 + \dots \leq 1$ , each element of the integral will occur  $2^n$  times for once under the first hypothesis. Therefore the former result must be multiplied by  $2^n$ .

**2832** If  $n$  positive variables,  $x, y, z, \&c.$ , are limited by the condition  $x^2 + y^2 + z^2 + \&c. \nless 1$ , then

$$\iiint \phi(ax + by + cz + \&c.) \, dx \, dy \, dz \dots = \frac{\pi^{\frac{1}{2}(n-1)}}{2^{n-1} \Gamma\{\frac{1}{2}(n+1)\}} \int_0^1 \phi(k\xi)(1-\xi^2)^{\frac{1}{2}(n-1)} d\xi,$$

where  $k^2 = a^2 + b^2 + c^2 + \&c.$

PROOF.—Change the variables to  $\xi, \eta, \zeta$  by the orthogonal transformation (1799), so that

$$a^2 + b^2 + c^2 + \&c. = k^2, \text{ and } ax + by + cz + \&c. = k\xi.$$

The integral then takes the form

$$\iiint \phi(k\xi) \, d\xi \, d\eta \, d\zeta \dots \text{ with } \xi^2 + \eta^2 + \zeta^2 + \&c. \nless 1.$$

Now, integrate for  $\eta, \zeta, \&c.$ , considering  $\xi$  constant, by adapting formula (2826). The limiting equation is

$$\left(\frac{\eta}{\sqrt{1-\xi^2}}\right)^2 + \left(\frac{\zeta}{\sqrt{1-\xi^2}}\right)^2 + \&c. \text{ to } n-1 \text{ terms,} = 1.$$

Therefore put  $l = m = \&c. = 1$ ;  $p = q = \&c. = 2$ ;  $a = \beta = \&c. = \sqrt{1-\xi^2}$ .

$$\text{The result is } \int_0^1 \phi(k\xi) \frac{(1-\xi^2)^{\frac{n-1}{2}}}{2^{n-1}} \frac{\{\Gamma(\frac{1}{2})\}^{n-1}}{\Gamma\{\frac{1}{2}(n-1)+1\}} d\xi,$$

which is equivalent to the value above.

	$x$	$y$	$z$	$\dots$
$\xi$	$\frac{a}{k}$	$\frac{b}{k}$	$\frac{c}{k}$	$\dots$
$\eta$	$\frac{a'}{k}$	$\frac{b'}{k}$	$\frac{c'}{k}$	$\dots$
$\zeta$	$\frac{a''}{k}$	$\frac{b''}{k}$	$\frac{c''}{k}$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

**2833** With the same limiting equation for  $n$  variables and the same value of  $k$ ,

$$\iiint \frac{\phi(ax + by + cz + \&c.)}{\sqrt{1-x^2-y^2-z^2-\&c.}} \, dx \, dy \, dz \dots = \frac{\pi^{\frac{1}{2}n}}{2^{n-1} \Gamma(\frac{1}{2}n)} \int_0^1 \phi(k\xi)(1-\xi^2)^{\frac{1}{2}n-1} d\xi.$$

PROOF.—Making the same orthogonal transformation as in (2832), the integral changes to

$$\iiint \dots \frac{\phi(k\xi) \, d\xi \, d\eta \, d\zeta \dots}{\sqrt{1-\xi^2-\eta^2-\&c.}},$$

Considering  $\xi$  constant, the integration for the remaining variables is effected by (2830). Adapting the integral to that formula, we have

$$\frac{\phi(k\xi) \, d\xi}{\sqrt{1-\xi^2}} \iint \dots \frac{d\eta \, d\zeta \dots}{\sqrt{\left\{1 - \frac{\eta^2}{1-\xi^2} - \frac{\zeta^2}{1-\xi^2} - \&c.\right\}}},$$

with  $\left(\frac{\eta}{\sqrt{1-\xi^2}}\right)^2 + \left(\frac{\zeta}{\sqrt{1-\xi^2}}\right)^2 + \&c., \text{ to } n-1 \text{ terms, } \nless 1$   
for the limiting equation.

Here  $l = m = \&c. = 1$ ;  $p = q = \&c. = 2$ ;  $\alpha = \beta = \&c. = \sqrt{1-\xi}$ ;  
 $f(h) = \frac{1}{\sqrt{1-h}}$ ;  $c = 1$ ; and the reductions are similar to those in (2832).

**2834** If in (2832-3) negative values of the variables are admitted (since the limiting equation is satisfied by such), each element of the integral with respect to  $\eta$ ,  $\zeta$ , &c. will then occur  $2^{n-1}$  times, and therefore the result in each case must be multiplied by  $2^{n-1}$ , and the limits of the integration for  $\xi$  will be  $-1$  and  $1$  instead of  $0$  and  $1$ .

## EXPANSIONS OF FUNCTIONS IN CONVERGING SERIES.

The expansion of a function by Maclaurin's theorem (1507) can be at once effected if the  $n^{\text{th}}$  derivative of the function is known, or if merely the value of the same, when the independent variable vanishes, is known. Some  $n^{\text{th}}$  derivatives of different functions, in addition to those given at (1461-71), are therefore here collected. When the general value would be too complicated, the value for the origin zero alone is given.

### DERIVATIVES OF THE $n^{\text{th}}$ ORDER.

The following is a general formula for calculating the  $n^{\text{th}}$  derivative of a function of a function.

If  $y$  be a function of  $z$ , and  $z$  a function of  $x$ ,

$$\mathbf{2852} \quad \frac{d^n y}{dx^n} = \sum \frac{z^r}{\boxed{r}} \frac{d^r y}{dz^r} \frac{d^n}{dx^n} \left( \frac{z}{a} - 1 \right)_{a=z}^r,$$

where  $r = 1, 2, 3, \dots n$  successively, and  $a$  is put  $= z$  in each term of the expanded binomial, *after* differentiation.

$$\text{PROOF.}—\text{Assume } y_{nz} = A_1 y_z + \frac{A_2}{\boxed{2}} y_{zz} + \frac{A_3}{\boxed{3}} y_{zzz} + \dots + \frac{A_n}{\boxed{n}} y_{nnz}.$$

To determine any coefficient  $A_r$ , form  $r$  equations from this by making  $y = z, z^2, z^3, \dots z^r$  in succession: multiply these  $r$  equations respectively by  $rz^{-1}, -C(r, 2)z^{-2}, C(r, 3)z^{-3}, \dots (-1)^{r-1}z^{-r}$ , and add the results. All the coefficients excepting  $A_r$  disappear. This is shown by differentiating the equation  $(1-x)^r = 1 - rz + C(r, 2)x^2 - C(r, 3)x^3 + \dots \pm x^r$

successively for  $x$ , and making  $x$  zero after each differentiation. Thus, finally,

$$\frac{A_r}{z^r} = (-1)^{r+1} \left\{ \frac{rz_{nx}}{z} - \frac{C(r, 2)(z^2)_{nx}}{z^2} + \frac{C(r, 3)(z^3)_{nx}}{z^3} - \dots (-1)^{r+1} \frac{(z^r)_{nx}}{z^r} \right\} \\ = d_{nx} \left( \frac{z}{a} - 1 \right),$$

with  $a$  put  $= z$ , after expanding and differentiating the binomial.

**2853** EXAMPLES.—The formula may be applied to verify equations (1416–19).

Jacobi's formula (1471) may also be obtained by it.

**2854**

$$d_{(n+1)x} \sin^{-1} x \\ = \frac{1.3 \dots (2n-1)}{2^n (1-x)^n (1-x^2)^{\frac{1}{2}}} \left\{ 1 - \frac{n}{2n-1} Z + \frac{1.3 C(n, 2)}{2n-1.2n-3} Z^2 \right. \\ \left. - \frac{1.3.5 C(n, 3)}{2n-1.2n-3.2n-5} Z^3 + \dots \pm Z^n \right\}, \text{ where } Z = \frac{1-x}{1+x}.$$

PROOF.  $(\sin^{-1} x)_{(n+1)x} = \{(1-x)^{-\frac{1}{2}} (1+x)^{-\frac{1}{2}}\}_{nx}$  (1434).

Expand the right member by (1460).

**2855**  $d_{nx} \tan^{-1} x$ . This derivative is obtained in (1468). The following is another method, which also includes the result in (1469).

$$d_x \tan^{-1} x = \frac{1}{1+x^2} = \frac{i}{2} \left\{ \frac{1}{x+i} - \frac{1}{x-i} \right\};$$

$$\therefore \text{ by (1425) } d_{nx} \tan^{-1} x = \frac{n-1}{2i} \frac{(-1)^n}{2i} \left\{ \frac{1}{(x+i)^n} - \frac{1}{(x-i)^n} \right\} \dots\dots\dots(1).$$

Put  $x = \cot \theta$ , therefore  $x \pm i = \sqrt{1+x^2} (\cos \theta \pm i \sin \theta)$ , which values substituted in (1) convert the equation, by (757), into

$$(\tan^{-1} x)_{nx} = (-1)^{n-1} \frac{n-1}{2} \sin^n \theta \sin n\theta.$$

**2856**  $d_{nx} \{e^{x \cos a} \cos(x \sin a)\} = e^{x \cos a} \cos(x \sin a + na)$ .

PROOF.—By Induction.

#### LAGRANGE'S METHOD.

**2857** Lemma.—The  $n^{\text{th}}$  derivative of a function  $u = f(x)$  will, by Taylor's theorem (1500), be equal to  $1.2 \dots n$  times the coefficient of  $h^n$  in the expansion of  $f(x+h)$  in powers of  $h$  by any known method.



Let  $u = a + bx + cx^2$ , and therefore  $u_x = b + 2cx$ ; then  $d_{rx}(a + bx + cx^2)^n$  is equal to either of the following series, with the notation of (2451-2).

$$2858 \quad 1^{(r)} u^{n-r} u_x^r \left\{ \frac{n_{-1}^{(r)}}{1^{(r)}} + \frac{n_{-1}^{(r-1)} cu}{1^{(r-2)} u_x^2} + \dots + \frac{n_{-1}^{(r-p)} c^p u^p}{1^{(p)} 1^{(r-2p)} u_x^{2p}} + \dots \right\};$$

2859 or, putting  $2n = m$  and  $4ac - b^2 = q^2$ ,

$$\frac{m_{-1}^{(r)}}{2^r} u^{n-r} \left\{ u_x^r + n \frac{r_{-1}^{(2)}}{m_{-1}^{(2)}} q^2 u_x^{r-2} + \dots + C(n, p) \frac{r_{-1}^{(2p)}}{m_{-1}^{(2p)}} q^{2p} u_x^{r-2p} + \dots \right\}.$$

PROOF.—Changing  $x$  into  $x + h$  in  $u^n$ , it becomes  $(u + u_x h + ch^2)^n$ . Then, by (2857),  $d_{rx} u^n$  will be =  $\lfloor r$  times the coefficient of  $h^r$  in the expansion of this trinomial. (2858) is the result, and it may be obtained by expanding  $\{(u + u_x h) + ch^2\}^n$  as a binomial, and collecting the coefficients of  $h^r$  from the subsequent expansions. The value (2859) is found by taking

$$(u + u_x h + ch^2)^n = u^n \left\{ \left(1 + \frac{u_x h}{2u}\right)^2 + \left(\frac{qh}{2u}\right)^2 \right\}^n,$$

expanding, collecting coefficients of  $h^r$ , and multiplying by  $1.2 \dots r$ , as before.

2860 Ex.—To find  $d_{nx}(a^2 + x^2)^n$ . Applying formula (2859), we have  $u = a^2 + x^2$ ,  $u_x = 2x$ ,  $q = 2a$ ,  $r = n$ . Therefore

$$d_{nx}(a^2 + x^2)^n = (2n)_{-1}^{(n)} \left\{ x^n + \frac{n^2(n-1)}{2n(2n-1)} a^2 x^{n-2} + \frac{n^2(n-1)^2(n-2)(n-3)}{1.2.2n \dots (2n-3)} a^4 x^{n-4} + \&c. \right\}.$$

$$2861 \quad d_{nx} e^{ax^2} = e^{ax^2} \left\{ a^n (2x)^n + \dots + \frac{n_{-1}^{(2r)}}{1^{(r)}} a^{n-r} (2x)^{n-2r} + \&c. \right\},$$

with  $r = 1, 2, 3, \&c.$  in succession.

PROOF.—By the method of (2857). Putting  $e^{a(x+h)^2} = e^{ax^2} e^{2axh} e^{ah^2}$ , expand the factors containing  $h$  by (150), and from the product of the two series collect the coefficients of  $h^n$ .

$$2862 \quad d_{nx} \frac{\sin}{\cos} x^2 = (2x)^n \frac{\sin}{\cos} \left( x^2 + \frac{n\pi}{2} \right) + \dots \\ \dots + \frac{n_{-1}^{(2r)}}{1^{(r)}} (2x)^{n-2r} \frac{\sin}{\cos} \left( x^2 + \frac{(n-r)\pi}{2} \right) + \&c.$$

PROOF.— $d_{nx}(\cos x^2 + i \sin x^2) = d_{nx}e^{ix^2}$ . Expand the right by (2861), putting  $i^{n-r} = e^{i(n-r)\frac{1}{2}\pi}$ , since, by (766),  $e^{i\frac{1}{2}\pi} = i \sin \frac{\pi}{2} = i$ . Also put

$$e^{ix^2 + i(n-r)\frac{1}{2}\pi} = \cos \left\{ x^2 + \frac{(n-r)\pi}{2} \right\} + i \sin \left\{ x^2 + \frac{(n-r)\pi}{2} \right\},$$

and then equate real and imaginary parts.

$$\begin{aligned} 2864 \quad d_{nx} \frac{1}{e^x + 1} &= (-1)^n \left[ e^{nx} + \{n+1-2^n\} e^{(n-1)x} \right. \\ &+ \{C(n+1, 2) - 2^n(n+1) + 3^n\} e^{(n-2)x} \\ &+ \{C(n+1, 3) - 2^n C(n+1, 2) + 3^n(n+1) - 4^n\} e^{(n-3)x} + \&c. \left. \right] \end{aligned}$$

PROOF.—Let  $u$  be the function. By differentiating  $u$  it is seen that

$$(e^x + 1)^{n+1} u_{nx} = A_n e^{nx} + A_{n-1} e^{(n-1)x} + A_{n-2} e^{(n-2)x} + \dots + A_1 e^x,$$

the  $A$ 's being constants. To determine their values, expand  $u = (e^x + 1)^{-1}$ , and also  $(e^x + 1)^{n+1}$ , by the Binomial theorem; thus

$$u_{nx} = (-1)^n \{e^{-x} - 2^n e^{-2x} + 3^n e^{-3x} - 4^n e^{-4x} + \&c.\},$$

$$(e^x + 1)^{n+1} = e^{(n+1)x} + (n+1) e^{nx} + C(n+1, 2) e^{(n-1)x} + C(n+1, 3) e^{(n-2)x} + \&c.$$

From the product of the two expansions the coefficients  $A_n, A_{n-1}, \&c.$  may be selected.

$$2865 \quad d_{nx0} \tan^{-1} x = (-1)^{\frac{n-1}{2}} \lfloor n-1 \rfloor \text{ or zero,}$$

according as  $n$  is *odd* or *even*.

PROOF.—By Rule IV. (1534). The first and last differential equations (see Example 1535) are, in this case,

$$(1+x^2) y_{2x} + 2xy_x = 0 \dots (i.); \quad y_{(n+2)x0} + n(n+1) y_{nx0} = 0 \dots (ii.);$$

with  $y_{x0} = 1$  and  $y_{2x0} = 0$ .

Otherwise.—By (1468), putting  $x = 0$ .

$$2867 \quad d_{nx0} \sin^{-1} x = 1.3^2.5^2 \dots (n-2)^2 \text{ or zero,}$$

according as  $n$  is *odd* or *even*.

PROOF.—By differentiating (1528).

Otherwise.—As in (2865) where equations (i.) and (ii.) will become in this case

$$(1-x^2) y_{2x} = xy_x \dots (i.) \quad y_{(n+2)x0} = n^2 y_{nx0} \dots (ii.)$$

$$2869 \quad d_{nx0} (\sin^{-1} x)^2 = 2.2^2.4^2.6^2 \dots (n-2)^2 \text{ or zero,}$$

according as  $n$  is *even* or *odd*.

PROOF.—As in (2865); equations (i.) and (ii.) being identical with those in (2867).

**2871** 
$$d_{nx0} \cos(m \sin^{-1} x)$$

$$= (-1)^{\frac{n}{2}} m^2 (m^2 - 2^2)(m^2 - 4^2) \dots [m^2 - (n-2)^2],$$
or zero ; according as  $n$  is *even* or *odd* and  $> 0$  if even.

**2873** 
$$d_{nx0} \sin(m \sin^{-1} x)$$

$$= (-1)^{\frac{n-1}{2}} m (m^2 - 1)(m^2 - 3^2) \dots [m^2 - (n-2)^2],$$
or zero ; according as  $n$  is *odd* or *even* and  $> 1$  if odd.

**2875** 
$$d_{nx0} \cos(m \cos^{-1} x)$$

$$= (-1)^{\frac{n-1}{2}} m (m^2 - 1)(m^2 - 3^2) \dots [m^2 - (n-2)^2] \sin \frac{m\pi}{2},$$

**2876** or 
$$= (-1)^{\frac{n}{2}} m^2 (m^2 - 2^2)(m^2 - 4^2) \dots [m^2 - (n-2)^2] \cos \frac{m\pi}{2},$$
according as  $n$  is *odd* and  $> 1$ , or *even* and  $> 0$ .

**2877** 
$$d_{nx0} \sin(m \cos^{-1} x)$$

$$= (-1)^{\frac{n}{2}} m^2 (m^2 - 2^2)(m^2 - 4^2) \dots [m^2 - (n-2)^2] \sin \frac{1}{2} m\pi,$$

**2879** or 
$$= (-1)^{\frac{n+1}{2}} m (m^2 - 1)(m^2 - 3^2) \dots [m^2 - (n-2)^2] \cos \frac{1}{2} m\pi,$$
according as  $n$  is *even* and  $> 0$ , or *odd* and  $> 1$ .

Observe that, in (2871-3),  $\sin^{-1} 0 = 0$ , and in (2875-9),  $\cos^{-1} 0 = \frac{\pi}{2}$ , are the only values admitted.

PROOF.—For (2871-9). As in (2865); equations (i.) and (ii.) now becoming in each case

$$(1-x^2) y_{2x} - xy_x + m^2 y = 0 \dots (i.) \quad y_{(n+2)x0} = (n^2 - m^2) y_{nx0} \dots (ii.)$$

Otherwise.—By the method of (1533).

**2880** Let  $y = x \cot x$ , then

$$n \cos \frac{n-1}{2} \pi = y_0 \sin \frac{n}{2} \pi + \dots + y_{rx0} C(n, r) \sin \frac{n-r}{2} \pi + \dots$$

$$\dots + y_{(n-1)x0} n \sin \frac{\pi}{2},$$

with integral values of  $r$ , from 0 to  $n-1$  inclusive.

**2881** Thus, denoting  $y_{nx0}$  shortly by  $y_n$ , we find, by making  $n = 1, 2, 3$ , &c. successively in the formula,

$$y_2 = -\frac{2}{3}, \quad y_4 = -\frac{8}{15}, \quad y_6 = -\frac{32}{21}, \quad y_8 = -\frac{128}{15}.$$

PROOF.—Take the  $n$ th derivative of the equation  $x \cos x = y \sin x$  by (1460), reducing the coefficients by (1461-2), and putting  $x$  finally  $= 0$ .

**2882** The derivatives of an odd order all vanish. This may be shown independently, as follows:—

Let  $y = \phi(x)$ , then  $\phi(x)$  is an even function of  $x$  (1401); therefore

$$\begin{aligned} \phi^{2n+1}(x) &= -\phi^{2n+1}(-x); \\ \therefore \phi^{2n+1}(0) &= -\phi^{2n+1}(0); \quad \therefore \phi^{2n+1}(0) = 0. \end{aligned}$$

**2883**  $d_{nx0} \left\{ (1+x^2)^{\frac{m}{2}} \sin(m \tan^{-1} x) \right\} = (-1)^{\frac{n-1}{2}} m_{-1}^{(n)}$  or zero, according as  $n$  is odd or even.

**2885**  $d_{nx0} \left\{ (1+x^2)^{\frac{m}{2}} \cos(m \tan^{-1} x) \right\} = (-1)^{\frac{n-2}{2}} m_{-1}^{(n)}$  or zero, according as  $n$  is even or odd.

PROOF.—As in (2865). Equations (i.) and (ii.), both for (2883) and (2885), are now  $(1+x^2) y_{2x} - 2(m-1)xy_x + m(m-1)y = 0$ .....(i.), and

$$y_{(n+2)x0} = -(m-n)(m-n-1)y_{nx0} \text{ ..... (ii.)}$$

Formula (ii.) gives the factors in succession, starting with  $y_0 = 0$ ,  $y_x = m$  in (2883); and with  $y_0 = 1$ ,  $y_x = 0$  in (2885).

**2887**  $d_{nx0} \left\{ (1+x^2)^{-\frac{m}{2}} \cos(m \tan^{-1} x) \right\} = (-1)^{\frac{n}{2}} m^{(n)}$  or zero, according as  $n$  is even or odd.

PROOF.—Change the sign of  $m$  in (2885).

NOTE.—In formulæ (2883-7) zero is the only admitted value of  $\tan^{-1} 0$ .

**2889**  $d_{nx0} \left( \frac{x}{e^x - 1} \right) = (-1)^{\frac{n}{2}-1} B_n$  or zero,

according as  $n$  is even or odd; by (1539).

**2891** When  $p$  is a positive integer,

$$d_{nx0} (x^p e^{ax} \cos bx) = n_{-1}^{(p)} (a^2 + b^2)^{\frac{n-p}{2}} \cos \left\{ (n-p) \tan^{-1} \frac{b}{a} \right\}$$

or zero, according as  $n$  is  $>$  or  $<$   $p$ .

PROOF.—Put  $y = e^{ax} \cos bx$  and  $z = x^p$  in (1460), employing (1465).

## MISCELLANEOUS EXPANSIONS.

The following series are placed here for the sake of reference, many of them being of use in evaluating definite integrals by Rule V. (2249). Other series and methods of expansion will be found in Articles (125–129), (149–159), (248–295), (756–817), (1460), (1471–1472), (1500–1573). For tests of convergency, see (239–247).

Numerous expansions may be obtained by differentiating or integrating known series or their logarithms. These and other methods are exemplified below.

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$$\begin{aligned} 2911 \quad \cot x = \frac{1}{x} - \frac{1}{\pi-x} + \frac{1}{\pi+x} - \frac{1}{2\pi-x} \\ + \frac{1}{2\pi+x} - \frac{1}{3\pi-x} + \frac{1}{3\pi+x} - \&c. \end{aligned}$$

PROOF.—By differentiating the logarithm of equation (815).

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$$\begin{aligned} 2912 \quad \pi \cot \pi x = \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1} + \frac{1}{x-2} \\ + \frac{1}{x+2} + \frac{1}{x-3} + \frac{1}{x+3} + \&c. \end{aligned}$$

PROOF.—By changing  $x$  into  $\pi x$  in (2911).

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$$\begin{aligned} 2913 \quad \tan x = \frac{1}{\frac{1}{2}\pi-x} - \frac{1}{\frac{1}{2}\pi+x} + \frac{1}{\frac{3}{2}\pi-x} \\ - \frac{1}{\frac{3}{2}\pi+x} + \frac{1}{\frac{5}{2}\pi-x} - \frac{1}{\frac{5}{2}\pi+x} + \&c. \end{aligned}$$

PROOF.—By changing  $x$  into  $\frac{1}{2}\pi-x$  in (2911).

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$$\begin{aligned} 2914 \quad \operatorname{cosec} x = \frac{1}{x} + \frac{1}{\pi-x} - \frac{1}{\pi+x} - \frac{1}{2\pi-x} \\ + \frac{1}{2\pi+x} + \frac{1}{3\pi-x} - \frac{1}{3\pi+x} - \frac{1}{4\pi-x} + \&c. \end{aligned}$$

PROOF.—By adding together equations (2911, 2913), and changing  $x$  into  $\frac{1}{2}x$ .

$$2915 \quad \frac{\pi}{\sin m\pi} = \frac{1}{m} + \frac{1}{1-m} - \frac{1}{1+m} \\ - \frac{1}{2-m} + \frac{1}{2+m} + \frac{1}{3-m} - \frac{1}{3+m} - \&c.$$

PROOF.—By putting  $x = m\pi$  in (2914).

$$2916 \quad \cot x = \frac{1}{x} - \frac{2^2 B_2 x}{2} - \frac{2^4 B_4 x^3}{4} - \frac{2^6 B_6 x^5}{6} - \&c.$$

For proof see (1545). The reference in that article (first edition) should be to (1541) not (1540).

$$2917 \quad \tan x = \frac{2^2(2^2-1)}{2} B_2 x + \frac{2^4(2^4-1)}{4} B_4 x^3 + \frac{2^6(2^6-1)}{6} B_6 x^5 + \&c.$$

$$2918 \quad \operatorname{cosec} x = \frac{1}{x} + \frac{2(2-1)}{2} B_2 x \\ + \frac{2(2^3-1)}{4} B_4 x^3 + \frac{2(2^5-1)}{6} B_6 x^5 + \&c.$$

PROOF.—By (2916) and the relations

$$\tan x = \cot x - 2 \cot 2x, \quad \operatorname{cosec} x = \cot \frac{1}{2}x - \cot x.$$

$$2919 \quad \frac{1-a^2}{1-2a \cos x + a^2} = 1 + 2a \cos x + 2a^2 \cos 2x + 2a^3 \cos 3x + \&c.$$

$$2920 \quad \frac{\cos x}{1-2a \cos x + a^2} \\ = \frac{a}{1-a^2} + \frac{1+a^2}{1-a^2} (\cos x + a \cos 2x + a^2 \cos 3x + a^3 \cos 4x + \&c...)$$

$$2921 \quad \frac{\sin x}{1-2a \cos x + a^2} = \sin x + a \sin 2x + a^2 \sin 3x + a^3 \sin 4x + \&c.$$

PROOF.—By (784-6) making  $a = \beta = x$  and  $c = a$ .

When  $a$  is less than unity and either positive or negative,



**2922**

$$\frac{1}{2} \log(1 + 2a \cos x + a^2) = a \cos x - \frac{a^2}{2} \cos 2x + \frac{a^3}{3} \cos 3x - \&c.$$

$$\mathbf{2923} \quad \tan^{-1} \frac{a \sin x}{1 + a \cos x} = a \sin x - \frac{a^2}{2} \sin 2x + \frac{a^3}{3} \sin 3x - \&c.$$

PROOF.—Putting  $z = a(\cos x + i \sin x)$ , we have

$$\begin{aligned} \log(1+z) &= \log(1+a \cos x + ia \sin x) \\ &= \frac{1}{2} \log(1+2a \cos x + a^2) + i \tan^{-1} \frac{a \sin x}{1+a \cos x} \quad (2214), \end{aligned}$$

and also 
$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \&c.$$

Substitute the value of  $z$  and equate real and imaginary parts.

**2924** *Otherwise.*—To obtain (2922),

$$\log(1+2a \cos x + a^2) = \log(1+ae^{ix}) + \log(1+ae^{-ix}).$$

Expanding by (154), the series is at once obtained by (768).

**2925** *Otherwise.*—Integrate the equation in (786) with respect to  $a$ , after changing  $a$  and  $\beta$  into  $x$ , and  $c$  into  $-a$ .

**2926** When  $a$  is greater than unity, put

$$\log(1+2a \cos x + a^2) = \log a^2 + \log(1+2a^{-1} \cos x + a^{-2}),$$

and the last term can be expanded in a converging series by (2922).

**2927**

$$\log 2 \cos \frac{1}{2}x = \cos x - \frac{1}{2} \cos 2x + \frac{1}{3} \cos 3x - \frac{1}{4} \cos 4x + \&c.$$

**2928**

$$\log 2 \sin \frac{1}{2}x = -\cos x - \frac{1}{2} \cos 2x - \frac{1}{3} \cos 3x - \frac{1}{4} \cos 4x - \&c.$$

$$\mathbf{2929} \quad \frac{1}{2}x = \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \&c.$$

$$\mathbf{2930} \quad \frac{1}{2}(\pi - x) = \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \frac{1}{4} \sin 4x + \&c.$$

PROOF.—(2927–30) Make  $a = \pm 1$  in (2922–3).

$$\mathbf{2931} \quad \frac{1}{4}\pi = \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \&c.$$

$$\mathbf{2932} \quad \pi = 2\sqrt{2} \left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \&c.\right).$$

PROOF.—Add together (2929–30), and put  $x = \frac{1}{4}\pi$ .

When  $n$  is less than unity, and  $a = 1 \pm \sqrt{1+n^2}$ ,

$$\mathbf{2933} \quad \log(1+n \cos x) = \log(1+2a \cos x + a^2) - \log(1+a^2),$$

and is therefore equal to twice the series in (2922), minus  $\log(1+a^2)$ . But if  $a$  be greater than unity, expand, as in (2926), by

$$\begin{aligned} 2934 \quad & \log(1+n \cos x) \\ &= \log(1+2a^{-1} \cos x + a^{-2}) + \log a^2 - \log(1+a^2). \end{aligned}$$

2935

$$(1+2a \cos x)^n = A + A_1 \cos x + A_2 \cos 2x + A_3 \cos 3x + \&c.,$$

where

$$\begin{aligned} A &= 1 + C(n, 2) 2a^2 + \dots + C(n, 2p) C(2p, p) a^{2p} + \dots, \\ A_1 &= 2a \{n + C(n, 3) 3a^2 + \dots + C(n, 2p+1) C(2p+1, p) a^{2p} + \dots\}, \end{aligned}$$

and

$$A_{r+1} = \frac{A_{r-1}(n-r+1)a - rA_r}{(n+r+1)a}.$$

If  $n$  be a positive integer, the series terminates with the  $n+1^{\text{th}}$  term, and the values of  $A$  and  $A_1$  are also finite.

PROOF.—Differentiate the logarithm of the first equation; multiply up and equate coefficients of  $\sin rz$  after transforming by (666); thus  $A_{r+1}$  is obtained.

To find  $A$  and  $A_1$ , expand  $(1+2a \cos x)^n$  by the Binomial Theorem, and the powers of  $\cos x$  afterwards by (772).

#### LEGENDRE'S FUNCTION $X_n$ .

$$2936 \quad (1-2ax+a^2)^{-\frac{1}{2}} = 1 + X_1 a + X_2 a^2 + \dots + X_n a^n +$$

with

$$X_n = \frac{1}{2^n \lfloor n \rfloor} \frac{d^n}{dx^n} (x^2-1)^n.$$

PROOF.—Expand by the Binomial Theorem, and in the numerical part of each coefficient of  $a^n$  express  $1.3.5 \dots 2n-1$  as  $\lfloor 2n \rfloor \div 2^n \lfloor n \rfloor$ .

Consecutive functions are connected by the relation

$$2937 \quad d_x X_{n+1} = (2n+1) X_n + d_x X_{n-1}.$$

PROOF.—Differentiate the factor once under the sign of differentiation in the values of  $X_{n+1}$  and  $X_{n-1}$  given by the formula for  $X_n$  in (2936).

A differential equation for  $X_n$  is

$$2938 \quad (1-x^2) d_{xx} X_n - 2x d_x X_n + n(n+1) X_n = 0.$$

**2939** When  $p$  is any positive integer,

$$1^p + 2^p + 3^p + \dots + (n-1)^p \\ = [p] \left\{ \frac{n^{p+1}}{[p+1]} - \frac{n^p}{2[p]} + \frac{B_2 n^{p-1}}{2[p-1]} - \frac{B_4 n^{p-3}}{4[p-3]} + \frac{B_6 n^{p-5}}{6[p-5]} - \&c. \right.$$

concluding, according as  $n$  is even or odd, with

$$(-1)^{\frac{1}{2}p+1} \frac{B_p n}{[p]} \quad \text{or} \quad (-1)^{\frac{1}{2}(p+1)} \frac{B_{p-1} n^2}{[p-1][2]} \Big\}.$$

**PROOF.**  $\frac{e^{nx}-1}{e^x-1} = \frac{e^{nx}-1}{x} \cdot \frac{x}{e^x-1}$ . Expand the left side by division, and each term subsequently by (150). Again, expand the first factor of the right side by (150), and the second by (1539), and equate the coefficients of  $x^p$  in the two results.

See (276) for the values of the series when  $p$  is 1, 2, 3, or 4. But the general formula there is incorrectly printed.

Let the series (2940–4) be denoted by  $S_{2n}$ ,  $S'_{2n}$ ,  $s_{2n}$ ,  $s'_{2n+1}$ , as under,  $n$  being any positive integer; then

$$\mathbf{2940} \quad S_{2n} \equiv 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \&c. \dots = \frac{2^{2n-1}}{[2n]} \pi^{2n} B_{2n}.$$

$$\mathbf{2941} \quad S'_{2n} \equiv 1 - \frac{1}{2^{2n}} + \frac{1}{3^{2n}} - \frac{1}{4^{2n}} + \&c. \dots = \frac{2^{2n-1}-1}{[2n]} \pi^{2n} B_{2n}.$$

$$\mathbf{2942} \quad s_{2n} \equiv 1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \&c. \dots = \frac{2^{2n}-1}{2[2n]} \pi^{2n} B_{2n}.$$

**PROOF.**—(i.)  $S_{2n}$  is obtained in (1545).

$$(ii.) \quad S_{2n} - S'_{2n} = 2 \left( \frac{1}{2^{2n}} + \frac{1}{4^{2n}} + \&c. \right) = 2 \frac{S_{2n}}{2^{2n}}. \quad \text{This give } S_{2n}.$$

$$(iii.) \quad s_{2n} = \frac{1}{2} (S_{2n} + S'_{2n}).$$

$$\mathbf{2943} \quad s_{2n} \equiv 1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \&c. = - \frac{\pi d_{(2n-1)x} \cot \pi x}{4^{2n} [2n-1]}, \\ x = \frac{1}{4}.$$

$$\mathbf{2944} \quad s'_{2n+1} \equiv 1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \&c. = \frac{\pi d_{2nx} \cot \pi x}{4^{2n+1} [2n]}, \\ x = \frac{1}{4}.$$

**PROOF.**—By differentiating equation (2912) successively, and putting  $x = \frac{1}{4}$  in the result. To compute  $d_{nx} \cot \pi x$ , see (1525).

**2945** The following values have been calculated by formulæ (2940-4).

$$\begin{aligned} S_2 &= \frac{\pi^2}{6}, & S_4 &= \frac{\pi^4}{90}, & S_6 &= \frac{\pi^6}{945}, & S_8 &= \frac{\pi^8}{9450}; \\ S_2' &= \frac{\pi^2}{12}, & S_4' &= \frac{7\pi^4}{720}, & S_6' &= \frac{31\pi^6}{30240}, & S_8' &= \frac{127\pi^8}{1209600}; \\ s_2 &= \frac{\pi^2}{8}, & s_4 &= \frac{\pi^4}{96}, & s_6 &= \frac{\pi^6}{960}, & s_8 &= \frac{17\pi^8}{161280}; \\ s_1' &= \frac{\pi}{4}, & s_3' &= \frac{\pi^3}{32}, & s_5 &= \frac{5\pi^5}{1536}, & s_7' &= \frac{61\pi^7}{184320}. \end{aligned}$$

**2946** 
$$\frac{\cos x - \cos a}{1 - \cos a} = \left(1 - \frac{x^2}{a^2}\right) \left[1 - \frac{x^2}{(2\pi - a)^2}\right] \left[1 - \frac{x^2}{(2\pi + a)^2}\right] \\ \times \left[1 - \frac{x^2}{(4\pi - a)^2}\right] \left[1 - \frac{x^2}{(4\pi + a)^2}\right] \dots \&c.,$$

**2947** 
$$\frac{\cos x + \cos a}{1 + \cos a} = \left[1 - \frac{x^2}{(\pi - a)^2}\right] \left[1 - \frac{x^2}{(\pi + a)^2}\right] \left[1 - \frac{x^2}{(3\pi - a)^2}\right] \\ \times \left[1 - \frac{x^2}{(3\pi + a)^2}\right] \left[1 - \frac{x^2}{(5\pi - a)^2}\right] \dots \&c.$$

PROOF.  $\frac{\cos x - \cos a}{1 - \cos a} = \frac{\sin \frac{1}{2}(a+x) \sin \frac{1}{2}(a-x)}{\sin^2 \frac{1}{2}a}$ . Expand the sines by (815). The two  $n+1^{\text{th}}$  factors of the numerator divided by the corresponding ones of the denominator reduce to

$$\begin{aligned} &\left(1 - \frac{2ax + x^2}{4n^2\pi^2 - a^2}\right) \left(1 + \frac{2ax - x^2}{4n^2\pi^2 - a^2}\right) \\ &= \left(1 - \frac{x}{2n\pi - a}\right) \left(1 + \frac{x}{2n\pi + a}\right) \left(1 + \frac{x}{2n\pi - a}\right) \left(1 - \frac{x}{2n\pi + a}\right) \\ &= \left(1 - \frac{x^2}{(2n\pi - a)^2}\right) \left(1 - \frac{x^2}{(2n\pi + a)^2}\right). \end{aligned}$$

Similarly with (2947) employing (816).

**2948** 
$$\cos x + \tan \frac{a}{2} \sin x = \left(1 + \frac{2x}{\pi - a}\right) \left(1 - \frac{2x}{\pi + a}\right) \left(1 + \frac{2x}{3\pi - a}\right) \\ \times \left(1 - \frac{2x}{3\pi + a}\right) \left(1 + \frac{2x}{5\pi - a}\right) \left(1 - \frac{2x}{5\pi + a}\right) \dots \&c.,$$

**2949** 
$$\cos x - \cot \frac{a}{2} \sin x = \left(1 - \frac{2x}{a}\right) \left(1 + \frac{2x}{2\pi - a}\right) \left(1 - \frac{2x}{2\pi + a}\right) \\ \times \left(1 + \frac{2x}{4\pi - a}\right) \left(1 - \frac{2x}{4\pi + a}\right) \left(1 + \frac{2x}{6\pi - a}\right) \dots \&c.$$

PROOF.  $\cos x + \tan \frac{a}{2} \sin x = \frac{\cos(\frac{1}{2}a - x)}{\cos \frac{1}{2}a}$ . Expand the cosines by (816), and reduce. Similarly with (2949), employing 815.

$$2950 \quad \frac{e^x - e^{-x}}{2} = x \left[ 1 + \left( \frac{x}{\pi} \right)^2 \right] \left[ 1 + \left( \frac{x}{2\pi} \right)^2 \right] \left[ 1 + \left( \frac{x}{3\pi} \right)^2 \right] \dots$$

$$2951 \quad \frac{e^x + e^{-x}}{2} = \left[ 1 + \left( \frac{2x}{\pi} \right)^2 \right] \left[ 1 + \left( \frac{2x}{3\pi} \right)^2 \right] \left[ 1 + \left( \frac{2x}{5\pi} \right)^2 \right] \dots$$

PROOF.—Change  $\theta$  into  $ix$  in (815) and (816).

$$2952 \quad \frac{e^x - 2 \cos a + e^{-x}}{2(1 - \cos a)} \\ = \left[ 1 + \left( \frac{x}{a} \right)^2 \right] \left[ 1 + \left( \frac{x}{2\pi - a} \right)^2 \right] \left[ 1 + \left( \frac{x}{2\pi + a} \right)^2 \right] \left[ 1 + \left( \frac{x}{4\pi - a} \right)^2 \right] \dots \&c.$$

$$2953 \quad \frac{e^x + 2 \cos a + e^{-x}}{2(1 + \cos a)} \\ = \left[ 1 + \left( \frac{x}{\pi - a} \right)^2 \right] \left[ 1 + \left( \frac{x}{\pi + a} \right)^2 \right] \left[ 1 + \left( \frac{x}{3\pi - a} \right)^2 \right] \left[ 1 + \left( \frac{x}{3\pi + a} \right)^2 \right] \dots \&c.$$

PROOF.—Change  $x$  into  $ix$  in (2946-7).

## FORMULÆ FOR THE EXPANSION OF FUNCTIONS IN TRIGONOMETRICAL SERIES.

2955 When  $x$  has any value between  $l$  and  $-l$ ,

$$\phi(x) = \frac{1}{2l} \int_{-l}^l \phi(v) dv + \frac{1}{l} \sum_{n=1}^{\infty} \int_{-l}^l \phi(v) \cos \frac{n\pi(v-x)}{l} dv \dots (i.),$$

where  $n$  must have all positive integral values in succession from 1 upwards

But, if  $x = l$  or  $-l$ , the left side becomes  $l\phi(l) + l\phi(-l)$ .

PROOF.—By (2919) we have, when  $h$  is  $< 1$ ,

$$\frac{1-h^2}{1-2h \cos \theta + h^2} = 1 + 2h \cos \theta + 2h^2 \cos 2\theta + 2h^3 \cos 3\theta + \&c....$$

Put  $\theta = \frac{\pi(v-x)}{l}$ ; multiply each side by  $\phi(v)$ , and integrate for  $v$  from  $-l$  to  $l$ ; then make  $h = 1$ . The left side becomes, by substituting  $z = v - x$ ,

$$\int_{-l}^l \frac{(1-h^2) \phi(v) dv}{1-2h \cos \frac{\pi(v-x)}{l} + h^2} = \int_{-l-x}^{l-x} \frac{(1-h^2) \phi(x+z) dz}{(1-h)^2 + 4h \sin^2 \frac{\pi z}{2l}}.$$

When  $h = 1$  each element of the integral *vanishes*, excepting for values of  $v$  which lie near to  $x$ . Therefore the only appreciable value of the integral arises from such elements, and in these  $z$  will have values near to zero, both positive and negative, since  $x$  has a fixed value between  $l$  and  $-l$ . Let these values of  $z$  range from  $-\beta$  to  $\alpha$ . Then between these small limits we shall

have  $\frac{\sin^2 \pi z}{2l} = \frac{\pi^2 z^2}{4l^2}$ , and  $\phi(x+z) = \phi(x)$ ,

and the integral takes the form

$$(1-h^2)\phi(x)\int_{-\beta}^{\alpha}\frac{dz}{(1-h)^2+\frac{h\pi^2z^2}{l^2}}$$

$$=\frac{(1+h)l}{\pi\sqrt{h}}\phi(x)\left(\tan^{-1}\frac{\alpha\pi\sqrt{h}}{l(1-h)}+\tan^{-1}\frac{\beta\pi\sqrt{h}}{l(1-h)}\right)=2l\phi(x),$$

when  $h$  is made equal to unity, which establishes the formula.

In the case, however, in which  $x=l$ ,  $\sin^2\frac{\pi z}{2l}$  vanishes at both limits, that is, when  $z=0$  and when  $z=-2l$ . We have therefore to integrate for  $z$  from  $-\beta$  to  $0$ , and also from  $-2l$  to  $-2l+\alpha$ ,  $\alpha$  and  $\beta$  being any small quantities. The first integration gives  $l\phi(l)$  as above, putting  $\alpha=0$ . The second integration, by substituting  $y=z+2l$ , produces a similar form with limits  $0$  to  $\alpha$ , and with  $\phi(x-2l)$  in the place of  $\phi(x)$  giving  $l\phi(-l)$  when  $x=l$ . Thus the total value of the integral is  $l\phi(l)+l\phi(-l)$ . The result is the same when  $x=-l$ .

That the right side of equation (i.) forms a converging series appears by integrating the general terms by Parts; thus

$$\int_{-l}^l \phi(v) \cos \frac{n\pi(v-x)}{l} dv = \frac{l}{n\pi} \left\{ \phi(v) \sin \frac{n\pi(v-x)}{l} \right\}_{-l}^l$$

$$- \frac{l}{n\pi} \int_{-l}^l \phi'(v) \sin \frac{n\pi(v-x)}{l} dv,$$

which vanishes when  $n$  is infinite, provided  $\phi'(v)$  is not infinite.

Hence the multiplication of such terms by  $h^n$  when  $n$  is infinite produces no finite result when  $h$  is made  $=1$ , although  $1^n$  is a factor of indeterminate value.

**2955a** A function of the form  $\phi(x)\cos nx$ , with  $n$  infinitely great, has been called "*a fluctuating function*," for the reason that between any two finite limits of the variable  $x$ , the function changes sign infinitely often, oscillating between the values  $\phi(x)$  and  $-\phi(x)$ . The preceding demonstration shows that the sum of all these values, as  $x$  varies continuously between the assigned limits, is zero.

By similar reasoning, the two following equations are obtained.

**2956** If  $x$  has any value between  $0$  and  $l$ ,

$$\phi(x) = \frac{1}{2l} \int_0^l \phi(v) dv + \frac{1}{l} \sum_1^\infty \int_0^l \phi(v) \cos \frac{n\pi(v-x)}{l} dv \dots (2).$$

But if  $x=0$ , write  $\frac{1}{2}\phi(0)$  on the left; and if  $x=l$ , write  $\frac{1}{2}\phi(l)$ .

If  $x$  has any value between  $0$  and  $l$ ,

$$\mathbf{2957} \quad 0 = \frac{1}{2l} \int_0^l \phi(v) dv + \frac{1}{l} \sum_1^\infty \int_0^l \phi(v) \cos \frac{n\pi(v+x)}{l} dv \dots (3).$$

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But if  $x=0$ , write  $\frac{1}{2}\phi(0)$  on the left; and if  $x=l$ , write  $\frac{1}{2}\phi(l)$ .

$$2958 \quad \phi(x) = \frac{1}{l} \int_0^l \phi(v) dv + \frac{2}{l} \sum_1^{\infty} \cos \frac{n\pi x}{l} \int_0^l \cos \frac{n\pi v}{l} \phi(v) dv \dots\dots (4).$$

This formula is true for any value of  $x$  between 0 and  $l$ , both inclusive.

But if  $x$  be  $>l$ , write  $\phi(x \sim 2ml)$  instead of  $\phi(x)$  on the left, where  $2ml$  is that even multiple of  $l$  which is nearest to  $x$  in value.

If the sign of  $x$  be changed on the right, the left side of the equation remains unaltered in every case.

$$2959 \quad \phi(x) = \frac{2}{l} \sum_1^{\infty} \sin \frac{n\pi x}{l} \int_0^l \sin \frac{n\pi v}{l} \phi(v) dv \dots\dots (5).$$

This formula holds for any value of  $x$  between 0 and  $l$  exclusive of those values.

If  $x$  be  $>l$ , write  $\pm \phi(x \sim 2ml)$  instead of  $\phi(x)$  on the left, + or - according as  $x$  is  $>$  or  $<$   $2ml$ , the even multiple of  $l$  which is nearest to  $x$  in value.

But if  $x$  be 0 or  $l$ , or any multiple of  $l$ , the left side of this equation vanishes.

If the sign of  $x$  be changed on the right, the left side is numerically the same in every case, but of opposite sign.

PROOF.—For (2958-9). To obtain (4) take the sum, and to obtain (5) take the difference, of equations (2) and (3). To determine the values of the series when  $x$  is  $>l$ , put  $x = 2ml \pm x'$ , so that  $x'$  is  $<l$ .

#### EXAMPLES.

For all values of  $x$ , from 0 to  $\pi$  inclusive,

$$2960 \quad x = \frac{\pi}{2} - \frac{4}{\pi} \left\{ \cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \&c. \right\}.$$

PROOF.—In formula (4) put  $\phi(x) = x$  and  $l = \pi$ , then

$$\int_0^{\pi} v \cos nv dv = \left[ \frac{v \sin nv}{n} + \frac{\cos nv}{n^2} \right]_0^{\pi} = -\frac{2}{n^2} \text{ or } 0,$$

according as  $n$  is odd or even.

Similarly, by formula (5), equation (2929) is reproduced.

For all values of  $x$ , from  $-\frac{1}{2}\pi$  to  $\frac{1}{2}\pi$  inclusive.

$$2961 \quad x = \frac{4}{\pi} \left\{ \sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \&c. \right\}.$$

PROOF.—Change  $x$  into  $\frac{1}{2}\pi - x$  in (2961).

$$2962 \quad \frac{\pi}{2} \frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} = \frac{\sin x}{a^2 + 1} - \frac{2 \sin 2x}{a^2 + 2^2} + \frac{3 \sin 3x}{a^2 + 3^2} - \&c.$$

PROOF.—In formula (5) put  $\phi(x) = e^{ax} - e^{-ax}$  and  $l = \pi$ ; then

$$\int_0^\pi (e^{av} - e^{-av}) \sin nv dv = (-1)^{n+1} \frac{n(e^{a\pi} - e^{-a\pi})}{a^2 + n^2}.$$

2963 If  $\phi(x)$  be not a continuous function between  $x=0$  and  $x=l$ , let the function be  $\phi(x)$  from  $x=0$  to  $x=a$ , and  $\psi(x)$  from  $x=a$  to  $x=l$ ; then, in formulæ (4) and (5), we shall have  $\phi(x)$  or  $\psi(x)$  respectively on the left side, according to the situation of  $x$  between 0 and  $a$ , or between  $a$  and  $l$ . But, if  $x=a$ , we must write  $\frac{1}{2}\{\phi(a) + \psi(a)\}$  for the left member.

PROOF.—In ascertaining the value of the integral in the demonstration of (2955), we are only concerned with the form of the function *close to* the value of  $x$  in question. Hence the result is not affected by the discontinuity unless  $x=a$ . In this case the integration for  $z$  is from  $-\beta$  to 0 with  $\phi(x)$  for the function, and from 0 to  $a$  with  $\psi(x)$  for the function, producing  $\frac{1}{2}\phi(a) + \frac{1}{2}\psi(a)$ .

2964 Hence an expression involving  $x$  in an infinite series of sines of consecutive multiples of  $\frac{\pi x}{l}$  may be found, such that, when  $x$  lies between any of the assigned limits (0 and  $a$ ,  $a$  and  $b$ ,  $b$  and  $c$ , ...  $k$  and  $l$ ), the series shall be equal respectively to the corresponding assigned functions

$$f_1(x), \quad f_2(x), \quad f_3(x) \dots f_n(x),$$

provided that the integrals

$$\int_0^a \sin\left(\frac{n\pi x}{l}\right) f_1(x) dx, \quad \int_a^b \sin\frac{n\pi x}{l} f_2(x) dx, \dots \int_k^l \sin\left(\frac{n\pi x}{l}\right) f_n(x) dx$$

can all be determined.

2965 The same is true, reading *cosine* for *sine* throughout, with the additional proviso [as appears from formula (4)] that the integrals

$$\int_0^a f_1(x) dx, \quad \int_a^b f_2(x) dx, \dots \int_k^l f_n(x) dx$$

can also be determined.

2966 Ex. 1.—To find in the form of a series of cosines of multiples of  $x$  a function of  $x$  which shall be equal to the constants  $\alpha$ ,  $\beta$ , or  $\gamma$ , according as  $x$  lies between 0 and  $a$ ,  $a$  and  $b$ , or  $b$  and  $\pi$ .

Formula (4) produces, putting  $l = \pi$ ,

$$\begin{aligned} & \frac{1}{\pi} \left\{ \int_0^a a \, dx + \int_a^b \beta \, dx + \int_b^\pi \gamma \, dx \right\} \\ & + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \cos nx \left\{ \int_0^a a \cos nx \, dx + \int_a^b \beta \cos nx \, dx + \int_b^\pi \gamma \cos nx \, dx \right\} \\ = & \frac{1}{\pi} \{ a(a-\beta) + b(\beta-\gamma) + \pi\gamma \} \\ & + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx \{ (a-\beta) \sin na + (\beta-\gamma) \sin nb + \gamma \sin n\pi \}. \end{aligned}$$

**2967** Ex. 2.—To find a function of  $x$  having the value  $c$ , when  $x$  lies between 0 and  $a$ , and the value zero when  $x$  lies between  $a$  and  $l$ .

By formula (4), we shall have

$$\int_0^l \cos \frac{n\pi v}{l} \phi(v) \, dv = c \int_0^a \cos \frac{n\pi v}{l} \, dv = \frac{cl}{n\pi} \sin \frac{n\pi a}{l},$$

since  $\phi(v) = c$  from 0 to  $a$ , and zero from  $a$  to  $l$ .

$$\begin{aligned} \text{Therefore } \phi(x) = \frac{ca}{l} + \frac{2c}{\pi} \left\{ \sin \frac{\pi a}{l} \cos \frac{\pi x}{l} + \frac{1}{2} \sin \frac{2\pi a}{l} \cos \frac{2\pi x}{l} \right. \\ \left. + \frac{1}{3} \sin \frac{3\pi a}{l} \cos \frac{3\pi x}{l} + \&c. \right\}. \end{aligned}$$

When  $x = a$ , the value is  $\frac{1}{2} [\phi(a) + 0] = \frac{1}{2}c$ , by the rule in (2963). This may be verified by putting  $a = -1$  in (2923).

**2968** Ex. 3.—To find a function of  $x$  which becomes equal to  $kx$  when  $x$  lies between 0 and  $\frac{1}{2}l$ , and equal to  $k(l-x)$  when  $x$  lies between  $\frac{1}{2}l$  and  $l$ .

By formula (4),

$$\int_0^l \phi(v) \cos \frac{n\pi v}{l} \, dv = \int_0^{\frac{1}{2}l} kv \cos \frac{n\pi v}{l} \, dv + \int_{\frac{1}{2}l}^l k(l-v) \cos \frac{n\pi v}{l} \, dv.$$

This reduces to  $\frac{kl^2}{\pi^2 n^2} \left( 2 \cos \frac{n\pi}{2} - \cos n\pi - 1 \right) = -\frac{4kl^2}{\pi^2 n^2}$  or 0,

according as  $n$  is, or is not, of the form  $4m+2$ . Also

$$\int_0^l \phi(v) \, dv = k \int_0^{\frac{1}{2}l} v \, dv + k \int_{\frac{1}{2}l}^l (l-v) \, dv = \frac{kl^2}{4};$$

$$\therefore \phi(x) = \frac{kl}{4} - \frac{8kl}{\pi^2} \left\{ \frac{1}{2^2} \cos \frac{2\pi x}{l} + \frac{1}{6^2} \cos \frac{6\pi x}{l} + \frac{1}{10^2} \cos \frac{10\pi x}{l} + \&c. \right\}.$$

## APPROXIMATE INTEGRATION.

**2991** Let  $\int_a^b f(x) \, dx$  be the integral, and let the curve  $y = f(x)$  be drawn. By summing the areas of the trapezoids, whose parallel sides are the  $n+1$  equidistant ordinates

$y_0, y_1, \dots, y_n$ , we find, for a first approximation,

$$\int_a^b f(x) dx = \frac{b-a}{2n} (y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n) \dots\dots (i.)$$

#### SIMPSON'S METHOD.

**2992** If  $y_1$  be the ordinate intermediate between  $y_0 = f(a)$  and  $y_2 = f(b)$ , then, approximately,

$$\int_a^b f(x) dx = \frac{b-a}{6} (y_0 + 4y_1 + y_2) \dots\dots\dots (ii.)$$

PROOF.—Take  $n = 3$  in formula (i.); write  $y_2$  for  $y_3$ , and suppose two intermediate ordinates each equal to  $y_1$ . The area thus obtained is equal to what it would be if the bounding curve were a parabola having for ordinates  $y_0, y_1, y_2$  parallel to its axis. Otherwise by Cotes's formula (2995).

**2993** A closer approximation, in terms of  $2n+1$  equidistant ordinates, is given by Simpson's formula,

$$\int_0^1 f(x) dx = \frac{1}{6n} [y_0 + y_{2n} + 4(y_1 + y_3 + \dots + y_{2n-1}) + 2(y_2 + y_4 + \dots + y_{2n-2})] \dots\dots (iii.)$$

PROOF.—We have

$$\int_0^1 f(x) dx = \int_0^{\frac{1}{n}} f(x) dx + \int_{\frac{1}{n}}^{\frac{2}{n}} f(x) dx + \dots + \int_{\frac{n-1}{n}}^1 f(x) dx.$$

Apply formula (ii.) to each integral and add the results, denoting by  $y_r$  the value of  $y$  corresponding to  $x = \frac{r}{2n}$ .

**2994** When the limits are  $a$  and  $b$ , the integral can be changed into another having the limits 0 and 1, by substituting  $x = a + (b-a)y$ .

#### COTES'S METHOD.

Let  $n$  equidistant ordinates, and the corresponding abscissæ, be

$$y_0, y_1, y_2 \dots y_{n-1}, y_n \quad \text{and} \quad 0, \frac{1}{n}, \frac{2}{n} \dots \frac{n-1}{n}, 1.$$

**2995** A formula for approximation will then be

$$\int_0^1 f(x) dx = A_0 y_0 + A_1 y_1 + \dots + A_r y_r + \dots + A_n y_n \quad (iv.),$$

where

$$A_r = \frac{(-1)^{n+r}}{r! (n-r)!} \int_0^1 \frac{(nx)^{(n)}_{-1}}{nx-r} dx. \quad (2452)$$

PROOF.—The method consists in substituting for  $f(x)$  the integral function

$$\psi(x) = (-1)^n \left[ \frac{(nx)^{(n)}_{-1}}{nx \mid n} y_0 \dots + (-1)^r \frac{(nx)^{(n)}_{-1}}{(nx-r) \mid r \mid n-r} y_r \dots \right. \\ \left. \dots + (-1)^n \frac{(nx)^{(n)}_{-1}}{(nx-n) \mid n} y_n \right],$$

$r$  taking all integral values from 0 to  $n$  inclusive. When  $x = r$ , we have  $\psi(r) = y_r$ ; so that  $\psi(x)$  has  $n+1$  values in common with  $f(x)$ . The approximate value of the integral is therefore  $\int_0^1 \psi(x) dx$ , and may be written as in (iv.)

By substituting  $1-x$ , it appears that

$$\int_0^1 \frac{(nx)^{(n)}_{-1}}{nx-r} dx = (-1)^n \int_0^1 \frac{(nx)^{(n)}_{-1}}{nx-(n-r)} dx;$$

and therefore  $A_r = A_{n-r}$ . Consequently it is only necessary to calculate half the number of coefficients in (iv.)

**2996** The coefficients corresponding to the values of  $n$  from 1 to 10 are as follows. Every number has been carefully verified, and two misprints in Bertrand corrected; namely, 2089 for 2989 in line 8, and 89500 for 89600 in line 11.

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$$n=1: A_0 = A_1 = \frac{1}{2}.$$


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$$n=2: A_0 = A_2 = \frac{1}{6}, \quad A_1 = \frac{2}{3}.$$


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$$n=3: A_0 = A_3 = \frac{1}{8}, \quad A_1 = A_2 = \frac{3}{8}.$$


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$$n=4: A_0 = A_4 = \frac{7}{90}, \quad A_1 = A_3 = \frac{16}{45}, \quad A_2 = \frac{2}{15}.$$


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$$n=5: A_0 = A_5 = \frac{19}{288}, \quad A_1 = A_4 = \frac{25}{96}, \quad A_2 = A_3 = \frac{25}{144}.$$


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$$n=6: A_0 = A_6 = \frac{41}{840}, \quad A_1 = A_5 = \frac{9}{35}, \quad A_2 = A_4 = \frac{9}{280}, \quad A_3 = \frac{34}{105}.$$


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$$n=7: A_0 = A_7 = \frac{751}{17280}, \quad A_1 = A_6 = \frac{3577}{17280}, \\ A_2 = A_5 = \frac{49}{640}, \quad A_3 = A_4 = \frac{2989}{17280}.$$


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$$n=8: A_0 = A_8 = \frac{989}{28350}, \quad A_1 = A_7 = \frac{2944}{14175}, \quad A_2 = A_6 = -\frac{464}{14175}, \\ A_3 = A_5 = \frac{5248}{14175}, \quad A_4 = -\frac{454}{2835}.$$


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$$n=9: A_0 = A_9 = \frac{2857}{89600}, \quad A_1 = A_8 = \frac{15741}{89600}, \quad A_2 = A_7 = \frac{27}{2240},$$

$$A_3 = A_6 = \frac{1209}{5600}, \quad A_4 = A_5 = \frac{2889}{44800}.$$


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$$n=10: A_0 = A_{10} = \frac{16067}{598752}, \quad A_1 = A_9 = \frac{26575}{149688}, \quad A_2 = A_8 = -\frac{16175}{199584},$$

$$A_3 = A_7 = \frac{5675}{12474}, \quad A_4 = A_6 = -\frac{4825}{11088}, \quad A_5 = \frac{17807}{24948}.$$


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## GAUSS'S METHOD.

**2997** When  $f(x)$  is an integral algebraic function of degree  $2n$ , or lower, Gauss's formula of approximation is

$$\int_0^1 f(x) dx = A_0 f(x_0) + \dots + A_r f(x_r) + \dots + A_n f(x_n) \quad (\text{v.}),$$

where  $x_0 \dots x_r \dots x_n$  are the  $n+1$  roots of the equation

$$\psi(x) \equiv d_{(n+1)x} \{x^{n+1}(x-1)^{n+1}\} = 0 \quad \dots\dots\dots (\text{vi.}),$$

$$\text{and } A_r = \int_0^1 \frac{(x-x_0) \dots (x-x_{r-1})(x-x_{r+1}) \dots (x-x_n)}{(x_r-x_0) \dots (x_r-x_{r-1})(x_r-x_{r+1}) \dots (x_r-x_n)} dx \quad \dots\dots\dots (\text{vii.})$$

The formula is evidently applicable to a function of any form which can be expanded in a converging algebraic series not having a fractional index in the first  $2n$  terms. The result will be the approximate value of those terms.

PROOF.—Let  $\psi(x) = (x-x_0)(x-x_1) \dots (x-x_n)$ ,  
and let  $f(x) = Q\psi(x) + R \quad \dots\dots\dots (\text{viii.}),$   
where  $f(x)$  is of the  $2n^{\text{th}}$  degree,  $Q$  of the  $n-1^{\text{th}}$ , and  $R$  of the  $n^{\text{th}}$ , since  $\psi(x)$  is of the  $n+1^{\text{th}}$  degree.

Then the method consists in choosing a function  $\psi(x)$  of the  $n+1^{\text{th}}$  degree, so that  $\int_0^1 Q\psi(x) dx$  shall vanish; and a function  $R$  of the  $n^{\text{th}}$  degree, which shall coincide with  $f(x)$  when  $x$  is any one of the  $n+1$  roots of  $\psi(x) = 0$ .

(i.) To ensure that  $\int_0^1 Q\psi(x) dx = 0$ . We have, by Parts, successively, writing  $N$  for  $\psi(x)$ , and with the notation of (2148),

$$\begin{aligned} \int_x x^p N &= x^p \int_x N - p \int_x \left( x^{p-1} \int_x N \right) \\ &= x^p \int_x N - p x^{p-1} \int_{2x} N + p(p-1) \int_x \left( x^{p-2} \int_{2x} N \right) \\ &= \quad \quad \quad \&c. \quad \quad \quad \&c. \\ &= x^p \int_x N - p x^{p-1} \int_{2x} N + p(p-1) x^{p-2} \int_{3x} N - \dots \pm p \int_{(p+1)x} N \quad (\text{ix.}) \end{aligned}$$

Now  $Q\psi(x)$  is made up of terms like  $x^p \psi(x)$  with integral values of  $p$  from 0 to  $n-1$  inclusive. Hence, if the value (vi.) be assumed for  $\psi(x)$ , we



see, by (ix.), that  $\int_0^1 Q\psi(x) dx$  will vanish at both limits, because the factors  $x$  and  $x-1$  will appear in every term.

(ii.) Let  $R$  be the function on the right of equation (v.) Then, when  $x = x_r$ , we see, by (vii.), that  $A_r = 1$ , and that the other coefficients all vanish. Hence  $R$  becomes  $f(x)$  whenever  $x$  is a root of  $\psi(x) = 0$ .

The values of the constants corresponding to the first six values of  $n$ , according to Bertrand, are as follows. The abscissæ values, only, have been recalculated by the author.

$n = 0 :$	$x_0 = \cdot 5,$	$A_0 = 1.$	
$n = 1 :$	$x_0 = \cdot 21132487,$ $x_1 = \cdot 78867513.$	$A_0 = A_1 = \cdot 5,$	$\log = 9\cdot 6989700 ;$
$n = 2 :$	$x_0 = \cdot 11270167,$ $x_1 = \cdot 5 ;$ $x_2 = \cdot 88729833,$	$A_0 = A_1 = \frac{1}{3},$ $A_1 = \frac{1}{3},$	$\log = 9\cdot 4436975 ;$ $\log = 9\cdot 6478175.$
$n = 3 :$	$x_0 = \cdot 06943184,$ $x_1 = \cdot 33000948,$ $x_2 = \cdot 66999052 ;$ $x_3 = \cdot 93056816.$	$A_0 = A_1 = \cdot 1739274,$ $A_1 = A_2 = \cdot 3260726,$	$\log = 9\cdot 2403681 ;$ $\log = 9\cdot 5133143 ;$
$n = 4 :$	$x_0 = \cdot 04691008,$ $x_1 = \cdot 23076534,$ $x_2 = \cdot 5,$ $x_3 = \cdot 76923466 ;$ $x_4 = \cdot 95308992.$	$A_0 = A_1 = \cdot 1184634,$ $A_1 = A_2 = \cdot 2393143,$ $A_2 = \cdot 2844444,$	$\log = 9\cdot 0735834 ;$ $\log = 9\cdot 3789687 ;$ $\log = 9\cdot 4539975 ;$
$n = 5 :$	$x_0 = \cdot 03376524,$ $x_1 = \cdot 16939531,$ $x_2 = \cdot 38069041,$ $x_3 = \cdot 61930959 ;$ $x_4 = \cdot 83060469 ;$ $x_5 = \cdot 96623476.$	$A_0 = A_1 = \cdot 0856622,$ $A_1 = A_2 = \cdot 1803808,$ $A_2 = A_3 = \cdot 2339570,$	$\log = 8\cdot 9327895 ;$ $\log = 9\cdot 2561903 ;$ $\log = 9\cdot 3691360 ;$

As a criterion of the relative degrees of approximation obtained by the foregoing methods, Bertrand gives the following values of

$$\int_0^1 \frac{\log(1+x)}{1+x^2} dx = \frac{\pi}{8} \log 2 = \cdot 2721982613.$$

Method of Trapezoids,	$n = 10,$	$\cdot 2712837.$
Simpson's method,	$n = 10,$	$\cdot 2722012.$
Cotes's           ,,	$n = 5,$	$\cdot 2722091.$
Gauss's           ,,	$n = 4,$	$\cdot 2721980.$

For other formulæ of approximation, see also p. 357.

# CALCULUS OF VARIATIONS.



## FUNCTIONS OF ONE INDEPENDENT VARIABLE.

**3028** Let  $y = f(x)$ , and let  $V$  be a known function of  $x, y$ , and a certain number of the derivatives  $y_x, y_{2x}, y_{3x}$ , &c. The chief object of the Calculus of Variations is to find the *form* of the function  $f(x)$  which will make

$$U = \int_{x_0}^{x_1} V dx \dots\dots\dots(i.)$$

a maximum or minimum. See (3084).

Denote  $y_x, y_{2x}, y_{3x}$ , &c. by  $p, q, r$ , &c.

For a maximum or minimum value of  $U$ ,  $\delta U$  must vanish. To find  $\delta U$ , let  $\delta y$  be the change in  $y$  caused by a change in the form of the function  $y = f(x)$ , and let  $\delta p, \delta q$ , &c. be the consequent changes in  $p, q$ , &c.

Now,  $p = y_x.$

Therefore the new value of  $p$ , when a change takes place in the form of the function  $y$ , is

$$p + \delta p = (y + \delta y)_x = y_x + (\delta y)_x,$$

therefore  $\delta p = (\delta y)_x$ ; that is,  $\delta \left( \frac{dy}{dx} \right) = \frac{d(\delta y)}{dx}.$

Similarly,  $\delta q = (\delta p)_x,$   
 $\delta r = (\delta q)_x$ , &c. ....(ii.)

Now  $\delta U = \int_{x_0}^{x_1} \delta V dx$  (1483). Expand by Taylor's theorem, rejecting the squares of  $\delta y, \delta p, \delta q$ , &c., and we find

$$\delta U = \int_{x_0}^{x_1} (V_y \delta y + V_p \delta p + V_q \delta q + \dots) dx,$$

or, denoting  $V_y, V_p, V_q, \dots$  by  $N, P, Q, \dots$ ,

$$\delta U = \int_{x_0}^{x_1} (N \delta y + P \delta p + Q \delta q + \dots) dx \dots\dots\dots(iii.)$$

Integrate each term after the first by Parts, observing that by (ii.)  $\int \delta p dx = \delta y$ , &c., and repeat the process until the final integrals involve  $\delta y dx$ . Thus

$$\int N \delta y dx \text{ is unaltered,}$$

$$\int P \delta p dx = P \delta y - \int P_x \delta y dx,$$

$$\int Q \delta p dx = Q \delta p - Q_x \delta y + \int Q_{xx} \delta y dx,$$

$$\int R \delta r dx = R \delta q - R_x \delta p + R_{xx} \delta y - \int R_{xx} \delta y dx,$$

... ..

**3029** Hence, collecting the coefficients of  $\delta y$ ,  $\delta p$ ,  $\delta q$ , &c.,

$$\begin{aligned} \delta U = & \int_{x_0}^{x_1} (N - P_x + Q_{xx} - R_{xx} + \dots) \delta y dx \\ & + \delta y_1 (P - Q_x + R_{xx} - \dots)_1 - \delta y_0 (P - Q_x + R_{xx} - \dots)_0 \\ & + \delta p_1 (Q - R_x + S_{xx} - \dots)_1 - \delta p_0 (Q - R_x + S_{xx} - \dots)_0 \\ & + \delta q_1 (R - S_x + T_{xx} - \dots)_1 - \delta q_0 (R - S_x + T_{xx} - \dots)_0 + \&c. \text{ (iv.)} \end{aligned}$$

The terms affected by the suffixes 1 and 0 must have  $x$  made equal to  $x_1$  and  $x_0$  respectively after differentiation.

Observe that  $P_x$ ,  $Q_x$ , &c. are here *complete* derivatives;  $y$ ,  $p$ ,  $q$ ,  $r$ , &c., which they involve, being functions of  $x$ .

Equation (iv.) is written in the abbreviated form,

$$\mathbf{3030} \quad \delta U = \int_{x_0}^{x_1} K \delta y dx + H_1 - H_0 \dots \dots \dots \text{ (v.)}$$

The condition for the vanishing of  $\delta U$ , that is, for minimum value of  $U$ , is

$$\mathbf{3031} \quad K = N - P_x + Q_{xx} - R_{xx} + \&c. = 0 \dots \dots \dots \text{ (vi.)},$$

$$\mathbf{3032} \quad \text{and } H_1 - H_0 = 0 \dots \dots \dots \text{ (vii.)}$$

PROOF.—For, if not, we must have

$$\int_{x_0}^{x_1} K \delta y dx = H_0 - H_1;$$

that is, the integral of an arbitrary function (since  $y$  is arbitrary in form) can be expressed in terms of the limits of  $y$  and its derivatives; which is impossible. Therefore  $H_1 - H_0 = 0$ . Also  $K = 0$ ; for, if the integral could vanish without  $K$  vanishing, the *form* of the function  $\delta y$  would be restricted, which is inadmissible.

The order of  $K$  is twice that of the highest derivative contained in  $V$ . Let  $n$  be the order of  $K$ , then there will be  $2n$  constants in the solution of equation (vi.) and the same number of equations for determining them. For there are  $2n$  terms in equation (vii.) involving  $\delta y_1, \delta y_0, dp_1, \&c.$  If any of these quantities are arbitrary, their coefficients must vanish in order that equation (vii.) may hold; and if any are not arbitrary, they will be fixed in their values by given equations which, together with the equations furnished by the coefficients which have to be equated to zero, will make up, in all,  $2n$  equations.

#### PARTICULAR CASES.

**3033** I.—When  $V$  does not involve  $x$  explicitly, a first integral of the equation  $K = 0$  can always be found. Thus, if, for example,

$$V = \phi(y, p, q, r, s),$$

a first integral will be

$$\begin{aligned} V = & Pp \\ & + Qp_x - Q_x p \\ & + Rp_{2x} - R_x p_x + R_{2x} p \\ & + Sp_{3x} - S_x p_{2x} + S_{2x} p_x - S_{3x} p + C. \end{aligned}$$

The order of this equation is less by one than that of (vi.)

PROOF.—We have  $V_x = Np + Pq + Qr + Rs$ .

Substitute the value of  $N$  from (vi.), and it will be found that each pair of terms involving  $P, Q, R, \&c.$  is an exact differential.

**3034** II.—When  $V$  does not involve  $y$ , a first integral can be found at once, for then  $N = 0$ , and therefore  $K = 0$ , and

we have  $P_x - Q_{2x} + R_{3x} - \&c. = 0$ ;

and therefore  $P - Q_x + R_{2x} - \&c. = A$ .

**3035** III.—When  $V$  involves only  $y$  and  $p$ ,

$$V = Pp + A, \quad \text{by Case I.}$$

**3036** IV.—When  $V$  involves only  $p$  and  $q$ ,

$$V = Qq + Ap + B. \quad \text{See also (3046).}$$

PROOF.  $K = -P_x + Q_{2x} = 0$ , giving, by integration,  $P = Q_x + A$ .

Also  $V_x = Pq + Qr = Aq + Q_x q + Qr$ .

Integrating again, we find  $V = Qq + Ap + B$ ,

a reduction from the fourth to the second order of differential equations.

**3037** Ex.—To find the brachistochrone, or curve of quickest descent, from a point  $O$  taken as origin to a point  $x_1y_1$ , measuring the axis of  $y$  downwards.\*

Velocity at a depth  $y = \sqrt{2gy}$ .

Therefore time of descent  $= \int_{x_0}^{x_1} \frac{\sqrt{1+p^2}}{\sqrt{2gy}} dx$ .

Here  $V = \sqrt{\frac{1+p^2}{y}} = \frac{p^2}{\sqrt{y(1+p^2)}} + A$ , by Case III.

By reduction,  $y(1+p^2) = \frac{1}{A^2} = 2a$ , an arbitrary constant.

That is, since  $p = \tan \theta$ ,  $y = 2a \cos^2 \theta$ , the defining property of a cycloid having its vertex downwards and a cusp at the origin

$$H_1 - H_0 \text{ reduces to } \frac{1}{\sqrt{2a}} \{ (p\delta y)_1 - (p\delta y)_0 \} = 0.$$

If the extreme points are fixed,  $\delta y_1$  and  $\delta y_0$  both vanish.

The values  $x_1, y_1$ , at the lower point, determine  $a$ .

Suppose  $x_1$ , but not  $y_1$ , is fixed. Then  $\delta y_1$  is arbitrary; therefore its coefficient in (3)  $(P - Q_x + \&c.)_1$  must vanish; that is,  $(V_p)_1 = 0$ , or  $\left\{ \frac{p}{\sqrt{y(1+p^2)}} \right\}_1 = 0$ , therefore  $p_1 = 0$ , which means that the tangent at the lower point is horizontal, and the curve is therefore a complete half cycloid.

**3038** In the example of the brachistochrone, it is useful to notice that—

(i.) If the extreme points are fixed,  $\delta y_0, \delta y_1$  both vanish.

(ii.) If the tangents at the extreme points have fixed directions,  $\delta p_0, \delta p_1$  both vanish.

(iii.) If the curvature at each extremity is fixed in value,  $\delta p_0, \delta q_0, \delta p_1, \delta q_1$  all vanish.

(iv.) If the abscissæ  $x_0, x_1$  only have fixed values,  $\delta y_0, \delta y_1$  are then arbitrary, and their coefficients in  $H_1 - H_0$  must vanish.

**3039** When the limits  $x_0, x_1$  are variable, add to the value of  $\delta U$  in (3029)  $V_1 dx_1 - V_0 dx_0$ .

PROOF.—The partial increment of  $U$ , due to changes in  $x_1$  and  $x_0$ , is

$$\frac{dU}{dx_1} dx_1 + \frac{dU}{dx_0} dx_0 = V_1 dx_1 - V_0 dx_0. \quad \text{By (2253).}$$

**3040** When  $x_1$  and  $y_1, x_0$  and  $y_0$  are connected by given equations,  $y_1 = \psi(x_1), y_0 = \chi(x_0)$ .

RULE.—Put

$$\delta y_1 = \{ \psi'(x_1) - p_1 \} dx_1 \quad \text{and} \quad \delta y_0 = \{ \chi'(x_0) - p_0 \} dx_0,$$

\* The Calculus of Variations originated with this problem, proposed by John Bernoulli in 1696.

and afterwards equate to zero the coefficients of  $dx_1$  and  $dx_0$ , because the values of the latter are arbitrary.

PROOF.— $y_1 + \delta y_1$  being a function of  $x_1$ ,

$$(y_1 + \delta y_1) + d_{x_1} (y_1 + \delta y_1) dx_1 = \psi(x_1 + dx_1) = \psi(x_1) + \psi'(x_1) dx_1;$$

therefore  $\delta y_1 + p_1 dx_1 = \psi'(x_1) dx_1$ , neglecting  $\delta p dx_1$ .

Ex.—In the brachistochrone problem (3037), the result thus arrived at signifies that the cycloid is at right angles to each of the given curves at its extremities.

**3041** If  $V$  involves the limits  $x_0, x_1, y_0, y_1, p_0, p_1$ , &c., the terms to be added to  $\delta U$  in (3029), on account of the variation of any of these quantities, are

$$\begin{aligned} & dx_1 \int_{x_0}^{x_1} \{ V_{x_1} + V_{y_1} p_1 + V_{p_1} q_1 + \dots \} dx \\ & + dx_0 \int_{x_0}^{x_1} \{ V_{x_0} + V_{y_0} p_0 + V_{p_0} q_0 + \dots \} dx \\ & + \int_{x_0}^{x_1} \{ V_{y_0} \delta y_0 + V_{y_1} \delta y_1 + V_{p_0} \delta p_0 + V_{p_1} \delta p_1 + \&c. \} dx. \end{aligned}$$

In the last integral,  $\delta y_0, \delta y_1, \delta p_0, \&c.$  may be placed outside the symbol of integration since, they are not functions of  $x$ .

Hence, when  $V$  involves the limits  $x_0, x_1, y_0, y_1, p_0, p_1$ , &c., and those limits are variable, the complete expression for  $\delta U$  is

$$\begin{aligned} \text{3042 } \delta U &= \int_{x_0}^{x_1} \{ N - P_x + Q_{2x} - R_{3x} + \&c. \} \delta y dx \\ &+ \{ V_1 + \int_{x_0}^{x_1} (V_{x_1} + V_{y_1} p_1 + V_{p_1} q_1 + \dots) dx \} dx_1 \\ &- \{ V_0 - \int_{x_0}^{x_1} (V_{x_0} + V_{y_0} p_0 + V_{p_0} q_0 + \dots) dx \} dx_0 \\ &+ \{ (P - Q_x + R_{2x} - \dots)_1 + \int_{x_0}^{x_1} V_{y_1} dx \} \delta y_1 \\ &- \{ (P - Q_x + R_{2x} - \dots)_0 - \int_{x_0}^{x_1} V_{y_0} dx \} \delta y_0 \\ &+ \{ (Q - R_x + S_{2x} - \dots)_1 + \int_{x_0}^{x_1} V_{p_1} dx \} \delta p_1 \\ &- \{ (Q - R_x + S_{2x} - \dots)_0 - \int_{x_0}^{x_1} V_{p_0} dx \} \delta p_0 + \&c. \end{aligned}$$



**3043** Also, if  $y_1 = \psi(x_1)$  and  $y_0 = \chi(x_0)$  be equations restricting the limits, put

$$\delta y_1 = \{\psi'(x_1) - p_1\} dx_1 \quad \text{and} \quad \delta y_0 = \{\chi'(x_0) - p_0\} dx_0. \quad (3040)$$

The relation  $K = 0$  is unaltered, and, by means of it, the additional integrals which appear in the value of  $H_1 - H_0$  become definite functions of  $x$ .

**3044** Ex.—To find the curve of quickest descent of a particle from some point on the curve  $y_0 = \chi(x_0)$  to the curve  $y_1 = \psi(x_1)$ .

As in (3037),  $t = \frac{1}{\sqrt{(2g)}} \int_{x_0}^{x_1} \sqrt{\frac{1+p^2}{y-y_0}} dx$ ,  $V = \sqrt{\frac{1+p^2}{y-y_0}}$ , and contains  $y$ ,  $p$ , and  $y_0$ . Equation (3042) now reduces to

$$\begin{aligned} \delta U = \int_{x_0}^{x_1} (N - P_x) \delta y dx + V_1 dx_1 - \left\{ V_0 - \int_{x_0}^{x_1} V_{y_0} p_0 dx \right\} dx_0 \\ + P_1 \delta y_1 - \left\{ P_0 - \int_{x_0}^x V_{y_0} dx \right\} \delta y_0 \dots \dots (1). \end{aligned}$$

Now  $K = 0$  gives  $N - P_x = 0$ ; therefore  $V = Pp + A$  (3035);

therefore 
$$\sqrt{\frac{1+p^2}{y-y_0}} = \frac{p^2}{\sqrt{(y-y_0)(1+p^2)}} + A.$$

Clearing of fractions, and putting  $A = \frac{1}{\sqrt{(2a)}}$ , this becomes

$$(y-y_0)(1+p^2) = 2a \dots \dots \dots (2).$$

Also 
$$P = V_p = \frac{p}{\sqrt{(y-y_0)(1+p^2)}} = \frac{p}{\sqrt{(2a)}} \dots \dots \dots (3).$$

Hence 
$$V = \frac{1+p^2}{\sqrt{(2a)}}; \quad V_{y_0} = -V_p = -N = -P_x \text{ (by } K = 0),$$

therefore 
$$\int_{x_0}^{x_1} V_{y_0} dx = P_0 - P_1 = \frac{p_0 - p_1}{\sqrt{(2a)}} \dots \dots \dots (4).$$

Substituting the values (2), (3), (4), in (1), the condition  $H_1 - H_0$  produces

$$(1+p_1^2) dx_1 - (1+p_0 p_1) dx_0 + p_1 \delta y_1 - p_1 \delta y_0 = 0.$$

Next, put for  $\delta y_1$  and  $\delta y_0$  the values in (3040); thus the equation becomes

$$\{1+p_1 \psi'(x_1)\} dx_1 - \{1+p_1 \chi'(x_0)\} dx_0 = 0 \dots \dots \dots (5);$$

$dx_1$ ,  $dx_0$  being arbitrary, their coefficients must vanish; therefore

$$p_1 \psi'(x_1) = -1 \quad \text{and} \quad p_1 \chi'(x_0) = -1.$$

That is, the tangents of the given curves  $\psi$  and  $\chi$  at the points  $x_0 y_0$  and  $x_1 y_1$  are both perpendicular to the tangent of the brachistochrone at the point  $x_1 y_1$ .

Equation (2) shews that the brachistochrone is a cycloid with a cusp at the starting-point, since there  $y = y_0$ , and therefore  $p = \infty$ .

OTHER EXCEPTIONAL CASES.

(Continued from 3036.)

**3045** V.—Denoting  $y, y_x, y_{2x} \dots y_{nx}$  by  $y, p_1, p_2 \dots p_n$ ;

and  $V_y, V_{p_1}, V_{p_2} \dots V_{p_n}$  by  $N, P_1, P_2 \dots P_n$ ;

let the first  $m$  of the quantities  $y, p_1, p_2$ , &c. be wanting in the function  $V$ ; so that

$$V = f(x, p_m, p_{m+1} \dots p_n).$$

Then  $K = d_{mx} P_m - d_{(m+1)x} P_{m+1} + \dots (-1)^{n-m} d_{nx} P_n = 0$ ,

which equation, being integrated  $m$  times, becomes

$$P_m - d_x P_{m+1} + d_{2x} P_{m+2} - \dots (-1)^{n-m} d_{(n-m)x} P_n \\ = c_0 + c_1 x + \dots + c_{m-1} x^{m-1} \dots \dots (i.),$$

a differential equation of the order  $2n - m$ .

**3046** VI.—Let  $x$  also be wanting in  $V$ , so that

$$V = f(p_m, p_{m+1} \dots p_n);$$

then  $K = 0$  is the same as before, and produces the same differential equation (i.) From that equation take the value of  $P_m$ , and substitute it in

$$V_x = P_m p_{m+1} + P_{m+1} p_{m+2} + \dots + P_n p_{n+1}.$$

Each pair of terms, such as  $P_{m+2} p_{m+3} - d_{2x} P_{m+2} p_{m+1}$ , is an exact differential; and we thus find

$$V = c + P_{m+1} p_{m+1} + (P_{m+2} p_{m+2} - d_x P_{m+2} p_{m+1}) + \dots \\ + (P_n p_n - d_x P_n p_{n-1} + d_{2x} P_n p_{n-2}) - \dots (-1)^{n-m-1} d_{(n-m-1)x} P_n p_{m+1} \\ + \int (c_0 + c_1 x + \dots + c_{m-1} x^{m-1}) p_{m+1} dx.$$

The resulting equation will be of the order  $2n - m - 1$ , or  $m + 1$  degrees lower than the original equation.

**3047** VII.—If  $V$ . be a *linear* function of  $p_n$ , that being the highest derivative it contains,  $P_n$  will not then contain  $p_n$ . Therefore  $d_{nx} P_n$  will be, at most, of the order  $2n - 1$ . Indeed, in this case, the equation  $K = 0$  cannot be of an order higher than  $2n - 2$ . (Jellet, p. 44.)

**3048** VIII.—Let  $p_m$  be the lowest derivative which  $V$  involves; then, if  $P_m = f(x)$ , and if only the limiting values of  $x$  and of derivatives higher than the  $m^{\text{th}}$  be given, the problem cannot generally be solved. (Jellet, p. 49.)

**3049** IX.—Let  $N = 0$ , and let the limiting values of  $x$  alone be given; then the equation  $K = 0$  becomes

$$P_x - Q_{2x} + R_{3x} - \&c. = 0,$$

or, by integration,  $P - Q_x + R_{2x} - \&c. = c$ ,

and the two conditions furnished by equating to zero the coefficients of  $\delta y_1, \delta y_0$ , viz.,

$$(P - Q_x + \&c.)_1 = 0, \quad (P - Q_x + \&c.)_0 = 0,$$

are equivalent to the single equation  $c = 0$ , and therefore  $H_1 - H_0 = 0$  supplies but  $2n - 1$  equations instead of  $2n$ , and the problem is indeterminate.

**3050** Let  $U = \int_{x_0}^{x_1} V dx + V'$ , where

$$V = F(x, y, p, q \dots) \text{ and } V' = f(x_0, x_1, y_0, y_1, p_0, p_1, \&c.)$$

The condition for a maximum or minimum value of  $U$  arising from a variation in  $y$ , is, as before,  $K = 0$ ; and the terms to be added to  $H_1 - H_0$  are

$$V'_{x_0} dx_0 + V'_{y_0} \delta y_0 + V'_{p_0} \delta p_0 + \dots + V'_{x_1} dx_1 + V'_{y_1} \delta y_1 + \&c.$$

If the order of  $V$  be  $n$ , and the number of increments  $dx_0, \delta y_0, \&c.$  be greater than  $n + 1$ , the number of independent increments will exceed the number of arbitrary constants in  $K$ , and no maximum or minimum can be found.

Generally,  $U$  does not in this case admit of a maximum or minimum if either  $V$  or  $V'$  contains either of the limiting values of a derivative of an order  $=$  or  $>$  than that of the highest derivative found in  $V$ . (Jellet, p. 72.)

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## FUNCTIONS OF TWO DEPENDENT VARIABLES.

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**3051** Let  $V$  be a function of two dependent variables  $y, z$ , and their derivatives with respect to  $x$ ; that is, let

$$V = f(x, y, p, q \dots z, p', q' \dots) \dots \dots \dots (1),$$

where  $p, q, \dots$ , as before, are the successive derivatives of  $y$ , and  $p', q', \dots$  those of  $z$ .

Then, if the *forms* of the functions  $y, z$  vary, the condition for a maximum or minimum value of  $U$  or  $\int_{x_0}^{x_1} V dx$  is

$$\delta U = \int_{x_0}^{x_1} (K \delta y + K' \delta z) dx + H_1 - H_0 + H'_1 - H'_0 = 0 \dots (2).$$

Here  $K', H'$  involve  $z, p', q', \dots$ , precisely as  $K, H$  involve  $y, p, q, \dots$ ; the values of the latter being given in (3029).

**3052** First, if  $y$  and  $z$  are independent, equation (2) necessitates the following conditions:

$$K = 0, \quad K' = 0, \quad H_1 - H_0 + H'_1 - H'_0 = 0 \dots (3).$$

The equations  $K = 0, K' = 0$  give  $y$  and  $z$  in terms of  $x$ , and the constants which appear in the solution must be determined by equating to zero the coefficients of the arbitrary quantities  $\delta y_0, \delta y_1, \delta p_0, \delta p_1 \dots \delta z_0, \delta z_1, \delta p'_0, \delta p'_1, \dots$ ,

which are found in the equation

$$H_1 - H_0 + H'_1 - H'_0 = 0 \dots (4).$$

**3053** The number of equations so obtained is equal to the number of constants to be determined.

PROOF.—Let  $V = f(x, y, y_x, y_{2x} \dots y_{nx}, z, z_x, z_{2x} \dots z_{mx})$ ,

$K$  is of order  $2n$  in  $y$ , and  $\therefore$  of form  $\phi(x, y, y_x \dots y_{2nx}, z, z_x \dots z_{(m+n)x}) \dots$  (i.),

$K'$  is of order  $2m$  in  $z$ , and  $\therefore$  of form  $\phi(x, y, y_x \dots y_{(m+n)x}, z, z_x \dots z_{2mx}) \dots$  (ii.)

Differentiating (i.)  $2m$  times, and (ii.)  $m+n$  times,  $3m+n+2$  equations are obtained, between which, if we eliminate  $z, z_x \dots z_{(3m+n)x}$ , we get a resulting equation in  $y$ , of order  $2(m+n)$ , whose solution will therefore contain  $2(m+n)$  arbitrary constants. The equations for finding these are also  $2(m+n)$  in number, viz.,  $2n$  in  $H_1 - H_0$  and  $2m$  in  $H'_1 - H'_0$ .

**3054** NOTE.—The number of equations for determining the constants is not generally affected by any auxiliary equations introduced by restricting the limits. For every such equation either removes a term from (4) by annulling some variation ( $\delta y, \delta p$ , &c.), or it makes two terms into one; in each case diminishing by one the number of equations, and adding one equation, namely itself.

**3055** Secondly, let  $y$  and  $z$  be connected by some equation

$\phi(xyz) = 0$ .  $y$  and  $z$  are then found by solving simultaneously the equations

$$\phi(x, y, z) = 0 \quad \text{and} \quad K : \phi_y = K' : \phi_z.$$

PROOF.— $\phi(x, y, z) = 0$ , and therefore  $\phi(x, y + \delta y, z + \delta z) = 0$ , when the forms of  $y$  and  $z$  vary. Therefore  $\phi_y \delta y + \phi_z \delta z = 0$  (1514). Also  $K \delta y + K' \delta z = 0$ , by (2). Hence the proportion.

**3056** Thirdly, let the equation connecting  $y$  and  $z$  be of the more general form

$$\phi(x, y, p, q \dots z, p', q' \dots) = 0 \dots \dots \dots (5).$$

By differentiation, we obtain

$$\phi_y \delta y + \phi_p \delta p + \phi_q \delta q + \dots \phi_z \delta z + \phi_{p'} \delta p' + \phi_{q'} \delta q' + \dots = 0 \dots (6).$$

If (which rarely happens) this equation can be integrated so as to furnish a value of  $\delta z$  in terms of  $\delta y$ , then  $\delta p'$ ,  $\delta q'$ , &c. may be obtained, by simple differentiation, in terms of  $\delta y$ ,  $\delta p$ . Generally, we proceed as follows:—

$$\delta V = N \delta y + P \delta p + Q \delta q + \dots + N' \delta z + P' \delta p' + Q' \delta q' + \dots \dots (7).$$

Multiply (6) by  $\lambda$ , and add it to (7), thus

$$\begin{aligned} \delta V = (N + \lambda \phi_y) \delta y + (P + \lambda \phi_p) \delta p + \dots \\ \dots + (N' + \lambda \phi_z) \delta z + (P' + \lambda \phi_{p'}) \delta p' + \dots \dots \dots (8). \end{aligned}$$

The expression for  $\delta U$  will therefore be the same as in (2), if we replace  $N$  by  $N + \lambda \phi_y$ ,  $P$  by  $P + \lambda \phi_p$ , &c., thus

$$\begin{aligned} \mathbf{3057} \quad \delta U = \int_{x_0}^{x_1} [ \{ (N + \lambda \phi_y) - (P + \lambda \phi_p)_x + \dots \} \delta y \\ + \{ (N' + \lambda \phi_z) - (P' + \lambda \phi_{p'})_x + \dots \} \delta z ] dx \\ + \{ P + \lambda \phi_p - (Q + \lambda \phi_q)_x + \dots \}_1 \delta y_1 \\ - \{ P + \lambda \phi_p - (Q + \lambda \phi_q)_x + \dots \}_0 \delta y_0 \\ + \{ Q + \lambda \phi_q - (R + \lambda \phi_r)_x + \dots \}_1 \delta p_1 \\ - \{ Q + \lambda \phi_q - (R + \lambda \phi_r)_x + \dots \}_0 \delta p_0 \\ \quad \quad \quad \&c. \quad \quad \quad \&c. \\ + \text{similar terms in } P, Q \dots p', q' \dots \&c. \dots (9). \end{aligned}$$

**3058** To render  $\delta U$  independent of the variation  $\delta z$ , we must

then equate to zero the coefficient of  $\delta z$  under the sign of integration; thus

$$N' + \lambda \phi_z - (P' + \lambda \phi_p)_x + (Q' + \lambda \phi_q)_{2x} - \&c. = 0 \dots (10),$$

the equation for determining  $\lambda$ .

**3059** Ex. (i.)—Given  $V = F(x, y, p, q \dots z)$ , where

$$z = \int v dx \quad \text{and} \quad v = F(x, y, p, q \dots).$$

The equation  $\phi$  is now  $z - \int v dx = 0$  or  $v - z_x = 0$ ,

$$\begin{aligned} \phi_y &= v_y, & \phi_p &= v_p, & \phi_q &= v_q, \&c., \\ \phi_z &= 0, & \phi_{p'} &= -1, & \phi_{q'} &= 0, \text{ the rest vanishing.} \end{aligned}$$

Substituting these values in (9), we obtain

$$\begin{aligned} \delta U &= \int_{x_0}^{x_1} [\{N + \lambda v_y - (P + \lambda v_p)_x + (Q + \lambda v_q)_{2x} - \dots\} \delta y + \{N' + \lambda_x\} \delta z] dx \\ &\quad + \{P + \lambda v_p - (Q + \lambda v_q)_x + \dots\}_1 \delta y_1 - \{P + \lambda v_p - (Q + \lambda v_q)_x + \dots\}_0 \delta y_0 \\ &\quad + \{Q + \lambda v_q - (R + \lambda v_r)_x + \dots\}_1 \delta p_1 - \{Q + \lambda v_q - (R + \lambda v_r)_x + \dots\}_0 \delta p_0 + \&c. \end{aligned}$$

For the complete variation  $DU$  add  $V_1 dx_1 - V_0 dx_0$ . To reduce the above so as to remove  $\delta z$ , we must put  $N' + \lambda_x = 0$ , and therefore  $\lambda = -\int N' dx$ . Let  $\lambda = u$  be the solution,  $u$  being a function of  $x, y, p, q \dots z$ . Substituting this expression for  $\lambda$ , the value of  $\delta U$  becomes independent of  $\delta z$ .

Ex. (ii.)—Similarly, if  $z$  in the last example be  $= \int_{p_x} v$  (2148),  $\phi$  becomes  $v - z_{p_x} = 0$ ; and, to make  $N' + \lambda_{p_x}$  vanish, we must put  $\lambda = -\int_{p_x} N'$ .

**3061** Ex. (iii.)—Let  $U = \int_{x_0} \sqrt{1 + y_x^2 + z_x^2} dx \dots (1)$ .

Here  $N = 0$ ;  $N' = 0$ ;  $P = \frac{p}{\sqrt{1 + p^2 + p'^2}}$ ;  $P' = \frac{p'}{\sqrt{1 + p^2 + p'^2}}$ ;  $Q = 0$ ;

$Q' = 0$ ; and the equations  $K = 0$ ,  $K' = 0$  become

$$P_x = 0, \quad P'_x = 0, \quad \text{or} \quad \frac{p}{\sqrt{1 + p^2 + p'^2}} = a, \quad \frac{p'}{\sqrt{1 + p^2 + p'^2}} = b.$$

Solving these equations, we get

$$y_x = m; \quad z_x = n; \quad \text{or} \quad y = mx + A; \quad z = nx + B.$$

**3062** First, if  $x_1, y_1, z_1, x_0, y_0, z_0$  be given, there are four equations to determine  $m, n, A$ , and  $B$ .

This solves the problem, to find a line of minimum length on a given curved surface between two fixed points on the surface.

**3063** Secondly, if the limits  $x_1, x_0$  only are given, then the equations

$$(P)_1 = 0, \quad (P)_0 = 0, \quad (P')_1 = 0, \quad (P')_0 = 0,$$

are only equivalent to the two equations  $m = 0$ ,  $n = 0$ , and  $A$  and  $B$  remain undetermined.



**3064** *Thirdly, let the limits be connected by the equations*

$$\phi(x_1, y_1, z_1) = 0, \quad \psi(x_0, y_0, z_0) = 0.$$

We shall have  $(\phi_x + \phi_y p_1 + \phi_z p'_1) dx_1 + \phi_y \delta y_1 + \phi_z \delta z_1 = 0$ .

Substitute  $\phi_y = m_1 \phi_x$ ,  $\phi_z = n_1 \phi_x$ ,  $p_1 = m$ ,  $p'_1 = n$ ; thus

$$(1 + mm_1 + nn_1) dx_1 + m_1 \delta y_1 + n_1 \delta z_1 = 0.$$

Eliminate  $dx_1$  by this equation from

$$V_1 dx_1 + (P)_1 \delta y_1 + (P')_1 \delta z_1 = 0,$$

and equate to zero the coefficients of  $\delta y_1$  and  $\delta z_1$ ; then

$$m_1 V_1 = (P)_1 (1 + mm_1 + nn_1); \quad n_1 V_1 = (P')_1 (1 + mm_1 + nn_1).$$

Replacing  $V_1$ ,  $P_1$  by their values, and solving these equations for  $m$  and  $n$ , we find  $m = m_1$ ,  $n = n_1$ .

Similarly from the equation  $\psi(x_0, y_0, z_0) = 0$  we derive  $m = m_0$ ,  $n = n_0$ .

Eliminating  $x_1, y_1, z_1, x_0, y_0, z_0$  between these equations, and

$$y_1 = mx_1 + A; \quad z_1 = nx_1 + B; \quad y_0 = mx_0 + A; \quad z_0 = nx_0 + B;$$

$$\phi(x_1, y_1, z_1) = 0; \quad \psi(x_0, y_0, z_0) = 0;$$

four equations remain for determining  $m$ ,  $n$ ,  $A$ , and  $B$ .

**3065** *On determining the constants in the solution of (3056).*

Denoting  $p, q, r \dots$  by  $p_1, p_2, p_3 \dots$ , we have

$$V = F(x, y, p_1, p_2 \dots p_n, z, p'_1, p'_2 \dots p'_m);$$

and for the limiting equation,

$$\phi(x, y, p_1, p_2, \dots p_n, z, p'_1, p'_2, \dots p'_m) = 0.$$

$V$  is of the order  $n$  in  $y$  and  $m$  in  $z$ .

$\phi$  is of the order  $n'$  in  $y$  and  $m'$  in  $z$ .

**3066** **RULE I.**—If  $m$  be  $> m'$ , and  $n$  either  $>$  or  $< n'$ , the order of the final differential equation will be the greater of the two quantities  $2(m+n')$ ,  $2(m'+n)$ ; and there will be a sufficient number of subordinate equations to determine the arbitrary constants.

**3067** **RULE II.**—If  $m$  be  $< m'$ , and  $n < n'$ , the order of the final equation will generally be  $2(m'+n')$ ; and its solution may contain any number of constants not greater than the least of the two quantities  $2(m'-m)$ ,  $2(n'-n)$ .

For the investigation, see *Jellet*, pp. 118—127.

**3068** If  $V$  does not involve  $x$  explicitly, a single integral of order  $2(m+n)-1$  may be found. The value of  $V$  is that given in (3033), with corresponding terms derived from  $z$ .

PROOF.—  $dV = Ndy + P_1dp_1 + \dots + P_ndp_n + N'dz + P'_1dp'_1 + \dots + P'_ndp'_n$ .  
 Substitute for  $N$  and  $N'$  from the equations  $K = 0$ ,  $K' = 0$ , as in (3033), and integrate for  $V$ .

### RELATIVE MAXIMA AND MINIMA.

**3069** In this class of problems, a maximum or minimum value of an integral,  $U_1 = \int_{x_0}^{x_1} V_1 dx$ , is required, subject to the condition that another integral,  $U_2 = \int_{x_0}^{x_1} V_2 dx$ , involving the same variables, has a constant value.

RULE.—Find the maximum or minimum value of the function  $U_1 + aU_2$ ; that is, take  $V = V_1 + aV_2$ , and afterwards determine the constant  $a$  by equating  $U_2$  to its given value.

For examples, see (3074), (3082).

### GEOMETRICAL APPLICATIONS.

**3070** PROPOSITION I.—To find a curve  $s$  which will make  $\int F(x, y) ds$  a maximum or minimum,  $F$  being a given function of the coordinates  $x, y$ .

The equation (5), in (3056), here becomes

$$p^2 + p'^2 = 1;$$

where  $p = x_s$ ,  $p' = y_s$ ,  $x$  and  $y$  being the dependent variables, and  $s$  the independent variable.

In (3057), we have now, writing  $u$  for  $F(x, y)$ ,

$$N = u_x, \quad N' = u_y, \quad \phi_p = 2p, \quad \phi_{p'} = 2p';$$

the rest zero. The equations of condition are therefore

$$* \quad u_x - d_s(\lambda x_s) = 0 \quad \text{and} \quad u_y - d_s(\lambda y_s) = 0 \dots\dots\dots(1).$$

Multiplying by  $x_s, y_s$  respectively, adding and integrating, the result is

$$\lambda = u,$$

the constant being zero.\*

Substituting this value in equations (1), differentiating  $u_x$  and  $u_y$ , and putting  $u_s = u_x x_s + u_y y_s$ , we get

$$y_s(u_x y_s - u_y x_s) = u x_s \dots\dots\dots(2),$$

$$x_s(u_y x_s - u_x y_s) = u y_s \dots\dots\dots(3).$$

\* See Todhunter's "History," p. 406.

Multiplying (2) by  $y$ , and (3) by  $x$ , and subtracting, we obtain finally

$$u(y, x_2 - x, y_2) = u_x y_2 - u_y x_2 \quad \text{or}$$

$$3071 \quad \frac{u}{\rho} = \frac{du}{dx} \frac{dy}{ds} - \frac{du}{dy} \frac{dx}{ds} \dots\dots\dots (4),$$

$\rho$  being the radius of curvature.

To integrate this equation, the form of  $u$  must be known, and, by assigning different forms, various geometrical theorems are obtained.

**3072** PROPOSITION II.—To find the curve which will make

$$\int F(x, y) ds + \int f(x, y) dx \dots\dots\dots (1)$$

a maximum or minimum, the functions  $F$  and  $f$  being of given form.

Let  $F(x, y) = u$  and  $f(x, y) = v$ .

Equation (1) is equivalent to  $\int (u + vx) ds$ .

In (3057) we now have  $V = u + vp$ ; and for  $\phi$ ,  $p^2 + p'^2 = 1$ , as in (3070).

Therefore  $N = u_x + pv_x$ ;  $P = V_p = v$ ;  $\phi_p = 2p$ ;  
 $N' = u_y + pv_y$ ;  $\phi_{p'} = 2p'$ ; the rest zero.

Therefore, equating to zero the coefficients of  $\delta x$  and  $\delta y$ , the result is the two equations

$$u_x + pv_x - (v + \lambda p)_s = 0,$$

$$u_y + pv_y - (\lambda p')_s = 0;$$

$$\text{or } d_s(\lambda x_s) + v_s = u_x + x_s v_x,$$

$$d_s(\lambda y_s) = u_y + x_s v_y.$$

Multiplying by  $x_s$ ,  $y_s$  respectively, adding, and integrating, we obtain, as in (3070),  $\lambda = u$ , and ultimately,

$$3073 \quad \frac{1}{\rho} = -\frac{1}{u} \left( \frac{du}{dx} \frac{dy}{ds} - \frac{du}{dy} \frac{dx}{ds} + \frac{dv}{dy} \right).$$

**3074** Ex.—To find a curve  $s$  of given length, such that the volume of the solid of revolution which it generates about a given line may be a maximum.

Here  $\int (y^2 x_s - a^2) ds$  must be a maximum, by (3069),  $a^2$  being the arbitrary constant. The problem is a case of (3072),

$$u = a^2, \quad u_x = 0, \quad u_y = 0, \quad v = y^2, \quad v_y = 2y.$$

Hence equation (3073) becomes  $\frac{1}{\rho} = \frac{2y}{a^2}$ .

Giving  $\rho$  its value,  $-\frac{(1+p^2)^{\frac{1}{2}}}{pp_y}$  (where  $p = \frac{dy}{dx}$ ), and integrating, the result

$$\frac{1}{\sqrt{1+p^2}} = \frac{y^2 + b^2}{a^2}; \quad \text{from which } x = \int \frac{(y^2 + b^2) dy}{\sqrt{a^4 - (y^2 + b^2)}}$$

FUNCTIONS OF TWO INDEPENDENT VARIABLES.

**3075** Let  $V = f(x, y, z, p, q, r, s, t)$ ,  
in which  $x, y$  are the independent variables, and  $p, q, r, s, t$   
stand for  $z_x, z_y, z_{2x}, z_{xy}, z_{2y}$  respectively (1815),  $z$  being an in-  
determinate function of  $x$  and  $y$ .

Let 
$$U = \int_{x_0}^{x_1} \int_{y_0}^{y_1} V dx dy,$$

and let the equation connecting  $x$  and  $y$  at the limits be  
 $\phi(x, y) = 0$ . The complete variation of  $U$ , arising solely from  
an infinitesimal change in the *form* of the function  $z$ , is as  
follows:—

Let  $V_x, V_p$ , &c. be denoted by  $Z, P, Q, R, S, T$ .

Let 
$$\begin{aligned}\phi &= (P - R_x - \tfrac{1}{2}S_y) \delta z + \tfrac{1}{2}S \delta q + R \delta p, \\ \psi &= (Q - T_y - \tfrac{1}{2}S_x) \delta z + \tfrac{1}{2}S \delta p + T \delta q, \\ \chi &= (Z - P_x - Q_y + R_{2x} + S_{xy} + T_{2y}) \delta z.\end{aligned}$$

The variation in question is then

**3076** 
$$\delta U = \int_{x_0}^{x_1} \left( \psi_{y=y_1} - \psi_{y=y_0} + \phi_{y=y_0} \frac{dy_0}{dx} - \phi_{y=y_1} \frac{dy_1}{dx} \right) dx$$
$$+ \left[ \int_{y_0}^{y_1} \phi dy \right]_{x=x_0}^{x=x_1} + \int_{x_0}^{x_1} \int_{y_0}^{y_1} \chi dx dy.$$

PROOF.— 
$$\begin{aligned}\delta \int_{x_0}^{x_1} \int_{y_0}^{y_1} V dx dy &= \int_{x_0}^{x_1} \int_{y_0}^{y_1} \delta V dx dy \\ &= \int_{x_0}^{x_1} \int_{y_0}^{y_1} (Z \delta z + P \delta p + Q \delta q + R \delta r + S \delta s + T \delta t) dx dy \\ &= \int_{x_0}^{x_1} \int_{y_0}^{y_1} \left\{ \frac{d\phi}{dx} + \frac{d\psi}{dy} + \chi \right\} dx dy,\end{aligned}$$

as appears by differentiating the values of  $\phi$  and  $\psi$ . But

$$\int_{y_0}^{y_1} \frac{d\phi}{dx} dy = \frac{d}{dx} \int_{y_0}^{y_1} \phi dy + \phi_{y=y_0} \frac{dy_0}{dx} - \phi_{y=y_1} \frac{dy_1}{dx},$$

by (2257), and 
$$\int_{y_0}^{y_1} \frac{d\psi}{dy} dy = \psi_{y=y_1} - \psi_{y=y_0}.$$

Hence the result.

**3077** The conditions for a maximum or minimum value of  
 $U$  are, by similar reasoning to that employed in (3032),

$$\phi = 0, \quad \psi = 0, \quad \chi = 0.$$

## GEOMETRICAL APPLICATIONS.

**3078** PROPOSITION I.—To find the surface,  $S$ , which will make  $\iint F(x, y, z) dS$  a maximum or minimum,  $F$  being a given function of the coordinates  $x, y, z$ . [Jellet, p. 276.]

Here, putting  $F(x, y, z) = u$ ,  $V = u\sqrt{1+p^2+q^2}$ ;

$$Z = \sqrt{1+p^2+q^2} \frac{du}{dz}; \quad P = \frac{up}{\sqrt{1+p^2+q^2}}; \quad Q = \frac{uq}{\sqrt{1+p^2+q^2}};$$

and  $V_x, V_y, V_z$  are all zero.

$$\frac{dP}{dx} = \frac{p}{\sqrt{1+p^2+q^2}} \left( \frac{du}{dx} + p \frac{du}{dz} \right) + u \frac{(1+q^2)r - pq^2s}{(1+p^2+q^2)^{\frac{3}{2}}},$$

$$\frac{dQ}{dy} = \frac{q}{\sqrt{1+p^2+q^2}} \left( \frac{du}{dy} + q \frac{du}{dz} \right) + u \frac{(1+p^2)t - pqs}{(1+p^2+q^2)^{\frac{3}{2}}}.$$

The equation  $\chi = 0$  or  $Z - P_x - Q_y = 0$  gives

$$\frac{(1+q^2)r - 2pq^2s + (1+p^2)t}{(1+p^2+q^2)^{\frac{3}{2}}} + \frac{1}{u\sqrt{1+p^2+q^2}} \left( p \frac{du}{dx} + q \frac{du}{dy} - \frac{du}{dz} \right) = 0.$$

If  $R, R'$  be the principal radii of curvature, and  $l, m, n$  the direction cosines of the normal, this equation may be written

**3079** 
$$\frac{1}{R} + \frac{1}{R'} + \frac{1}{u} \left( l \frac{du}{dx} + m \frac{du}{dy} + n \frac{du}{dz} \right) = 0,$$

and according to the nature of the function  $u$  different geometrical theorems may be deduced.

**3080** PROPOSITION II.—To find the surface  $S$  which will make

$$\iint F(x, y, z) dS + \iint f(x, y, z) dx dy$$

a maximum or minimum;  $F$  and  $f$  being given functions of the coordinates  $x, y, z$ .

Let  $F(x, y, z) = u$  and  $f(x, y, z) = v$ . Proceeding throughout as in (3078), we have

$$V = u\sqrt{1+p^2+q^2} + v,$$

$$Z = \sqrt{1+p^2+q^2} u_z + v_z,$$

and the remaining equations the same as in that article if we add to the resulting differential equation the term  $-\frac{v_z}{u}$  on the left.

This equation may then be put in the form

$$3081 \quad \frac{1}{R} + \frac{1}{R'} = -\frac{1}{u} \left( l \frac{du}{dx} + m \frac{du}{dy} + n \frac{du}{dz} - \frac{dv}{dz} \right),$$

where  $l, m, n$  are the direction cosines of the normal to the surface.

**3082** Ex.—To find a surface of given area such that the volume contained by it shall be a maximum.

By (3069), the integral  $\iint (z - a \sqrt{1 + p^2 + q^2}) dx dy$

must take a maximum or minimum value. The problem is a case of (3080).

We have  $u = -a, v = z, u_x = 0, u_y = 0, u_z = 0, v_z = 1$ ;

and the differential equation of the surface (3081) reduces to

$$(1 + q^2) r - 2pq s + (1 + p^2) t + \frac{1}{a} (1 + p^2 + q^2)^{\frac{3}{2}} = 0;$$

$$3083 \quad \text{or} \quad \frac{1}{R} + \frac{1}{R'} = \frac{1}{a}.$$

## APPENDIX.

### ON THE GENERAL OBJECT OF THE CALCULUS OF VARIATIONS.

**3084** DEFINITIONS.—A function whose form is invariable is called *determinate*, and one whose form is variable, *indeterminate*.

Let  $du$  be the increment of a function  $u$  due to a change in the magnitude of the independent variable,  $\delta u$  that due to a change in the form of the function,  $Du$  the total increment from both causes; then

$$Du = du + \delta u.$$

Thus, in (3042), the terms involving  $dx_1$  and  $dx_0$  constitute  $du$ , and the remaining terms  $\delta u$ ; the whole variation being  $Du$ .

$\delta u$  is called the *variation* of the function  $u$ .

**3085** A *primitive* indeterminate function,  $u$ , of any number of variables is a function whose variation is of arbitrary but constant form; in other words,  $\delta^2 u = 0$ .



**3086** Let  $v = F.u$  be a *derived* function,—that is, a function derived by some process from the function  $u$ ;  $F$  denoting a relation between the forms, but not between the magnitudes, of  $u$  and  $v$ .

The general object of the Calculus of Variations is to determine the change in a derived function  $v$ , caused by a change in the form of its primitive  $u$ .

The particular derived functions considered are those whose symbols are  $d$  and  $\int$ , denoting operations of differentiation and integration respectively.

#### SUCCESSIVE VARIATION.

**3087** Let the variation of the variation, or second variation of  $V$  due to a change in the form of the involved function,  $y = f(x)$ , be denoted by  $\delta(\delta V)$  or  $\delta^2 V$ ; the third variation by  $\delta^3 V$ , and so on.

By definition (3085),  $y$  being a primitive indeterminate function, and  $\delta y$  its variation,  $\delta^2 y = 0$  ..... (1).

**3088** The second variation of any derivative of  $y$  is also zero, i.e.,  $\delta^2 p$ ,  $\delta^2 q$ , &c. all vanish.

PROOF.—  $\delta^2(y_{xz}) = \delta(\delta y_{xz}) = \{\delta(\delta y)\}_{xz} = (\delta^2 y)_{xz} = 0$  by (1).

**3089** If  $V = f(x, y, p, q, r, \&c. \dots)$ , where  $y$  is a primitive indeterminate function of  $x$ , then

$$\delta^2 V = (\delta y d_y + \delta p d_p + \delta q d_q + \dots)^2 V,$$

where, in the formal expansion by the multinomial theorem,  $\delta y$ ,  $\delta p$ , &c. follow the law of involution, but the indices of  $d_y$ ,  $d_p$ , &c. indicate repetition of the operation  $d_y$ ,  $d_p$ , &c. upon  $V$ .

PROOF.—First,  $\delta V = (\delta y d_y + \delta p d_p + \delta q d_q + \dots) V$ .

In finding  $\delta^2 V$ , each product, such as  $\delta y d_y V$ , is differentiated again as a function of  $y$ ,  $p$ ,  $q$ , &c.; but, since the variations of  $\delta y$ ,  $\delta p$ , &c. vanish by (2), it is the same in effect as though  $\delta y$ ,  $\delta p$ , &c. were not operated upon at all. They accordingly rank as algebraic quantities merely, and therefore

$$\delta^2 V = (\delta y d_y + \delta p d_p + \delta q d_q + \dots)^2 V.$$

Similarly for a third differentiation; and so on.

#### IMMEDIATE INTEGRABILITY OF THE FUNCTION $V$ .

**3090** DEF.—When the function  $V$  (3028) is integrable without assigning the value of  $y$  in terms of  $x$ , and therefore

integrable whatever the form of the function  $y$  may be, it is said to be *immediately integrable*, or integrable *per se*.

**3091** The requisite condition for  $V$  to be immediately integrable is that  $K = 0$  shall be identically true.

PROOF.—  $\int_{x_0}^{x_1} V dx$  must be expressible in the form

$$\phi(x_1 y_1 p_1 q_1 \dots) - \phi(x_0 y_0 p_0 q_0 \dots),$$

where  $\phi$  is independent of the form of  $y$ . Hence, a change in the form of  $y$ , which leaves the values at the limits unaltered, will leave

$$\delta \int_{x_0}^{x_1} V dx = 0; \quad \text{that is, } \int_{x_0}^{x_1} K \delta y = 0.$$

But the last equation necessitates  $K = 0$ , since  $\delta y$  is arbitrary. And  $K = 0$  must be identically true, otherwise it would determine  $y$  as a function of  $x$ .

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# DIFFERENTIAL EQUATIONS.

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## GENERATION OF DIFFERENTIAL EQUATIONS.

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**3050** By differentiating ordinary algebraic equations, and eliminating constants or functions, differential equations are produced. Some methods are illustrated in the following examples.

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**3051** From an equation between two variables and  $n$  arbitrary constants, to eliminate the constants.

**RULE.**—Differentiate  $r$  times ( $r < n$ ), and from the  $r+1$  equations any  $r$  constants may be eliminated, and thus  $C(n, r)$  different equations of the  $r^{\text{th}}$  order (3060) obtained, involving  $\frac{d^r y}{dx^r}$ ,  $\frac{d^{r-1} y}{dx^{r-1}}$ , &c. Only  $r+1$ , however, of these equations will be independent. By differentiating  $n$  times and eliminating the constants, a single final differential equation of the  $n^{\text{th}}$  order free from constants may be obtained.

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**3052** Ex.—To eliminate the constants  $a$  and  $b$  from the equation

$$y = ax^3 + bx \dots\dots\dots (i.)$$

Differentiating, we find  $\frac{dy}{dx} = 2ax + b \dots\dots\dots (ii.)$

Eliminating  $a$  and  $b$  in turn, we get

$$x \frac{dy}{dx} + bx = 2y, \quad x \frac{dy}{dx} = ax^3 + y \dots\dots\dots (iii., iv.)$$

Now, differentiating (iii.) and eliminating  $b$  produces the final equation of the second order,

$$x^3 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0 \dots\dots\dots (v.)$$

The same equation is obtained by differentiating (iv.) and eliminating  $a$ .

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**3053** To eliminate the function  $\phi$  from the equation  $z = \phi(v)$ , where  $v$  is a function of  $x$  and  $y$ . We have

$$z_x = \phi'(v) v_x, \quad z_y = \phi'(v) v_y.$$

Therefore

$$z_x v_y = z_y v_x.$$

**3054** To eliminate  $\phi$  from  $u = \phi(v)$ , where  $u$  and  $v$  are functions of  $x, y, z$ .

Consider  $x$  and  $y$  the independent variables, and differentiate for each separately, thus

$$u_x + u_z z_x = \phi'(v)(v_x + v_z z_x),$$

$$u_y + u_z z_y = \phi'(v)(v_y + v_z z_y),$$

and, by division,  $\phi'(v)$  is eliminated.

**3055** To eliminate  $\phi_1, \phi_2, \dots \phi_n$  from the equation

$$F\{x, y, z, \phi_1(a_1), \phi_2(a_2), \dots \phi_n(a_n)\} = 0,$$

where  $a_1, a_2, \dots a_n$  are known functions of  $x, y, z$ .

**RULE.**—Differentiate for  $x$  and  $y$  as independent variables, forming the derivatives of  $F$  of each order, up to the  $(2n-1)^{\text{th}}$  in every possible way; that is,  $F; F_x, F_y; F_{2x}, F_{xy}, F_{2y}; \&c.$  There will be  $2n^2$  unknown functions, consisting of  $\phi_1, \phi_2, \dots \phi_n$  and their derivatives, and  $2n^2+n$  equations for eliminating them.

**3056** To eliminate  $\xi, \phi_1(\xi), \phi_2(\xi), \dots \phi_n(\xi)$  between the equations

$$F\{x, y, z, \xi, \phi_1(\xi), \phi_2(\xi) \dots \phi_n(\xi)\} = 0,$$

$$f\{x, y, z, \xi, \phi_1(\xi), \phi_2(\xi) \dots \phi_n(\xi)\} = 0.$$

**RULE.**—Consider  $z$  and  $\xi$  functions of the independent variables  $x, y$ , and form the derivatives of  $F$  and  $f$  up to the  $2n-1^{\text{th}}$  order in the manner described in (3055). There will be  $4n^2+n$  functions, and  $4n^2+2n$  equations for eliminating them.

**3057** To eliminate  $\phi$  from the equation

$$F\{x, y, z, w, \phi(a, \beta)\} = 0,$$

where  $a, \beta$  are known functions of  $x, y, z, w$ .

**RULE.**—Consider  $x, y, z$  the independent variables. Differentiate for each, and eliminate  $\phi, \phi_a, \phi_\beta$  between the four equations.

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 DEFINITIONS AND RULES.
 

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**3058** *Ordinary* differential equations involve the derivatives of a *single* independent variable.

**3059** *Partial* differential equations involve partial derivatives, and therefore two or more independent variables are concerned.

**3060** The *order* of a differential equation is the order of the highest derivative which it contains.

**3061** The *degree* of a differential equation is the power to which the highest derivative is raised.

**3062** A *Linear* differential equation is one in which the derivatives are all involved in the first degree.

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**3063** The *complete primitive* of a differential equation is that equation between the primitive variables from which the differential equation may be obtained by the process of differentiation.

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**3064** The *general solution* is the name given to the complete primitive when it has been obtained by solving the given differential equation.

Thus, reverting to the example in (3051), equation (i.) is the *complete primitive* of (v.) which is obtained from (i.) by differentiation and elimination.

The *differential equation* (v.) being given, the process is reversed. Equations (iii.) and (iv.) are called the *first integrals* of (v.), and equation (i.) the *final integral* or *general solution*.

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**3065** A *particular solution*, or *particular integral*, of a differential equation is obtained by giving particular values to the arbitrary constants in the general solution.

For the definition of a *singular solution*, see (3068).

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**3066** To find when two differential equations of the first order have a common primitive.

**RULE.**—Differentiate each equation, and eliminate its arbitrary constant. The two results will agree if there is a common primitive, which, in that case, will be found by eliminating  $y_1$  between the given equations.

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**Ex.**—Apply the rule to equations (iii.) and (iv.) in (3052).

**3067** To find when two solutions of a differential equation, each involving an arbitrary constant, are equivalent.

**RULE.**—*Eliminate one of the variables. The other will also disappear, and a relation between the arbitrary constants will remain.*

Otherwise, if  $V=C$ ,  $v=c$  be the two solutions:  $V$  and  $v$  being functions of the variables, and  $C$  and  $c$  constants; then

$$\frac{dV}{dx} \frac{dv}{dy} = \frac{dV}{dy} \frac{dv}{dx}$$

is the required condition.

**PROOF.**— $V$  must be a function of  $v$ . Let  $V = \phi(v)$ ; therefore  $V_x = \phi_v v_x$  and  $V_y = \phi_v v_y$ ; then eliminate  $\phi_v$ .

**Ex.**— $\tan^{-1}(x+y) + \tan^{-1}(x-y) = a$  and  $x^2 + 2bx = y^2 + 1$  are both solutions of  $2xyy_x = x^2 + y^2 + 1$ . Eliminating  $y$ ,  $x$  disappears, and the resulting equation is  $b \tan a = 1$ .

## SINGULAR SOLUTIONS.

**3068** **DEFINITION.**—“A *singular solution* of a differential equation is a relation between  $x$  and  $y$  which satisfies the equation by means of the values which it gives to the differential coefficients  $y_x$ ,  $y_{2x}$ , &c., but is not included in the complete primitive.” See examples (3132–3).

**3069** To find a singular solution from the complete primitive  $\phi(x, y, c) = 0$ .

**RULE I.**—*From the complete primitive determine  $c$  as a function of  $x$ , by solving the equation  $y_c = 0$ , or else by solving  $x_c = 0$ , and substitute this value of  $c$  in the primitive. The result is a singular solution, unless it can also be obtained by giving to  $c$  a constant value in the primitive.*

**3070** If the singular solution involves  $y$  only, it results from the equation  $y_c = 0$  only, and if it involves  $x$  only, it results from  $x_c = 0$  only. If it involves both  $x$  and  $y$ , the two equations  $x_c = 0$ ,  $y_c = 0$  give the same result.





**3071** When the primitive equation  $\phi(x, y, c) = 0$  is a rational integral function,  $\phi_c = 0$  may be used instead of  $x_c = 0$  or  $y_c = 0$ .

PROOF.—Let  $\phi(x, y, c) = 0$  be expressed in the form

$$y = f(x, c) \dots\dots\dots (1).$$

$$\text{Then, if } c \text{ be constant, } y_x = f_x \dots\dots\dots (2);$$

$$\text{and, if } c \text{ varies, } y_x = f_x + f_c c_x \dots\dots\dots (3).$$

When  $c$  is constant, the differential equation of which (1) is the primitive is satisfied by the value of  $y_x$  in (2). But it will also be satisfied by the same value of  $y_x$  when  $c$  is variable, provided that either  $f_c = 0$  or  $f_c = \infty$ , and in either case a solution is obtained which is not the result of giving to  $c$  a constant value in the complete primitive; that is, it is a singular solution. But  $f_c = 0$  is equivalent to  $y_c = 0$ , and  $f_c = \infty$  makes  $y_x = \infty$ , and therefore  $x = \text{constant}$ .

#### GEOMETRICAL MEANING OF A SINGULAR SOLUTION.

**3072** Since the process in Rule I. is identical with that employed in finding the envelope of the series of curves obtained by varying the parameter  $c$  in the equation  $\phi(x, y, c) = 0$ ; the singular solution so obtained is the equation of the envelope itself.

An exception occurs when the envelope coincides with one of the curves of the system.

**3073** Ex.—Let the complete primitive be

$$y = cx + \sqrt{1 - c^2}, \text{ therefore } y_c = x - \frac{c}{\sqrt{1 - c^2}}; \quad y_c = 0 \text{ gives } c = \frac{x}{\sqrt{1 + x^2}}.$$

Substituting this in the primitive gives  $y = \sqrt{1 + x^2}$ , a singular solution. It is the equation of the envelope of all the lines that are obtained by varying the parameter  $c$  in the primitive; for it is the equation of a circle, and the primitive, by varying  $c$ , represents all lines which touch the circle. See also (3132-3).

**3074** “The determination of  $c$  as a function of  $x$  by the solution of the equation  $y_c = 0$ , is equivalent to determining what particular primitive has contact with the envelop at that point of the latter which corresponds to a given value of  $x$ .

“The elimination of  $c$  between a primitive  $y = f(x, c)$  and the derived equation  $y_c = 0$ , does not necessarily lead to a singular solution in the sense above explained.

“For it is possible that the derived equation  $y_c = 0$  may neither, on the one hand, enable us to determine  $c$  as a function of  $x$ , so leading to a singular solution; nor, on the other hand, as an absolute constant, so leading to a particular primitive.

"Thus the particular primitive  $y = e^{cx}$  being given, the condition  $y_c = 0$  gives  $e^{cx} = 0$ , whence  $c$  is  $+\infty$  if  $x$  be negative, and  $-\infty$  if  $x$  be positive. It is a dependent constant. The resulting solution  $y = 0$  does not then represent an envelope of the curves of particular primitives, nor strictly one of those curves. It represents a curve formed of branches from two of them. It is most fitly characterised as a particular primitive marked by a singularity in the mode of its derivation from the complete primitive."

[Boole's "Differential Equations," Supplement, p. 13.]

#### DETERMINATION OF A SINGULAR SOLUTION FROM THE DIFFERENTIAL EQUATION.

**3075** RULE II.—*Any relation is a singular solution which, while it satisfies the differential equation, either involves  $y$  and makes  $p_y$  infinite, or involves  $x$  and makes  $\left(\frac{1}{p}\right)_x$  infinite.*

**3076** "One negative feature marks all the cases in which a solution involving  $y$  satisfies the condition  $p_y = \infty$ . It is, that the solution, while expressed by a single equation, is not connected with the complete primitive by a single and absolutely constant value of  $c$ .

"The relation which makes  $p_y$  infinite satisfies the differential equation only because it satisfies the condition  $y_c = 0$ , and this implies a connexion between  $c$  and  $x$ , which is the ground of a real, though it may be unimportant, singularity in the solution itself.

"In the first, or, as it might be termed, the envelope species of singular solutions,  $c$  receives an infinite number of different values connected with the value of  $x$  by a law. In the second, it receives a finite number of values also connected with the values of  $x$  by a law. In the third species, it receives a finite number of values, determinate, but not connected with the values of  $x$ ."

Hence the general inclusive definition—

**3077** "A singular solution of a differential equation of the first order is a solution the connexion of which with the complete primitive does not consist in giving to  $c$  a single constant value absolutely independent of the value of  $x$ ."

[Boole's "Differential Equations," p. 163, and Supplement, p. 19.]

#### RULES FOR DISCRIMINATING A SINGULAR SOLUTION OF THE ENVELOPE SPECIES.

**3078** RULE III.—*When  $p_y$  or  $\left(\frac{1}{p}\right)_x$  is made infinite by equating to zero a factor having a negative index, the solution "may be considered to belong to the envelope species."*

**3079** "In other cases, the solution is deducible from the

complete primitive by regarding  $c$  as a constant of multiple value,—its particular values being either, 1st, dependent in some way on the value of  $x$ , or, 2ndly, independent of  $x$ , but still such as to render the property a singular one.”

[Boole's "Differential Equations," p. 164.]

**3080** RULE IV.—*A solution which, while it makes  $p_y$  infinite and satisfies the differential equation of the first order, does not satisfy all the higher differential equations obtained from it, is a singular solution of the envelope species.*

Ex.:  $y_x = my^{\frac{m-1}{m}}$  has the singular solution  $y = 0$  when  $m$  is  $> 1$ .

Now  $y_{rx} = m(m-1) \dots (m-r+1)y^{\frac{m-r}{m}}$ ,

and, when  $r$  is  $> m$ , the value  $y = 0$  makes  $y_{rx}$  infinite. The solution is, therefore, by the rule of the envelope species.

**3081** RULE V.—“The proposed solution being represented by  $u = 0$ , let the differential equation, transformed by making  $u$  and  $x$  the variables, be  $u_x + f(x, u) = 0$ . Determine the integral  $\int_0^u \frac{du}{U}$  as a function of  $x$  and  $u$ , in which  $U$  is either equal to  $f(x, u)$  or to  $f(x, u)$  deprived of any factor which neither vanishes nor becomes infinite when  $u = 0$ . If that integral tends to zero with  $u$ , the solution is singular” and of the envelope species.

[Boole, Supplement, p. 30.]

**3082** Ex.—To determine whether  $y = 0$  is a singular solution or particular integral of

$$y_x = y(\log y)^2.$$

Here  $u = y$ , and

$$\int_0^u \frac{dy}{y(\log y)^2} = -\frac{1}{\log y}.$$

As this tends to zero with  $y$ , the solution is singular.

Verification.—The complete primitive is  $y = e^{\frac{1}{c-x}}$ , and no constant value assigned to  $c$  will produce the result  $y = 0$ .

**3083** Professor De Morgan has shown that any relation involving both  $x$  and  $y$ , which satisfies the conditions  $p_y = \infty$ ,  $p_x = \infty$ , will satisfy the differential equation when it does not make  $y_{2x}$ , as derived from it, infinite; that it may satisfy it even if it makes  $y_{2x}$  infinite; and that, if it does not satisfy the differential equation, the curve it represents is a locus of points of infinite curvature, usually cusps, in the curves of complete primitives.

[Boole, Supplement, p. 35.]

## FIRST ORDER LINEAR EQUATIONS.

**3084**  $M + N \frac{dy}{dx} = 0, \quad \text{or} \quad Mdx + Ndy = 0,$

$M$  and  $N$  being either functions of  $x$  and  $y$  or constants.

## SOLUTION BY SEPARATION OF THE VARIABLES.

**3085** This method of solution, when practicable, is the simplest, and is frequently involved in other methods.

Ex.  $xy(1+x^2)dy = (1+y^2)dx,$

therefore  $\frac{ydy}{1+y^2} = \frac{dx}{x(1+x^2)},$

and each member can be at once integrated.

## HOMOGENEOUS EQUATIONS.

**3086** Here  $M$  and  $N$ , in (3084), are homogeneous functions of  $x$  and  $y$ , and the solution is affected as follows:—

RULE.—Put  $y = vx$ , and therefore  $dy = vdx + xdv$ , and then separate the variables. For an example, see (3108).

## EXACT DIFFERENTIAL EQUATIONS.

**3087**  $Mdx + Ndy = 0$  is an exact differential when

$$M_y = N_x,$$

and the solution is then obtained by the formula

$$\int Mdx + \int \{N - d_y(\int Mdx)\} dy = C.$$

PROOF.—If  $V = 0$  be the primitive, we must have  $V_x = M$ ,  $V_y = N$ ; therefore  $V_{xy} = M_y = N_x$ . Also  $V = \int Mdx + \phi(y)$ ,  $\phi(y)$  being a constant with respect to  $x$ .

Therefore  $N = V_y = d_y \int Mdx + \phi'(y),$

therefore  $\phi(y) = \int \{N - d_y \int Mdx\} dy + C.$

**3088** Ex.  $(x^3 - 3x^2y)dx + (y^3 - x^3)dy = 0.$

Here  $M_y = -3x^2 = N_x$ . Therefore the solution is

$$\begin{aligned} C &= \frac{x^4}{4} - x^2y + \int \left\{ y^3 - x^3 - d_y \left( \frac{x^4}{4} - x^2y \right) \right\} dy \\ &= \frac{x^4}{4} - x^2y + \int y^3 dy = \frac{x^4 + y^4}{4} - x^2y. \end{aligned}$$

**3089** Observe that, if  $Mdx + Ndy$  can be separated into two parts, so that one of them is an exact differential, the other part must also be an exact differential in order that the whole may be such.

**3090** Also, if a function of  $x$  and  $y$  can be expressed as the product of two factors, one of which is a function of the integral of the other, the original function is an exact differential.

**3091** Ex.—  $\frac{1}{y} \cos \frac{x}{y} dx - \frac{x}{y^2} \cos \frac{x}{y} dy = \cos \frac{x}{y} \cdot \frac{y dx - x dy}{y^2} = 0.$

Here  $\frac{x}{y}$  is the integral of the second factor. Hence the solution is

$$\sin \frac{x}{y} = C.$$

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INTEGRATING FACTOR FOR  $Mdx + Ndy = 0$ .

When this equation is not an exact differential, a factor which will make it such can be found in the following cases.

**3092** I.—When one only of the functions  $Mx + Ny$  or  $Mx - Ny$  vanishes identically, the reciprocal of the other is an integrating factor.

**3093** II.—If, when  $Mx + Ny = 0$  identically, the equation is at the same time homogeneous, then  $x^{-(n+1)}$  is also an integrating factor.

**3094** III.—If neither  $Mx + Ny$  nor  $Mx - Ny$  vanishes identically, then, when the equation is homogeneous,  $\frac{1}{Mx + Ny}$  is an integrating factor; and when the equation can be put in the form  $\phi(xy) x dy + \chi(xy) y dx = 0$ ,  $\frac{1}{Mx - Ny}$  is an integrating factor.

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PROOF.—I. and III.—From the identity

$$Mdx + Ndy = \frac{1}{2} \left\{ (Mx + Ny) d \log xy + (Mx - Ny) d \log \frac{x}{y} \right\},$$



assuming the integrating factor in each case, and deducing the required forms for  $M$  and  $N$ , employing (3090).

II.—Put  $v = \frac{y}{x}$ ,  $M = x^n \phi(v)$ ,  $N = x^n \psi(v)$ , and  $dy = x dv + v dx$  in  $Mdx + Ndy$  and  $Mx + Ny$ .

**3095** The general form for an integrating factor of  $Mdx + Ndy = 0$  is

$$\mu = e^{\int \frac{M_y - N_x}{Nv_x - Mv_y} dv},$$

where  $v$  is some chosen function of  $x$  and  $y$ ; and the condition for the existence of an integrating factor under that hypothesis is that

**3096**  $\frac{M_y - N_x}{Nv_x - Mv_y}$  must be a function of  $v$ .

PROOF.—The condition for an exact differential of  $M\mu dx + N\mu dy = 0$  is  $(M\mu)_y = (N\mu)_x$  (3087). Assume  $\mu = \phi(v)$ , and differentiate out; we thus obtain

$$\frac{\phi'v}{\phi v} = \frac{M_y - N_x}{Nv_x - Mv_y}.$$

The following are cases of importance.

**3097** I.—If an integrating factor is required which is a function of  $x$  only, we put  $\mu = \phi(x)$ , that is,  $v = x$ ; and the necessary condition becomes

$$\frac{M_y - N_x}{N} \text{ must be a function of } x \text{ only.}$$

**3098** II.—If the integrating factor is to be a function of  $xy$ , the condition becomes, by putting  $xy = v$ ,

$$\frac{M_y - N_x}{Ny - Mx} \text{ must be a function of } xy \text{ only.}$$

**3099** III.—If the integrating factor is to be a function of  $\frac{y}{x}$ , the condition is

$$\frac{x^2(N_x - M_y)}{Mx + Ny} \text{ must be a function of } \frac{y}{x}.$$

If  $Mx + Ny$  vanishes, (3092) must be resorted to.



In this and similar cases, the expression found will be a function of  $v = \frac{y}{x}$  if it takes the form  $F(v)$  when  $y$  is replaced by  $vx$ .

**3100** IV.—*Theorem.*—The condition that the equation  $Mdx + Ndy = 0$  may have a homogeneous function of  $x$  and  $y$  of the  $n^{\text{th}}$  degree for an integrating factor, is

$$\frac{x^2(N_x - M_y) + nNx}{Mx + Ny} = F(u), \quad \text{where } u = \frac{y}{x}.$$

**3101** The integrating factor will then be obtained from

$$\mu = x^n e^{\int F(u) du}.$$

PROOF.—Put  $\mu = v = x^n \psi(u)$  in (3097), thus

$$\frac{1}{v} = \frac{M_y - N_x}{Nv_x - Mv_y}.$$

Perform the differentiations, and, by reduction, we get

$$\frac{\psi'(u)}{\psi(u)} = \frac{x^2(N_x - M_y) + nNx}{Mx + Ny}.$$

The right member must be a function of  $u$  in order that  $\psi(u)$  may be found by integration.

**3102** Ex.—To ascertain whether an integrating factor, which is a homogeneous function of  $x$  and  $y$ , exists for the equation

$$(y^3 + axy^2) dy - ay^2 dx + (x + y)(x dy - y dx) = 0.$$

Here  $M = -(ay^3 + xy + y^2)$ ,  $N = (y^3 + axy^2 + xy + x^2)$ .

Substituting in the formula of (3100), we find that, by choosing  $n = -3$ , the fraction reduces to  $\frac{ax^2y^2 - 3xy^3}{y^4}$ , and, by putting  $y = ux$ , it becomes  $\frac{a - 3u}{u^3}$ , a function of  $u$ .

$$\int \frac{a - 3u}{u^3} du = -\frac{a}{u} - 3 \log u,$$

$$\therefore \mu = x^{-3} e^{-\left(\frac{ax}{y} + 3 \log \frac{y}{x}\right)} = y^{-3} e^{-\frac{ax}{y}},$$

the integrating factor required. It is homogeneous, and of the degree  $-3$  in  $x$  and  $y$ , as is seen by expanding the second factor by (150).

**3103** If by means of the integrating factor  $\mu$  the equation  $\mu M dx + \mu N dy = 0$  is found to have  $V = C$  for its complete primitive, then the form for all other integrating factors will be  $\mu f(V)$ , where  $f$  is any arbitrary function.

PROOF.—The equation becomes

$$\mu Mf(V) dx + \mu Nf(V) dy = 0.$$

Applying the test of integrability (3087), we have

$$\{\mu Mf(V)\}_y = \{\mu Nf(V)\}_x.$$

Differentiate out, remembering that

$$(\mu M)_y = (\mu N)_x, \quad V_y = \mu N, \quad V_x = \mu M,$$

and the equality is established.

**3104** GENERAL RULE.—*Ascertain by the determination of an integrating factor that an equation is solvable, and then seek to effect the solution in some more direct way.*

#### SOME PARTICULAR EQUATIONS.

**3105**  $(ax+by+c) dx + (a'x+b'y+c') dy = 0.$

This equation may be solved in three ways.

I.—Substitute  $x = \xi - \alpha$ ,  $y = \eta - \beta$ ,

and determine  $\alpha$  and  $\beta$  so that the constant terms in the new equation in  $\xi$  and  $\eta$  may vanish.

II.—Or substitute  $ax+by+c = \xi$ ,  $a'x+b'y+c' = \eta$ .

**3106** But if  $a : a' = b : b'$ , the methods I. and II. fail. The equation may then be written as a function of  $ax+by$ .

Put  $z = ax+by$ , and substitute  $b dy = dz - a dx$ , and afterwards separate the variables  $x$  and  $z$ .

**3107** III.—A third method consists in assuming

$$(A\eta + C) d\xi + (A'\xi + C') d\eta = 0,$$

and equating coefficients with the original equation after substituting

$$\xi = x + m_1 y, \quad \eta = x + m_2 y.$$

$m_1, m_2$  are the roots of the quadratic

$$am^2 + (b+a')m + b' = 0.$$

The solution then takes the form

$$\frac{\{(am_1 - a')(x + m_1 y) + cm_1 - c'\}^{\frac{1}{am_1 - a'}}}{\{(am_2 - a')(x + m_2 y) + cm_2 - c'\}^{\frac{1}{am_2 - a'}}} = C.$$

**3108** Ex.  $(3y - 7x + 7) dx + (7y - 3x + 3) dy = 0$ .

Put  $x = \xi - \alpha$ ,  $y = \eta - \beta$ , thus

$$(3\eta - 7\xi) d\xi + (7\eta - 3\xi) d\eta = 0 \dots\dots\dots (i.),$$

with equations for  $\alpha$  and  $\beta$ ,  $7\alpha - 3\beta + 7 = 0$ ;  $3\alpha - 7\beta + 3 = 0$ ;

therefore  $\alpha = -1$ ,  $\beta = 0 \dots\dots\dots (ii.)$

(i.) being homogeneous, put  $\eta = v\xi$ , and therefore  $d\eta = v d\xi + \xi dv$  (3086);

$$\therefore (7v^2 - 7) \xi d\xi + (7v - 3) \xi^2 dv = 0, \text{ or } \frac{d\xi}{\xi} + \frac{7v - 3}{7v^2 - 7} dv = 0.$$

The second member is integrated, as in (2080), with  $b = 0$ , and, after reduction, we find  $5 \log(\eta + \xi) + 2 \log(\eta - \xi) = C$ .

Putting  $\xi = x - 1$  and  $\eta = y$ , by (ii.) the complete solution is  $(y + x - 1)^5 (y - x + 1)^2 = C$ .

**3109** When  $P$  and  $Q$  are functions of  $x$  only, the solution of the equation

$$\frac{dy}{dx} + Py = 0 \quad \text{is} \quad y = Ce^{-\int P dx} \dots\dots\dots (i.)$$

by merely separating the variables.

**3110** Secondly, the solution of

$$\frac{dy}{dx} + Py = Q \quad \text{is} \quad y = e^{-\int P dx} \left\{ C + \int Q e^{\int P dx} dx \right\}.$$

This result is obtained by the method of *variation of parameters*.

**RULE.**—Assume equation (i.) to be the form of the solution, considering the parameter  $C$  a function of  $x$ . Differentiate (i.) on this hypothesis, and put the value of  $y_x$  so obtained in the proposed equation to determine  $C$ .

Thus, differentiating (i.), we get  $y_x = C_x e^{-\int P dx} - Py$ ,

therefore  $Q = C_x e^{-\int P dx}$ , therefore  $C = \int Q e^{\int P dx} dx + C'$ .

Then substitute this expression for  $C$  in equation (i.).

Otherwise, writing the equation in the form  $(Py - Q) dx + dy = 0$ , the integrating factor  $e^{\int P dx}$  may be found by (3097).

**3111**  $y_x + Py = Qy^n$

is reduced to the last case by dividing by  $y^n$  and substituting

$$z = y^{1-n}.$$

**\*3212**  $P_1 dx + P_2 dy + Q(x dy - y dx) = 0.$

$P_1, P_2$  being homogeneous functions of  $x$  and  $y$  of the  $p^{\text{th}}$  degree, and  $Q$  homogeneous and of the  $q^{\text{th}}$  degree, is solved by assuming

$$P_1 = x^p \phi\left(\frac{y}{x}\right), \quad P_2 = x^p \psi\left(\frac{y}{x}\right), \quad Q = x^q \chi\left(\frac{y}{x}\right).$$

Put  $y = vx$ , and change the variables to  $x$  and  $v$ . The result may be reduced to

$$\frac{dv}{dx} + \frac{x\psi(v)}{\phi(v) + v\psi(v)} = -\frac{\chi(v) x^{q-p+2}}{\phi(v) + v\psi(v)},$$

which is identical in form with (3211), and may be solved accordingly.

**3213**  $(A_1 + B_1 x + C_1 y)(x dy - y dx) - (A_2 + B_2 x + C_2 y) dy - (A_3 + B_3 x + C_3 y) dx = 0.$

To solve this equation, put  $x = \xi + \alpha$ ,  $y = \eta + \beta$ , and determine  $\alpha$  and  $\beta$  so that the coefficients may become homogeneous, and the form of (3212) will be obtained.

#### RICCATI'S EQUATION.

**3214**  $u_x + bu^2 = cx^m \dots\dots\dots (A).$

Substitute  $y = ux$ , and this equation is reduced to the form of the following one, with  $n = m + 2$  and  $a = 1$ . It is solvable whenever  $m(2t \pm 1) = -4t$ ,  $t$  being 0 or a positive integer.

**3215**  $xy_x - ay + by^2 = cx^n \dots\dots\dots (B).$

I.—This equation is solvable, when  $n = 2a$ , by substituting  $y = vx^a$ , dividing by  $x^{2a}$ , and separating the variables. We

thus obtain 
$$\frac{dv}{c - bv^2} = x^{a-1} dx.$$

Integrating by (1937) or (1935), according as  $b$  and  $c$  in equation (B) have the same or different signs, and eliminating  $v$  by  $y = vx^a$ , we obtain the solution

**3216** 
$$y = \sqrt{\frac{c}{b}} x^a \frac{Ce^{\frac{2x^a \sqrt{bc}}{a}} + 1}{Ce^{\frac{2x^a \sqrt{bc}}{a}} - 1} \dots\dots\dots (1),$$

\* The preceding articles of this section are wrongly numbered. Each number and reference to it, up to this point, should be increased by 100. The sheets were printed off before the error was discovered.

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**3217** or  $y = \sqrt{\left(-\frac{c}{b}\right)x^a \tan \left\{C - \frac{x^a \sqrt{(-bc)}}{a}\right\}} \dots\dots (2).$

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**3218** II.—Equation (B) may also be solved whenever  $\frac{n-2a}{2n} = t$  a positive integer.

*RULE.*—Write  $z$  for  $y$  in equation (B), and  $nt+a$  for  $a$  in the second term, and transpose  $b$  and  $c$  if  $t$  be odd.

Thus, we shall have

$$xz_x - (nt+a)z + bz^2 = cx^n \text{ (when } t \text{ is even) } \dots\dots (3),$$

$$xz_x - (nt+a)z + cz^2 = bx^n \text{ (when } t \text{ is odd) } \dots\dots (4).$$

Either of these equations can be solved as in case (I.), when  $n = 2(nt+a)$ , that is, when  $\frac{n-2a}{2n} = t$ .  $z$  having been determined by such a solution, the complete primitive of (B) will be the continued fraction

$$y = \frac{a}{b} + \frac{\frac{x^n}{n+a}}{c} + \frac{\frac{x^n}{2n+a}}{b} + \dots + \frac{\frac{x^n}{(t-1)n+a}}{k} + \frac{x^n}{z} \dots (5),$$

where  $k$  stands for  $b$  or  $c$  according as  $t$  is odd or even.

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**3219** III.—Equation (B) can also be solved whenever  $\frac{n+2a}{2n} = t$  a positive integer. The method and result will be the same as in Case II., if the sign of  $a$  be changed throughout and the first fraction omitted from the value of  $y$ . Thus

$$y = \frac{\frac{x^n}{n-a}}{c} + \frac{\frac{x^n}{2n-a}}{b} + \dots + \frac{\frac{x^n}{(t-1)n-a}}{k} + \frac{x^n}{z} \dots\dots (6).$$


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*PROOF.*—Case II.—In equation (B), substitute  $y = A + \frac{x^n}{y_1}$ , and equate the absolute term to zero. This gives  $A = \frac{a}{b}$  or 0.

Taking the first value, the transformed equation becomes

$$x \frac{dy_1}{dx} - (n+a)y_1 + cy_1^2 = bx^n.$$

Next, put  $y_1 = \frac{n+a}{c} + \frac{x^n}{y_2}$ , and so on. In this way the  $t^{\text{th}}$  transformed equation (3) or (4) is obtained with  $z$  written for the  $t^{\text{th}}$  substituted variable  $y_t$ .

Case III.—Taking the second value,  $A = 0$ , the first transformed equation differs from the above only in the sign of  $a$ ; and consequently the same series of subsequent transformations arises, with  $-a$  in the place of  $a$ . The successive substitutions produce (5) and (6) in the respective cases for the values of  $y$ .

**3220** Ex.  $u_x + u^2 = cx^{-\frac{1}{2}}. \quad (3214)$

Putting  $u = \frac{y}{x}, \quad \frac{du}{dx} = \frac{xy_x - y}{x^2},$

and the equation is reduced to  $xy_x - y + y^2 = cx^{\frac{3}{2}}$  of the form (B). Here  $a = 1, b = 1, n = \frac{2}{3}$ , and  $\frac{n+2a}{2n} = 2$ , Case III. By the rule in (3218), changing the sign of  $a$  for Case III., equation (3) becomes

$$xz_x - \frac{1}{2}z + z^2 = cx^{\frac{3}{2}}.$$

Solving as in Case I., we put  $z = vx^{\frac{1}{2}}$ , &c.; or, employing formula (1) directly,

$$z = \sqrt{cx^{\frac{1}{2}}} \frac{Ce^{6\sqrt{cx^{\frac{1}{2}}}} + 1}{Ce^{6\sqrt{cx^{\frac{1}{2}}}} - 1}; \text{ and then, by (6), } y = \frac{x^{\frac{3}{2}}}{\frac{1}{3c} + \frac{x^{\frac{1}{2}}}{z}}$$

is the final solution.

## FIRST ORDER NON-LINEAR EQUATIONS.

**3221** *Type*

$$\left(\frac{dy}{dx}\right)^n + P_1\left(\frac{dy}{dx}\right)^{n-1} + \dots + P_{n-1}\frac{dy}{dx} + P_n = 0 \dots (1),$$

where the coefficients  $P_1, P_2, \dots P_n$  may be functions of  $x$  and  $y$ .

### SOLUTION BY FACTORS.

**3222** If (1) can be resolved into  $n$  equations,

$$y_x - p_1 = 0, \quad y_x - p_2 = 0, \quad \dots \quad y_x - p_n = 0 \quad \dots (2),$$

and if the complete primitives of these are

$$V_1 = c_1, \quad V_2 = c_2, \quad \dots \quad V_n = c_n \dots (3),$$

then the complete primitive of the original equation will be

$$(V_1 - c)(V_2 - c) \dots (V_n - c) = 0 \dots (4).$$



PROOF.—Taking  $n = 3$ , assume the last equation. Differentiate and eliminate  $c$ . The result is

$$(V_3 - V_2)^2 (V_3 - V_1)^2 (V_1 - V_2)^2 dV_1 dV_2 dV_3 = 0 \dots \dots \dots (5).$$

By (2),  $dV_1 = \mu_1 (y_3 - p_1) dx$ , &c., where  $\mu_1$  is an integrating factor. Substitute these values in (5), rejecting the factors which do not contain differential coefficients, and the result is

$$(y_3 - p_1)(y_3 - p_2)(y_3 - p_3) = 0,$$

which is the differential equation (1).

**3223** Ex.—Given  $y_x^2 + 3y_x + 2 = 0$ .

The component equations are  $y_x + 1 = 0$  and  $y_x + 2 = 0$ , giving for the complete primitive

$$(y + x - c)(y + 2x - c) = 0.$$

#### SOLUTION WITHOUT RESOLVING INTO FACTORS.

**3224** CLASS I.—Type  $\phi(x, p) = 0$ .

When  $x$  only is involved with  $p$ , and it is easier to solve the equation for  $x$  than for  $p$ , proceed as follows.

RULE.—Obtain  $x = f(p)$ . Differentiate and eliminate  $dx$  by means of  $dy = p dx$ . Integrate and eliminate  $p$  by means of the original equation.

Similarly, when  $y = f(p)$ , eliminate  $dy$ , &c.

**3225** Ex.—Given  $x = ay_x + by_x^2$ , i.e.,  $x = ap + bp^2 \dots \dots \dots (1)$ ,

$$dx = a dp + 2b p dp, \quad \text{therefore} \quad dy = p dx = ap dp + 2bp^2 dp,$$

therefore 
$$y = \frac{ap^2}{2} + \frac{2bp^3}{3} + C.$$

Eliminating  $p$  between this equation and (1), the result is the complete primitive

$$(ax + 6by - bc)^2 = (6ay - 4x^2 - ac)(a^2 + 4bx).$$

**3226** CLASS II.—Type

$$x\phi(p) + y\psi(p) = \chi(p).$$

RULE.—Differentiate and eliminate  $y$  if necessary. Integrate and eliminate  $p$  by means of the original equation.

If the equation be first divided by  $\psi(p)$ , the form is simplified into

**3227** 
$$y = x\phi(p) + \chi(p).$$

Differentiate, and an equation is obtained of the form  $x_p + Px = Q$ , where  $P$  and  $Q$  are functions of  $p$ .

This may be solved by (3210), and  $p$  afterwards eliminated.

**3228** Otherwise, a differential equation may be formed between  $y$  and  $p$ , instead of between  $x$  and  $p$ .

**3229** Or, more generally, a differential equation may be formed between  $x$  or  $y$  and  $t$ , any proposed function of  $p$ , after which  $t$  must be eliminated to obtain the complete primitive.

**3230** *Clairaut's equation*, which belongs to this class, is of the form

$$y = px + f(p).$$

**RULE.**—Differentiate, and two equations are obtained—

$$(1) \dots p_x = 0, \text{ and } \therefore p = c; \quad (2) \dots x + f'(p) = 0.$$

Eliminate  $p$  from the original equation by means of (1), and again by means of (2). The first elimination gives  $y = cx + f(c)$ , the complete primitive. The second gives a singular solution.

**PROOF.**—For, if Rule I. (3169) be applied to the primitive  $y = cx + f(c)$ , we have  $x + f'(c) = 0$ ; and to eliminate  $c$  between these equations is the elimination directed above,  $c$  being merely written for  $p$  in the two equations.

**3231** Ex. 1.  $y = px + x\sqrt{1+p^2}.$

This is of the form  $y = x\phi(p)$ , and therefore falls under (3227). Differ-

entiating, we obtain  $x dp + dx\sqrt{1+p^2} + \frac{xp dp}{\sqrt{1+p^2}} = 0,$

since  $dy = p dx$ ; thus

$$\left( \frac{1}{\sqrt{1+p^2}} + \frac{p}{1+p^2} \right) dp + \frac{dx}{x} = 0,$$

in which the variables are separated.

Integrating by (1928), and eliminating  $p$ , we find for the complete primitive  $x^2 + y^2 = Cx$ .

**3232** Ex. 2.  $y = px + \sqrt{b^2 - a^2 p^2}.$

This is Clairaut's form (3230). Differentiating, we have

$$\frac{dp}{dx} \left\{ x - \frac{a^2 p}{\sqrt{b^2 - a^2 p^2}} \right\} = 0.$$

The complete primitive is  $y = cx + \sqrt{b^2 - a^2 c^2};$

and the elimination of  $p$  by the other equation gives for the singular solution  $a^2 y^2 - b^2 x^2 = a^2 b^2$ , an hyperbola and the envelope of the lines obtained by varying  $c$  in the complete primitive, which is the equation of a tangent.

**3233** Ex. 3.—To find a curve having the tangent intercepted between the coordinate axes of constant length.

The differential equation which expresses this property is

$$\frac{y\sqrt{1+p^2}}{p} - x\sqrt{1+p^2} = a,$$

or  $y = px + \frac{ap}{\sqrt{1+p^2}} \dots\dots\dots (1).$

Differentiating gives  $\frac{dp}{dx} \left\{ x + \frac{a}{(1+p^2)^{\frac{3}{2}}} \right\} = 0 \dots\dots\dots (2).$

Eliminating  $p$  between (1) and (2) gives,

1st, the primitive  $y = cx + \frac{ac}{\sqrt{1+c^2}} \dots\dots\dots (3);$

2nd, the singular solution  $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \dots\dots\dots (4).$

(3) is the equation of a straight line; (4) is the envelope of the lines obtained by varying the parameter  $c$  in equation (3).

### 3234 CLASS III.—Homogeneous in $x$ and $y$ .

Type  $x^n \phi\left(\frac{y}{x}, p\right) = 0.$

RULE.—Put  $y = vx$ , and divide by  $x^n$ . Solve for  $p$ , and eliminate  $p$  by differentiating  $y = vx$ ; or solve for  $v$ , and eliminate  $v$  by putting  $v = \frac{Y}{X}$ ; and in either case separate the variables.

3235 Ex.  $y = px + x\sqrt{1+p^2}.$

Substitute  $y = vx$ , and therefore  $p = v + xv_x$ . This gives  $v = p + \sqrt{1+p^2}$ . Eliminate  $p$  between the last two equations, and then separate the variables.

The result is  $\frac{dx}{x} + \frac{2v dv}{1+v^2} = 0,$

from which  $x(v^2 + 1) = C$  or  $x^2 + y^2 = Cx.$

The same equation is solved in (3131) in another way.

### SOLUTION BY DIFFERENTIATION.

### 3236 To solve an equation of the form

$$F\{\phi(x, y, y_x), \psi(x, y, y_x)\} = 0.$$

RULE.—Equate the functions  $\phi$  and  $\psi$  respectively to arbitrary constants  $a$  and  $b$ . Differentiate each equation, and eliminate the constants. If the results agree, there is a common

primitive (3166), which may be found by eliminating  $y_x$  between the equations  $\phi = a$ ,  $\phi = b$ , and subsequently eliminating one of the constants by means of the relation  $F(a, b) = 0$ .

Ex. 
$$x - yy_x + f(y^3 - y^2y_x^2) = 0.$$

Here the two equations  $x - yy_x = a$ ,  $f(y^3 - y^2y_x^2) = b$ , on applying the test, are found to have a common primitive. Therefore, eliminating  $y_x$ , we obtain

$$f\{y^3 - (x - a)^2\} = b.$$

Also, by the given equation,  $a + b = 0$ .

Hence the solution is  $f\{y^3 - (x + b)^2\} = b$ .

## HIGHER ORDER LINEAR EQUATIONS.

**3237** Type 
$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + P_{(n-1)x} \frac{dy}{dx} + P_n y = Q,$$

where  $P_1 \dots P_n$  and  $Q$  are either functions of  $x$  or constants.

LEMMA.—If  $y_1, y_2, \dots y_n$  be  $n$  different values of  $y$  in terms of  $x$ , which satisfy (3237), when  $Q = 0$ , the solution in that case will be  $y = C_1 y_1 + C_2 y_2 + \dots + C_n y_n$ .

PROOF.—Substitute  $y_1, y_2, \dots y_n$  in turn in the given equation. Multiply the resulting equations by arbitrary constants,  $C_1, C_2, \dots C_n$  respectively; add, and equate coefficients of  $P_1, P_2, \dots P_n$  with those in the original equation.

## LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS.

**3238** 
$$y_{nx} + a_1 y_{(n-1)x} + \dots + a_{(n-1)} y_x + a_n y = Q \dots \dots \dots (1).$$

**3239** Case I.—When  $Q = 0$ .

The roots of the auxiliary equation

$$m^n + a_1 m^{n-1} + \dots + a_{n-1} m + a_n = 0 \dots \dots \dots (2)$$

being  $m_1, m_2, \dots m_n$ , the complete primitive of the differential equation will be

**3240** 
$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x} \dots \dots \dots (3).$$

If the auxiliary equation (2) has a pair of imaginary roots

$(a \pm ib)$ , there will be in the value of  $y$  the corresponding terms

$$\mathbf{3241} \quad Ae^{ax} \cos bx + Be^{ax} \sin bx \dots\dots\dots (4).$$

If any real root  $m'$  of equation (2) is repeated  $r$  times, the corresponding part of the value of  $y$  will be

$$\mathbf{3242} \quad (A_0 + A_1x + A_2x^2 + \dots + A_{r-1}x^{r-1}) e^{m'x}.$$

And if a pair of imaginary roots occurs  $r$  times, substitute for  $A$  and  $B$  in (3241) similar polynomials of the  $r-1^{\text{th}}$  degree in  $x$ .

PROOF.—(i.) Substituting  $y = Ce^{mx}$  in (1) as a particular solution, and dividing by  $Ce^{mx}$ , the auxiliary equation is produced, the roots of which furnish  $n$  particular solutions,  $y = C_1e^{m_1x}$ ,  $y = C_2e^{m_2x}$ , &c., and therefore, by the preceding lemma, the general solution will be equation (2).

(ii.) The imaginary roots  $a \pm ib$  give rise to the terms  $Ce^{ax+ibx} + C'e^{ax-ibx}$ , which, by the Exp. values (766), reduce to

$$(C + C')e^{ax} \cos bx + i(C - C')e^{ax} \sin bx.$$

(iii.) If there are two equal roots  $m_2 = m_1$ , put at first  $m_2 = m_1 + h$ . The two terms  $C_1e^{m_1x} + C_2e^{(m_1+h)x}$  become  $e^{m_1x}(C_1 + C_2e^{hx})$ . Expand  $e^{hx}$  by (150), and put  $C_1 + C_2 = A$ ,  $C_2h = B$  in the limit when  $h = 0$ ,  $C_1 = \infty$ ,  $C_2 = -\infty$ . By repeating this process, in the case of  $r$  equal roots, we arrive at the form

$$(A_0 + A_1x + A_2x^2 + \dots + A_{r-1}x^{r-1}) e^{m_1x};$$

and similarly in the case of repeated pairs of imaginary roots.

### **3243** Case II.—When $Q$ in (3238) is a function of $x$ .

*First method.*—By variation of parameters.

Putting  $Q = 0$ , as in Case I., let the complete primitive be

$$y = A\alpha + B\beta + C\gamma + \&c. \text{ to } n \text{ terms } \dots\dots\dots (6),$$

$\alpha, \beta, \gamma$  being functions of  $x$  of the form  $e^{mx}$ . The values of the parameters  $A, B, C, \dots$ , when  $Q$  has its proper value assigned, are determined by the  $n$  equations

$$\begin{array}{llll} \mathbf{3244} & A_x \alpha & + B_x \beta & + \text{to } n \text{ terms} = 0, \\ & A_x \alpha_x & + B_x \beta_x & + \quad \quad \quad = 0, \\ & A_x \alpha_{2x} & + B_x \beta_{2x} & + \quad \quad \quad = 0, \\ & \dots & \dots & \dots \quad \quad \dots \\ & A_x \alpha_{(n-1)x} & + B_x \beta_{(n-1)x} & + \quad \quad \quad = Q, \end{array}$$

$A_x, B_x, \&c.$  being found from these equations, their integrals must be substituted in (6) to form the complete primitive.

PROOF.—Differentiate (6) on the hypothesis that  $A, B, C, \&c.$  are functions of  $x$ ; thus

$$y_x = (A\alpha_x + B\beta_x + \dots) + (A_x\alpha + B_x\beta + \dots).$$

Now, in addition to equation (1),  $n-1$  relations may be assumed between the  $n$  arbitrary parameters. Equate then the last term in brackets to zero, and differentiate  $y$ , in all,  $n-1$  times, equating to zero the second part of each differentiation; thus we obtain

$$y_x = A\alpha_x + B\beta_x + \&c. \quad \text{and} \quad A_x\alpha + B_x\beta + \&c. = 0,$$

$$y_{2x} = A\alpha_{2x} + B\beta_{2x} + \&c. \quad \text{and} \quad A_x\alpha_x + B_x\beta_x + \&c. = 0,$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$y_{(n-1)x} = A\alpha_{(n-1)x} + B\beta_{(n-1)x} + \&c. \quad \text{and} \quad A_x\alpha_{(n-2)x} + B_x\beta_{(n-2)x} + \&c. = 0.$$

The  $n$  quantities  $A_x, B_x, \&c.$  are now determined by the  $n-1$  equations on the right and equation (1). For, differentiating the value of  $y_{(n-1)x}$ , we have

$$y_{nx} = \{A\alpha_{nx} + B\beta_{nx} + \&c.\} + \{A_x\alpha_{(n-1)x} + B_x\beta_{(n-1)x} + \&c.\},$$

and if these values of  $y_x, y_{2x}, \dots, y_{nx}$  be substituted in (1), it reduces to

$$A_x\alpha_{(n-1)x} + B_x\beta_{(n-1)x} + \&c. = Q,$$

for the other part vanishes by the hypothetical equation

$$y_{nx} + a_1 y_{(n-1)x} + \dots + a_n y = 0,$$

since the values of  $y_x, \dots, y_{(n-1)x}$ , and the first part of  $y_{nx}$ , are the true values in this equation.

**3245** *Case II.—Second Method.*—Differentiate and eliminate  $Q$ . The resulting equation can be solved as in Case I. Being of a higher order, there will be additional constants which may be eliminated by substituting the result in the given equation.

**3246** Ex.—Given  $y_{2x} - 7y_x + 12y = x \dots\dots\dots (1).$

*1st Method.*—Putting  $x = 0$ , the auxiliary equation is  $m^2 - 7m + 12 = 0$ ; therefore  $m = 3$  and  $4$ . Hence the complete primitive of  $y_{2x} - 7y_x + 12y = 0$  is

$$y = Ae^{3x} + Be^{4x} \dots\dots\dots (2).$$

The corrected values of  $A$  and  $B$  for the primitive of equation (1) are found from

$$\left. \begin{aligned} A_x e^{3x} + B_x e^{4x} &= 0 \\ 3A_x e^{3x} + 4B_x e^{4x} &= x \end{aligned} \right\}, \quad \therefore A_x = -xe^{-3x} \quad \text{and} \quad A = \frac{3x+1}{9} e^{-3x} + a.$$

$$B_x = xe^{-4x} \quad \text{and} \quad B = -\frac{4x+1}{16} e^{-4x} + b.$$

Substituting these values of  $A$  and  $B$  in (2), we find for its complete primitive

$$y = ae^{3x} + be^{4x} + \frac{12x+7}{144}.$$

**3247** *2nd Method.*  $y_{2x} - 7y_x + 12y = x \dots\dots\dots (1).$   
3 Q



Differentiating to eliminate the term on the right, we get

$$y_{4x} - 7y_{3x} + 12y_{2x} = 0.$$

The aux. equation is  $m^4 - 7m^3 + 12m^2 = 0$ ; therefore  $m = 4, 3, 0, 0$ .

Therefore  $y = Ae^{4x} + Be^{3x} + Cx + D \dots\dots\dots (2)$ ;

$$y_x = 4Ae^{4x} + 3Be^{3x} + C; \quad y_{2x} = 16Ae^{4x} + 9Be^{3x}.$$

Substitute these values in (1); thus  $C = \frac{1}{12}$ ;  $D = \frac{7}{144}$ ;

therefore, substituting in (2),  $y = Ae^{4x} + Be^{3x} + \frac{x}{12} + \frac{7}{144}$  as before.

**3248** When a particular integral of the linear equation (3238) is known in the form  $y = f(x)$ , the complete primitive may be obtained by adding to  $y$  that value which it would take if  $Q$  were zero.

Thus, in Ex. (3247),  $y = \frac{x}{12} + \frac{7}{144}$  is a particular integral of (1); and the complementary part  $Ae^{4x} + Be^{3x}$  is the value of  $y$  when the dexter is zero.

**3249** The order of the linear equation (3238) may always be depressed by unity if a particular integral of the same equation, when  $Q = 0$ , be known.

Thus, if  $y_{3x} + P_1y_{2x} + P_2y_x + P_3y = Q \dots\dots\dots (1)$ , and if  $y = z$  be a particular solution when  $Q = 0$ ; let  $y = vz$  be the solution of (1). Therefore, substituting in (1),

$$(z_{3x} + P_1z_{2x} + P_2z_x + P_3z)v + \&c. = Q,$$

the unwritten terms containing  $v_x, v_{2x}$ , and  $v_{3x}$ .

The coefficient of  $v$  vanishes, by hypothesis; therefore, if we put  $v_x = u$ , we have an equation of the *second* order for determining  $u$ .  $u$  being found,

$$v = \int u dx + C.$$

**3250** The linear equation

$$(a+bx)^n y_{nx} + A(a+bx)^{n-1} y_{(n-1)x} + B(a+bx)^{n-2} y_{(n-2)x} + \dots \dots + Ly = Q,$$

where  $A, B, \dots L$  are constants, and  $Q$  is a function of  $x$ , is solved by substituting  $a+bx = e^t$ , changing the variable by formula (1770), and in the complete primitive putting  $t = \log(a+bx)$ .

Otherwise, reduce to the form in (3446) by putting  $a+bx = \lambda\xi$ , and solve as in that article.

## HIGHER ORDER NON-LINEAR EQUATIONS.

**3251** *Type*  $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots \frac{d^ny}{dx^n}\right) = 0.$

## SPECIAL FORMS.

**3252**  $F(x, y_{rx}, y_{(r+1)x} \dots y_{nx}) = 0.$

When the dependent variable  $y$  is absent, and  $y_{rx}$  is the derivative of lowest order present, the equation may be depressed to the order  $n-r$  by putting  $y_{rx} = z$ . If the equation in  $z$  can be solved, the complete primitive will then be

$$y = \int_{rx} z + \int_{rx} 0 \quad (2149).$$

**3253**  $F(y, y_{rx}, y_{(r+1)x} \dots y_{nx}) = 0.$

If  $x$  be absent instead of  $y$ , change the independent variable from  $x$  to  $y$ , and proceed as before.

Otherwise, change the independent variable to  $y$ , and make  $p (= y_x)$  the dependent variable.

For example, let the equation be of the form

**3254**  $F(y, y_x, y_{2x}, y_{3x}) = 0 \dots\dots\dots (1).$

(i.) This may be changed into the form

$$F(y, x_y, x_{2y}, x_{3y}) = 0 \quad \text{by (1761, '63, and '66);}$$

and the order may then be depressed to the 2nd by (3252). The solution will thus give  $x$  in terms of  $y$ .

**3255** (ii.) Otherwise, equation (1) may be changed at once into one of the form

$$F(y, p, p_y, p_{2y}) = 0, \quad \text{by (1764 and '67),}$$

the order being here depressed from the 3rd to the 2nd. If the solution of this equation be  $p = \phi(y, c_1, c_2)$ , then, since  $dy = p dx$ , we get, for the complete primitive of (1),

$$x = \int \frac{dy}{\phi(y, c_1, c_2)} + c_3.$$

**3256**  $y_{nx} = F(x).$

Integrate  $n$  times successively, thus

$$y = \int_{nx} F(x) + c_1 x^{n-1} + c_2 x^{n-2} + \dots + c_n.$$

**3257**  $y_{2x} = F(y).$

Multiply by  $2y_x$  and integrate, thus

$$\left(\frac{dy}{dx}\right)^2 = 2 \int F(y) dy + c_1, \quad x = \int \frac{dy}{\sqrt{\{2 \int F(y) dy + c_1\}}} + c_2.$$

**3258**  $y_{nx} = F\{y_{(n-1)x}\},$

an equation between two successive derivatives.

Put  $y_{(n-1)x} = z$ , then  $z_x = F(z)$ , from which

$$x = \int \frac{dz}{F(z)} + c \dots\dots\dots (1).$$

If, after integrating, this equation can be solved for  $z$  so that  $z = \phi(x, c)$ , we have  $y_{(n-1)x} = \phi(x, c)$ , which falls under (3256).

**3259** But if  $z$  cannot be expressed in terms of  $x$ , proceed as follows:—

$$y_{(n-1)x} = z; \quad y_{(n-2)x} = \int z dx = \int \frac{dz}{F(z)} z;$$

$$y_{(n-3)x} = \int \frac{dz}{F(z)} \int \frac{dz}{F(z)} z; \quad \dots \quad \dots$$

Finally, 
$$y = \int \frac{dz}{F(z)} \int \frac{dz}{F(z)} \dots \int \frac{dz}{F(z)} z;$$

the number of integrations and arbitrary constants introduced being  $n-1$ .

**3260**  $y_{nx} = F\{y_{(n-2)x}\}.$

Put  $y_{(n-2)x} = z$ ; then  $z_{2x} = F(z)$ , which is (3257), the solution giving  $x$  in terms of  $z$  and two constants. If  $z$  can be found from this in terms of  $x$  and the two constants, we get

$$z \quad \text{or} \quad y_{(n-2)x} = \phi(x, c_1, c_2),$$

which may be solved by (3256).

**3261** But if  $z$  cannot be expressed in terms of  $x$ , proceed as in (3259).

#### DEPRESSION OF ORDER BY UNITY.

**3262** When  $F(x, y, y_x, y_{2x}, \dots) = 0$

is rendered homogeneous by considering

$$x, y, y_x, y_{2x}, y_{3x}, \&c.$$

to be of the respective dimensions 1, 1, 0, -1, -2, &c.; put

$$x = e^\theta, \quad y = ze^\theta, \quad \&c.$$

The transformed equation will contain the same power of  $e$  in every term, and will reduce to the form

$$F(z, z_\theta, z_{2\theta}, \dots) = 0,$$

the order of which is depressed by unity by putting  $z_\theta = u$ .

**3263** When the original equation is of the 2nd order, the transformed equation in  $u$  and  $z$  may be obtained at once by changing  $x, y, y_x, y_{2x}$  into  $1, z, u+z, u+uu_x$ , respectively. The solution is then completed, as in example (3264).

PROOF.—We have  $x = e^\theta; \quad y = ze^\theta;$

$$y_x = z_\theta + z; \quad y_{2x} = e^{-\theta}(z_{2\theta} + z_\theta); \quad y_{3x} = e^{-2\theta}(z_{3\theta} - z_\theta); \quad \text{and so on.}$$

The dimensions of  $x, y, y_x$ , &c. with respect to  $e$ , are 1, 1, 0, -1, -2, &c. Therefore the same power of  $e^\theta$  will occur in every term of the *homogeneous* equation.

**3264** Ex.:  $x^4 y_{2x} = (y - xy_x)^4.$

Making the above substitutions for  $x, y, y_x$ , and  $y_{2x}$ , the equation becomes

$$z_{2\theta} + z_\theta = -z_\theta^2.$$

Put  $z_\theta = u$ ; thus

$$u^2 + u = -u_\theta = -uu_x, \quad \text{therefore} \quad u^2 + 1 = -u_x, \quad \frac{du}{u^2 + 1} = -dz,$$

therefore  $\tan^{-1} u = a - z$  (1935), therefore  $z_\theta = u = \tan(a - z)$ ,

therefore  $d\theta = \cot(a - z) dz$ , therefore  $\theta = -\log b \sin(a - z)$  (1941).

But  $\theta = \log x$  and  $z = \frac{y}{x}$ ,

therefore  $bx = \operatorname{cosec}\left(a - \frac{y}{x}\right)$ , or  $bx = \sec\left(c + \frac{y}{x}\right)$ ,

by altering the arbitrary constant.

**3265** When  $F(x, y, y_x, y_{2x}, \dots) = 0$

is made homogeneous by considering  $x, y, y_x, y_{2x}$ , &c. to be of the respective dimensions 1,  $n$ ,  $n-1$ ,  $n-2$ , &c.; put

$$x = e^\theta, \quad y = ze^{n\theta},$$

and depress the order by putting  $z_\theta = u$ , as in (3262).

**3266** When the original equation is of the 2nd order, the

final equation between  $u$  and  $z$  may be obtained at once by changing

$x, y, y_x, y_{2x}$  into  $1, z, u+nz, uu_x+(2n-1)u+n(n-1)z$ , respectively.

**3267** Ex.:  $y_{2x}y_x = xy \dots\dots\dots (1).$

With the view of applying (3265), the assumed dimensions of each member of this equation, being equated, give

$$n-2+n-1 = 1+n, \quad \text{therefore } n = 4.$$

Thus  $x = e^z$ ;  $y = ze^u$ ;  $y_x = e^{2z}(z+4z)$ ;  $y_{2x} = e^{2z}(z_{2z}+7z_z+12z)$ .

Substituting in (1),  $e$  disappears; and by putting  $z_z = u$ ,  $z_{2z} = uu_z$ , the equation is reduced to

$$(u^3+4uz) du + (7u^2+40uz+48z^2-z) dz = 0,$$

which is linear and of the 1st order. This equation is also obtained at once by the rule in (3266).

**3268** When  $F(x, y, y_x, y_{2x}, \dots) = 0$

is homogeneous with respect to  $y, y_x, y_{2x}$ , &c., put  $y = e^{\int u dx}$ , and remove  $e$  as before by division. The equation between  $u$  and  $x$  will have its order less by unity than the order of  $F$ .

**3269** Ex.:  $y_{2x} + Py_x + Qy = 0 \dots\dots\dots (1),$

$P$  and  $Q$  being functions of  $x$ .

Here  $y = e^{\int u dx}$ ;  $y_x = uy$ ;  $y_{2x} = (u_x + u^2)y$ .

Substituting, the equation becomes  $u_x + u^2 + Pu + Q = 0$ , an equation of the 1st order. If the solution gives  $u = \phi(x, c)$ , then  $\int \phi(x, c) dx = \log y$  is the complete primitive of (1).

## EXACT DIFFERENTIAL EQUATIONS.

**3270** Let  $dU = \phi(x, y, y_x, y_{2x}, \dots, y_{nx}) dx = 0$

be an exact differential equation of the  $n^{\text{th}}$  order. The highest derivative involved will be of the 1st degree.

**3271** RULE FOR THE SOLUTION (Sarrus).—Integrate the term involving  $y_{nx}$  with respect to  $y_{(n-1)x}$  only, and call the result  $U_1$ . Find  $dU_1$ , considering both  $x$  and  $y$  as variables.  $dU - dU_1$  will be an exact differential of the  $n-1^{\text{th}}$  order.

Integrate this with respect to  $y_{(n-2)x}$  only, calling the result  $U_2$ , and so on.

The first integral of the proposed equation will be

$$U = U_1 + U_2 + \dots + U_n = C.$$

**3272** Ex.: Let  $dU = \{y^2 + (2xy - 1)y'_x + xy_{2x} + x^2y_{3x}\} dx = 0$ .

Here  $U_1 = x^2y_{2x}$ ,  $dU_1 = (2xy_{2x} + x^2y_{3x}) dx$ ;

$$dU - dU_1 = \{y^2 + (2xy - 1)y'_x - xy_{2x}\} dx = 0;$$

$$\therefore U_2 = -xy_x, \quad dU_2 = -(y_x + xy_{2x}) dx, \quad dU - dU_1 - dU_2 = (y^2 + 2xyy'_x) dx;$$

$$\text{therefore } U_3 = xy^2, \quad \text{and } U = x^2y_{2x} - xy_x + xy^2 = C$$

is the first integral.

**3273** Denoting equation (3270) by  $dU = Vdx$ , the series of steps in Rule (3171) involve and amount to the single condition that the equation

$$N - P_x + Q_{2x} - R_{3x} + \&c. = 0,$$

with the notation in (3028), shall be identically true. This then is the condition that  $V$  shall be an exact 1st differential.

**3274** Similarly, the condition that  $V$  shall be an exact 2nd differential is

$$P - 2Q_x + 3R_{2x} - 4S_{3x} + \&c. = 0.$$

**3275** The condition that  $V$  shall be an exact 3rd differential

$$\text{is } Q - 3R_x + \frac{3.4}{1.2} S_{2x} - \frac{3.4.5}{1.2.3} T_{3x} + \&c. = 0,$$

and so on.

[Euler, *Comm. Petrop.*, Vol. viii.]

## MISCELLANEOUS METHODS.

**3276**  $y_{2x} + Py_x + Qy_x^2 = 0 \dots\dots\dots (1),$

where  $P$  and  $Q$  are functions of  $x$  only.

The solution is  $y = \int e^{-\int Pdx} (2 \int Qe^{-2\int Pdx} dx)^{-1} dx.$

PROOF. — Put  $e^{\int Pdx} = z$ , and multiply (1) by  $z$ ; then, since  $z_x = Pz$ ,  $zy_x + Qzy_x^2 = 0$ . Put  $zy_x = u^{-1}$ ,  $\therefore uu_x = Qz^{-2}$ ,  $\therefore u = \sqrt{(2 \int Qz^{-2} dx)}$ , &c.

**3277**  $y_{2x} + Qy_x^2 + R = 0 \dots\dots\dots (1),$

where  $Q$  and  $R$  are functions of  $y$  only.



The solution is  $x = \int e^{\int Q dy} (2 \int R e^{2 \int Q dy} dy)^{-1} dy$ .

PROOF.—Put  $e^{\int Q dy} = z$ , and multiply (1) by  $z$ .

$$\therefore (zy_x)_x = Rz, \quad \therefore zy_x(zy_x)_x = Rz^3y_x, \quad \therefore (zy_x)^2 = 2 \int Rz^3 dy, \text{ \&c.}$$

**3278**  $y_{2x} + Py_x + Qy_x^n = 0,$

where  $P, Q$  involve  $x$  only.

Put  $y_x = z$ , and the form (3211) is arrived at.

**3279**  $y_{2x} + Py_x^2 + Qy_x^n = 0.$

This reduces to the last case by changing the variable from  $x$  to  $y$  by (1763).

**3280** For a few cases in which the equation

$$y_x + Py^2 + Qy + R = 0$$

can be integrated, see De Morgan's "Differential and Integral Calculus," p. 690.

**3281**  $y_{2x} = ax + by.$

Put  $ax + by = t$  (1762-3). Result  $t_{2x} = bt$ . Solve by (3239) or (3257).

**3282**  $(1-x^2)y_{2x} - xy_x + q^2y = 0.$

Put  $\sin^{-1} x = t$ , and obtain  $y_{2t} + q^2y = 0$ .

Solution,  $y = A \cos(q \sin^{-1} x) + B \sin(q \sin^{-1} x).$

**3283**  $(1+ax^2)y_{2x} + axy_x \pm q^2y = 0.$

Put  $\int \frac{dx}{\sqrt{1+ax^2}} = t$ , and obtain  $y_{2t} \pm q^2y = 0$  as above.

**3284** *Liouville's equation*,  $y_{2x} + f(x)y_x + F(y)y_x^2 = 0.$

Suppress the last term. Obtain a first integral by (3209), and vary the parameter. The complete primitive is

$$\int e^{\int F(y) dy} dy = A \int e^{-\int f(x) dx} dx + B.$$

**3285** *Jacobi's theorem*.—If one of the first integrals of the equation  $y_{2x} = f(x, y)$  is  $y_x = \phi(x, y, c) \dots \dots (i., ii.),$

the complete primitive will be

$$\int \phi_c(dy - \phi dx) = c.$$

PROOF.—Differentiating (ii.), we obtain  $\phi_x + \phi\phi_y = f(x, y)$ , and differentiating this for  $c$ ,  $\phi_{xc} + \phi_c\phi_y + \phi\phi_{yc} = 0$ . But, by (3187), this is the condition for ensuring that  $\phi_c dy - \phi_c\phi dx = 0$  shall be an exact differential; therefore  $\phi_c$  is an integrating factor for equation (ii.),  $y_x - \phi(x, y, c) = 0$ .

*Equations involving the arc s, having given*

**3286**  $ds^2 = dx^2 + dy^2 \quad \text{or} \quad s_x = \sqrt{1 + y_x^2}.$

**3287**  $s = ax + by.$

Here  $\sqrt{1 + y_x^2} = a + by_x$ . Find  $y_x$  from the quadratic equation.

**3288**  $x_{2x} = a.$

Change from  $s$  to  $x$  by (1763);  $\therefore -s_x^{-3}s_{2x} = a$ ,  $\therefore s_x^{-2} = 2ax + c$ ,

or  $1 + y_x^2 = \frac{1}{2ax + c}$ ,  $\therefore y = \int \sqrt{\left(\frac{1}{2ax + c} - 1\right)} dx + c'.$

#### APPROXIMATE SOLUTION OF DIFFERENTIAL EQUATIONS BY TAYLOR'S THEOREM.

**3289** The following example will illustrate the method:—

Given  $y_{2x} = xy_x + y$ ,  $\therefore y_{3x} = (x^2 + 2)y_x + xy.$

Generally, let  $y_{nx} = A_n y_x + B_n y$ ;  $y_{(n+1)x} = A_{n+1} y_x + B_{n+1} y.$

But, by differentiation,

$$y_{(n+1)x} = (A_n x + A'_n + B_n) y_x + (A_n + B'_n) y,$$

$$\therefore A_{n+1} = A_n x + A'_n + B_n \quad \text{and} \quad B_{n+1} = A_n + B'_n.$$

But  $A_2 = x$ ,  $B_2 = 1$ ,  $\therefore A_3 = x^2 + 2$ ,  $B_3 = x$ ;

$$A_4 = x^3 + 5x, \quad B_4 = x^2 + 3, \quad \&c.$$

Now, when  $x = a$ , let  $y = b$  and  $y_x = p$ ; then, by Taylor's theorem (1500),

$$y = a + p(x - a) + (A_2 p + B_2 b) \frac{(x - a)^2}{2!} + (A_3 p + B_3 b) \frac{(x - a)^3}{3!} + \&c.,$$

which converges when  $x - a$  is small.

[De Morgan, p. 692.]

## SINGULAR SOLUTIONS OF HIGHER ORDER EQUATIONS.

### DERIVATION FROM THE COMPLETE PRIMITIVE.

**3301** Let  $y_{nx} = \phi(x, y, y_x, y_{2x} \dots y_{(n-1)x}) \dots \dots \dots (1)$

be the differential equation, and let its complete primitive be

$$y = f(x, a, b, c \dots s) \dots \dots \dots (2),$$

containing  $n$  arbitrary constants.

**3302** RULE.—To find the general singular solution of (1), eliminate  $a, b, c \dots s$  between the equations

$$y = f, \quad y_x = f_x, \quad y_{2x} = f_{2x} \dots, \quad y_{(n-1)x} = f_{(n-1)x} \dots \dots (3)$$

and

$$\begin{vmatrix} f_a & f_{ax} & f_{a2x} & \dots & f_{a(n-1)x} \\ f_b & f_{bx} & f_{b2x} & \dots & f_{b(n-1)x} \\ \dots & \dots & \dots & \dots & \dots \\ f_s & f_{sx} & f_{s2x} & \dots & f_{s(n-1)x} \end{vmatrix} = 0 \dots \dots \dots (4).$$

The result is a differential equation of the  $n-1$ <sup>th</sup> order, and the integral of it, containing  $n-1$  arbitrary constants, is the singular solution.

PROOF.—Differentiate (2), considering the parameters  $a, b \dots s$  variable, thus  $y_s = f_s + f_a a_x + \dots + f_s s_x$ . Therefore, as in (3171),

$$y_s = f_s \quad \text{if} \quad f_a a_x + f_b b_x + \dots + f_s s_x = 0,$$

$$y_{2x} = f_{2x} \quad \text{if} \quad f_{ax} a_x + f_{bx} b_x + \dots + f_{sx} s_x = 0, \text{ as well;}$$

and so on up to  $y_{nx} = f_{nx}$ . Eliminating  $a_x, b_x, \dots s_x$  between the  $n$  equations on the right, the determinant equation (4) is produced with the rows and columns interchanged.

**3303** Ex.:  $y - xy_s + \frac{1}{2}x^2 y_{2x} - y_{2x}^2 - (y_s - xy_{2x})^2 = 0 \dots \dots \dots (1).$

The complete primitive is  $y = \frac{ax^2}{2} + bx + a^2 + b^2 \dots \dots \dots (2).$

From which  $y_s = ax + b \dots \dots \dots (3),$

and the determinant equation is

$$\begin{vmatrix} \frac{1}{2}x^2 + 2a, & x \\ x + 2b, & 1 \end{vmatrix} = 0 \quad \text{or} \quad \frac{x^2}{2} + 2bx = 2a \dots \dots \dots (4).$$

Eliminating  $a$  and  $b$  from (2), (3), and (4), we get the differential equation

$$\frac{4dy + (2x + x^3) dx}{\sqrt{(16y + 4x^3 + x^4)}} = \sqrt{(1 + x^2)} dx \dots\dots\dots (5),$$

the integral of which, and the singular solution of (1), is

$$\sqrt{(16y + 4x^3 + x^4)} = x\sqrt{(1 + x^2)} + \log \{x + \sqrt{(1 + x^2)}\} + C.$$

[Boole, *Sup.*, p. 49.]

**3304** Either of the two 'first integrals' (3064) of a second order differential equation leads to the same singular solution of that equation.

**3305** The complete primitive of a singular first integral of a differential equation of the second order is itself a singular solution of that equation; but a singular solution of a singular first integral is not generally a solution of the original equation.

Thus the singular first integral (5) of equation (1) in the last example has the singular solution  $16y + 4x^3 + x^4 = 0$ , which is not a solution of equation (1).

#### DERIVATION OF THE SINGULAR SOLUTION FROM THE DIFFERENTIAL EQUATION.

**3306** RULE.—Assuming the same form (3173), a singular solution of the first order of a differential equation of the  $n^{\text{th}}$  order will make  $\frac{d(y_{nx})}{d(y_{(n-1)x})}$  infinite; a singular solution of the second order will make  $\frac{d(y_{nx})}{d(y_{(n-1)x})}$ ,  $\frac{d(y_{nx})}{d(y_{(n-2)x})}$  both infinite; and so on.

[Boole, *Sup.*, p. 51.]

**3307** Ex.—Taking the differential equation (3303) again,

$$y - xy_x + \frac{1}{2}x^3y_{2x} - y_{2x}^2 - (y_x - xy_{1x})^2 = 0 \dots\dots\dots (1),$$

and differentiating for  $y_x$  and  $y_{2x}$  only,

$$\{\frac{1}{2}x^3 + 2x(y_x - xy_{2x}) - 2y_{2x}\} d(y_{2x}) - \{x + 2(y_x - xy_{2x})\} d(y_x) = 0.$$

The condition  $\frac{d(y_{2x})}{d(y_x)} = \infty$  requires

$$\frac{1}{2}x^3 + 2x(y_x - xy_{2x}) - 2y_{2x} = 0.$$

Substituting the value of  $y_{2x}$  obtained from this in equation (1), and rejecting the factor  $(x^2 + 1)$ , the same singular integral as before is produced (3303, equation 5).

## EQUATIONS WITH MORE THAN TWO VARIABLES.

**3320**  $P dx + Q dy + R dz = 0 \dots\dots\dots (1).$

$P, Q, R$  being here functions of  $x, y, z$ , the condition that this equation may be an exact differential of a single complete primitive is

**3321**  $P(Q_z - R_y) + Q(R_x - P_z) + R(P_y - Q_x) = 0.$

PROOF.—Let  $\mu$  be an integrating factor of  $P dx + Q dy + R dz = 0$ . Then  $-\mu P dx = \mu Q dy + \mu R dz$ , and, by (3187), for an exact differential, we must have  $(\mu Q)_x = (\mu R)_y$ . Write this symmetrically for  $P, Q$ , and  $R$ , differentiate out, and add the three equations after multiplying them respectively by  $P, Q$ , and  $R$ .

To find the single complete primitive of equation (1).

**3322** RULE.—Consider one of the variables  $z$  constant, and therefore  $dz = 0$ . Integrate, and add  $\phi(z)$  for the constant of integration. Then differentiate for  $x, y$ , and  $z$ , and compare with the given equation (1). If a primitive exists,  $\phi(z)$  will be determined in terms of  $z$  only by means of preceding equations.

The complete primitive so obtained is the equation of a system of surfaces, all of the same species, varying in position according to the value assigned to the arbitrary constant.

**3323** Ex.:  $(x - 3y - z) dx + (2y - 3x) dy + (z - x) dz = 0 \dots\dots\dots (1).$

Condition (3321) is satisfied; therefore, putting  $dz = 0$ , we have

$$(x - 3y - z) dx + (2y - 3x) dy = 0.$$

Applying (3187),  $M_y = -3 = N_x$ , and integration gives

$$\frac{1}{2}x^2 - 3xy - zx + y^2 + \phi(z) = 0.$$

Differentiating now for  $x, y$ , and  $z$ ,

$$(x - 3y - z) dx + (2y - 3x) dy + \{\phi'(z) - x\} dz = 0.$$

Equating coefficients with (1),  $\phi'(z) = z$ , therefore  $\phi(z) = \frac{1}{2}z^2 + C$ . Hence the single complete primitive is

$$x^2 + 2y^2 + z^2 - 6xy - 2zx = C,$$

the equation of a system of surfaces obtained by varying the constant  $C$ .

**3324** When the equation  $P dx + Q dy + R dz = 0$  is homogeneous, put  $x = uz, y = vz$ . The result, when the coefficient of  $dz$  vanishes, is of the form

**3325**  $M du + N dv = 0,$

solvable by (3184). Otherwise it is of the form

$$3326 \quad \frac{dz}{z} = M du + N dv,$$

and the right will be an exact differential if a complete primitive exists.

$$3327 \quad \text{Ex.:} \quad yz dx + zx dy + xy dz = 0 \dots\dots\dots (1).$$

Condition (3321) is satisfied. Putting

$$x = uz, \quad y = vz, \quad dx = u dz + z du, \quad dy = v dz + z dv,$$

$$(1) \text{ becomes} \quad \frac{dz}{z} + \frac{du}{3u} + \frac{dv}{3v} = 0,$$

$$\text{and the solution is} \quad \log (zu^{\frac{1}{3}}v^{\frac{1}{3}}) = C \quad \text{or} \quad xyz = C.$$

When the equation

$$P dx + Q dy + R dz = 0 \dots\dots\dots (1)$$

has no single primitive:

$$3328 \quad \text{RULE.}—\text{Assume } \phi(x, y, z) = 0 \dots\dots\dots (2)$$

and differentiate; thus

$$\phi_x dx + \phi_y dy + \phi_z dz = 0 \dots\dots\dots (3).$$

The form of  $\phi$  being given, eliminate  $z$  and  $dz$  from (1) by (2) and (3). The result, being of the form

$$M dx + N dy = 0,$$

can be integrated, and the solution taken with (2) constitutes a solution of equation (1), and represents a system of lines (by varying the constant of integration) drawn on the surface  $\phi(x, y, z) = 0$ .

$$3329 \quad \text{Ex.:} \quad (1+2m)x dx + (1-x)y dy + z dz = 0.$$

The condition (3321) not being satisfied, assume  $x^2 + y^2 + z^2 = r^2$  as the function  $\phi$ , therefore  $x dx + y dy + z dz = 0$ ; and by eliminating  $z$  and  $dz$ ,  $2mdx - y dy = 0$ , the integration of which gives  $y^2 - 4mx = C$ , a cylindrical surface intersecting the spherical surface in a system of curves (by varying  $C$ ), whose projections on the plane of  $xy$  are parabolas.

The condition that

$$3330 \quad X dx + Y dy + Z dz + T dt = 0,$$

where  $X, Y, Z, T$  are functions of  $x, y, z, t$ , may be an exact



differential, may be shewn, in a manner similar to that of (3321), to be expressed by any three of the equations

$$\begin{aligned} 3331 \quad & Y(Z_t - T_z) + Z(T_y - Y_t) + T(Y_z - Z_y) = 0, \\ & Z(T_x - X_t) + T(X_z - Z_x) + X(Z_t - T_z) = 0, \\ & T(X_y - Y_z) + X(Y_t - T_y) + Y(T_x - X_t) = 0, \\ & X(Y_z - Z_y) + Y(Z_x - X_z) + Z(X_y - Y_x) = 0, \end{aligned}$$

the fourth being always deducible from the other equations.

If this condition is fulfilled, the solution of equation (3330) is analogous to (3322).

Integrate as if  $z$  and  $t$  were constant, and therefore  $dz$  and  $dt$  zero, adding for the constant of integration  $\phi(z, t)$ .

Differentiate next for all the variables, and determine  $\phi$  by comparison with the original equation.

**3332** If a single primitive does not exist, the solution must be expressed by simultaneous equations in a manner similar to that of (3328).

## SIMULTANEOUS EQUATIONS WITH ONE INDEPENDENT VARIABLE.

### GENERAL THEORY.

**3340** Let the first of  $n$  equations between  $n+1$  variables be

$$Pdx + P_1dy + P_2dz + \dots + P_ndw = 0 \dots\dots\dots (1),$$

where  $P, P_1 \dots P_n$  may be functions of all the variables.

Let  $x$  be the independent variable. The solution depends upon a single differential equation of the  $n^{\text{th}}$  order between two variables.

Solving the  $n$  equations for the ratios  $dx:dy:dz:\&c.$ , let

$$\begin{aligned} \frac{dx}{Q} &= \frac{dy}{Q_1} = \frac{dz}{Q_2} = \dots = \frac{dw}{Q_n}, \\ \therefore \frac{dy}{dx} &= \frac{Q_1}{Q}, \quad \frac{dz}{dx} = \frac{Q_2}{Q}, \quad \dots \quad \frac{dw}{dx} = \frac{Q_n}{Q}. \end{aligned}$$

Differentiate the first of these equations  $n-1$  times, substituting from the others the values of  $z_x \dots w_x$ , and the result

is  $n$  equations in  $y_x, y_{2x} \dots y_{nx}$ , and the primitive variables  $x, y, z \dots w$ .

Eliminate all the variables but  $x$  and  $y$ , and let the differential equation obtained be

$$F(x, y, y_x \dots y_{nx}) = 0.$$

Find the  $n$  first integrals of this, each of the form  $F(x, y, y_x \dots y_{(n-1)x}) = C$ , and substitute in them the values of  $y_x, y_{2x}, \dots y_{nx}$ , in terms of  $x, y, z \dots w$ , found by solving the  $n$  equations last mentioned. Thus a system of  $n$  primitives is obtained, each of the form  $F(x, y, z \dots w) = C$ .

**3341** The same in the case of three variables.

Here  $n = 2$ . Let the given equations be

$$P_1 dx + Q_1 dy + R_1 dz = 0,$$

$$P_2 dx + Q_2 dy + R_2 dz = 0.$$

**3342** Therefore  $\frac{dx}{Q_1 R_2 - Q_2 R_1} = \frac{dy}{R_1 P_2 - P_2 R_1} = \frac{dz}{P_1 Q_2 - P_2 Q_1}$ .

From these let  $y_x = \phi(x, y, z)$ ,  $z_x = \psi(x, y, z)$ .

Therefore  $y_{2x} = \phi_x + \phi_y y_x + \phi_z z_x$ .

Substitute the value of  $z_x$ , and eliminate  $z$  by means of  $y_x = \phi(x, y, z)$ . An equation of the 2nd order in  $x, y, y_x, y_{2x}$  is the result. Let the complete primitive of this be  $y = \chi(x, a, b)$ . Then we also have  $\phi(x, y, z) = d_x \chi(x, a, b)$ . These two equations form the complete solution.

#### FIRST ORDER LINEAR SIMULTANEOUS EQUATIONS WITH CONSTANT COEFFICIENTS.

**3343** In equations of this class, the coefficients of the dependent variables are constants, but any function of the independent variable may exist in a separate term.

Such equations may be solved by the method of (3340), but more practically by indeterminate multipliers.

**3344** Ex. (1):  $\frac{dx}{dt} + 7x - y = 0$ ,  $\frac{dy}{dt} + 2x + 5y = 0$ .

Multiply the second equation by  $m$  and add. The result may be written

$$\frac{d(x + my)}{dt} + (2m + 7) \left\{ x + \frac{5m - 1}{2m + 7} y \right\} = 0 \dots\dots\dots (1).$$

To make the whole expression an exact differential, put  $\frac{5m-1}{2m+7} = m$ . This

gives  $m = \frac{-1+i}{2}, \quad m' = \frac{-1-i}{2} \dots \dots \dots (2);$

(1) now becomes  $\frac{d(x+my)}{dt} + (2m+7)(x+my) = 0,$

and the solution is  $x+my = ce^{-(2m+7)t}$  and  $x+m'y = c'e^{-(2m'+7)t}.$

Solving these equations, and substituting the values (2),

$$iy = ce^{-(6+i)t} - c'e^{-(6-i)t} = e^{-6t} \{ (c-c') \cos t - i(c+c') \sin t \},$$

$$ix = e^{-6t} \left\{ \left( \frac{c-c'}{2} + i \frac{c+c'}{2} \right) \cos t + \left( \frac{c-c'}{2} - i \frac{c+c'}{2} \right) \sin t \right\},$$

or  $y = e^{-6t}(C \cos t - C' \sin t), \quad x = \frac{1}{2}e^{-6t} \{ (C+C') \cos t + (C-C') \sin t \}.$

**3345** Ex. 2:  $x_t + 5x + y = e^t, \quad y_t + 3y - x = e^{2t}.$

Multiply the second equation by  $m$ , and add to the first

$$\frac{d(x+my)}{dt} + (5-m) \left\{ x + \frac{1+3m}{5-m} y \right\} = e^t + me^{2t}.$$

Put  $\frac{1+3m}{5-m} = m$ , thus determining two values of  $m$ , and put  $x+my = z$ ; thus  $z_t + (5-m)z = e^t + me^{2t}$ . This is of the form (3210).

NOTE.—The equations of this example, written in the symmetrical form of (3342), would be  $\frac{dx}{e^t - 5x - y} = \frac{dy}{e^{2t} + x - 3y} = dt.$

**3346** *General solution by indeterminate multipliers.*

Let  $\frac{dx}{P_1} = \frac{dy}{P_2} = \frac{dz}{P_3}$

be given with

$$P_1 = a_1x + b_1y + c_1z + d_1,$$

$$P_2 = a_2x + b_2y + c_2z + d_2,$$

$$P_3 = a_3x + b_3y + c_3z + d_3.$$

Assume a third variable  $t$  and indeterminate multipliers  $l, m, n$  such that

$$\frac{dt}{t} = \frac{l dx + m dy + n dz}{l P_1 + m P_2 + n P_3} = \frac{l dx + m dy + n dz}{\lambda (lx + my + nz + r)} \dots \dots \dots (1).$$

The last fraction is an exact differential, and, to determine  $\lambda, l, m, n, r$ , we have

$$\begin{array}{l} a_1 l + a_2 m + a_3 n = \lambda l, \\ b_1 l + b_2 m + b_3 n = \lambda m, \\ c_1 l + c_2 m + c_3 n = \lambda n, \\ d_1 l + d_2 m + d_3 n = \lambda r, \end{array} \quad \left| \begin{array}{ccc} a_1 - \lambda & a_2 & a_3 \\ b_1 & b_2 - \lambda & b_3 \\ c_1 & c_2 & c_3 - \lambda \end{array} \right| = 0.$$

The determinant is the eliminant of the first three equations in  $l, m, n$ . The roots of this cubic in  $\lambda$  furnish three sets of values of  $l, m, n, r$ , which, being substituted in the integral of (1), give rise to three equations involving three arbitrary constants; thus,

$$c_1 t = (l_1 x + m_1 y + n_1 z + r_1)^{\frac{1}{\lambda_1}}, \quad c_2 t = (l_2 x + m_2 y + n_2 z + r_2)^{\frac{1}{\lambda_2}}, \\ c_3 t = (l_3 x + m_3 y + n_3 z + r_3)^{\frac{1}{\lambda_3}}.$$

Eliminating  $t$ , we find for the solution two equations involving two arbitrary constants.

A similar solution may be obtained when there are more than three variables.

**3347** To solve  $\frac{dx}{P_1 - xP} = \frac{dy}{P_2 - yP} = \frac{dz}{P_3 - zP} = \&c. \dots (1),$

where  $P = ax + by + c, \quad P_1 = a_1 x + b_1 y + c_1, \quad \&c.$

Assume  $p = a\xi + b\eta + c\zeta, \quad p_1 = a_1 \xi + b_1 \eta + c_1 \zeta, \quad \&c.,$

and take  $\frac{d\xi}{p_1} = \frac{d\eta}{p_2} = \frac{d\zeta}{p} \dots \dots \dots (2),$

the solution of which is known by (3346). Substitute  $\xi = x\zeta, \eta = y\zeta$ , and these equations become

$$\frac{x d\zeta + \zeta dx}{p_1} = \frac{y d\zeta + \zeta dy}{p_2} = \frac{d\zeta}{p},$$

and therefore

$$\frac{\zeta dx}{p_1 - xp} = \frac{\zeta dy}{p_2 - yp} = \frac{d\zeta}{p}.$$

Dividing numerators and denominators by  $\zeta$ , the first equation in (1) is produced, and therefore its solution is obtained by changing  $\xi, \eta$  in the solution of (2) into  $x\zeta$  and  $y\zeta$ .

Certain simultaneous equations in which the coefficients are not constants may be solved by the method of multipliers. Thus,

**3348** Ex. (1):  $x_t + P(ax + by) = Q, \quad y_t + P(a'x + b'y) = R,$

$P, Q, R$  being functions of  $t$ . Multiply the second equation by  $m$ , add, and determine  $m$  as in (3344). The solution is obtained from

$$x + my = e^{-(a+ma') \int P dt} \left\{ C + \int e^{(a+ma') \int P dt} (Q + mR) dt \right\}, \quad (3210)$$

with two values of  $m$ .

**3349** Ex. (2):  $x_t + \frac{2}{t}(x - y) = 1, \quad y_t + \frac{1}{t}(x + 5y) = t$

are equations solvable in a similar manner, and the results are

$$x + y = \frac{1}{t^2} \left( C_1 + \frac{t^4}{4} + \frac{t^5}{5} \right), \quad x + 2y = \frac{1}{t^4} \left( C_2 + \frac{t^5}{5} + \frac{t^6}{6} \right).$$

[Boole, p. 307.]

## REDUCTION OF ORDER IN SIMULTANEOUS EQUATIONS.

**3350** THEOREM.— $n$  simultaneous equations of any orders between  $n$  dependent variables and 1 independent variable are reducible to a system of equations of the *first* order by substituting a new variable for every derivative except the highest.

**3351** The number of equations and dependent variables in the transformed system will be equal to the sum of the indices of order of the highest derivatives. This will, therefore, in general be the number of constants introduced in integrating those equations. If, after integrating, all the new variables be eliminated, there will remain  $n$  equations in the original variables and the above-named constants. These equations form the complete solution.

In practice, such reduction is unnecessary. The following are methods frequently adopted:—

**3352** RULE I.—*Differentiate until by elimination of a variable and its derivatives an equation of a higher order in one dependent variable only is obtained.*

**3353** RULE II.—*Employ indeterminate multipliers.*

**3354** Ex. (1):  $x_{2t} = ax + by, \quad y_{2t} = a'x + b'y.$

By Rule I., differentiating twice for  $t$  and eliminating  $y$  and  $y_{2t}$ , we obtain

$$x_{4t} - (a + b')x_{2t} + (ab' - a'b)x = 0,$$

which may be solved by (3239).

Otherwise by Rule II., exactly as in (3344), we find

$$a'm^3 + (a - b')m - b = 0,$$

and for the exact differential

$$(x + my)_{2t} = (a + ma')(x + my),$$

the solution of which, by (3239), is

$$x + my = C_1 e^{\sqrt{(a+ma')}t} + C_2 e^{-\sqrt{(a+ma')}t}$$

in duplicate with the two values of  $m$ .

**3355** Ex. (2):  $x_{2t} - 2ay_t + bx = 0, \quad y_{2t} + 2ax_t + by = 0.$

Differentiate, and eliminate  $y, y_t, y_{2t}$ ; thus

$$x_{4t} + 2(2a^2 + b)x_{2t} + b^2x = 0,$$

and solve by (3239). Otherwise assume

$$x = \xi \cos at + \eta \sin at, \quad y = \eta \cos at - \xi \sin at,$$

and the given equations reduce to

$$\xi_{2t} = -(a^2 + b)\xi, \quad \eta_{2t} = -(a^2 + b)\eta,$$

which are solved in (3257).

[Boole, p. 311.]

**3356** Ex. (3).—Let  $u = 0$ ,  $v = 0$ ,  $w = 0$  be three equations in  $x, y, z, t$ , involving derivatives of  $t$  up to  $x_{3t}, y_{3t}, z_{3t}$ .

To obtain an equation between  $x$  and  $t$ . Differentiate each equation  $6+7=13$  times, producing  $3+13 \times 3 = 42$  equations involving derivatives of  $t$  up to  $x_{13t}, y_{13t}, z_{13t}$ . Between these 42 equations eliminate  $y, y_t, \dots, y_{13t}, z, z_t, \dots, z_{13t}$ , in all 41 quantities, and an equation of the 16th order in  $x$  and  $t$  is the result. [De Morgan.]

**3357** If a number of equations involve the quantities  $x, x_{2t}, x_{4t}, \&c., y_t, y_{3t}, y_{5t}, \&c.$ , all in the first degree, these quantities may be eliminated by assuming

$$x = L \sin pt, \quad y = M \cos pt.$$

**3358** If there be  $n$  linear homogeneous equations in  $n$  variables  $x, y, z, \dots$  and their derivatives of the 2nd order only, the equations may be solved by putting

$$x = L \sin pt, \quad y = M \sin pt, \quad z = N \sin pt, \quad \&c.$$

**3359** Ex.:  $x_{2t} = ax + by, \quad y_{2t} = gx + fy.$   
 Putting  $x = L \sin pt, \quad y = M \sin pt,$   

$$\left. \begin{aligned} (a+p^2)L + bM &= 0 \\ gL + (f+p^2)M &= 0 \end{aligned} \right\}, \quad \therefore \begin{vmatrix} a+p^2 & b \\ g & f+p^2 \end{vmatrix} = 0,$$

$p$  and the ratios  $L:M$  are thus found.

Suppose  $L = -kb$  and  $M = k(p^2 + a),$   
 then  $x = -kb \sin pt, \quad y = k(p^2 + a) \sin pt,$   
 and  $k$  and  $t$  are arbitrary constants.

## PARTIAL DIFFERENTIAL EQUATIONS.

**3380** An equation is termed a *general primitive* or a *complete primitive* of a partial differential equation, according as the latter is obtained from it by eliminating arbitrary functions or arbitrary constants, as illustrated in (3150–7).

### LINEAR FIRST ORDER P. D. EQUATIONS.

**3381** To form the P. D. equation from the primitive  $u = \phi(v)$ , where  $u$  and  $v$  are functions of  $x, y, z$ .



**RULE.**—Differentiate for  $x$  and  $y$  in turn, and eliminate  $\phi'(v)$ . See (3054).

*Otherwise.*—Differentiate the equations  $u=a$ ,  $v=b$ ; thus

$$u_x dx + u_y dy + u_z dz = 0,$$

$$v_x dx + v_y dy + v_z dz = 0.$$

Therefore  $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ , where  $P = \frac{d(uv)}{d(yz)}$ , &c.

Then the P. D. equation will be

$$Pz_x + Qz_y = R.$$

**PROOF.**—Since  $z$  is a function of  $x$  and  $y$ ,  $z_x dx + z_y dy = dz$ . But  $dx = kP$ ,  $dy = kQ$ ,  $dz = kR$ , therefore  $kPz_x + kQz_y = kR$ .

**3382** Ex.—The general equation of a conical surface drawn through the point  $(a, b, c)$  is  $\frac{y-b}{x-a} = \phi\left(\frac{z-c}{x-a}\right)$ ,

the form of  $\phi$  being arbitrary.

Considering  $z$  as a function of two independent variables  $x$  and  $y$ , differentiate for  $x$  and  $y$  in turn, and eliminate  $\phi'$  as in (3154). The result is the partial differential equation

$$(x-a)z_x + (y-b)z_y + z - c = 0.$$

**3383** To obtain the complete primitive; that is, to solve the P. D. equation,  $Pz_x + Qz_y = R$ ,

$P, Q, R$  being either functions of  $x, y, z$  or constants.

**RULE.**—Solve the equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

Let the two integrals obtained be  $u=a$ ,  $v=b$ ;  
then  $u = \phi(v)$   
will be the complete primitive.

Propositions (3381) and (3383) extended to any number of variables.

**3384** To form the partial differential equation from the primitive  $\phi(u, v, \dots w) = 0$  ..... (1),

where  $u, v, \dots w$  are  $n$  given functions of  $n$  independent variables  $x, y, \dots z$  and one dependent  $t$ .

RULE.—Differentiate for all the variables thus,

$$\phi_u du + \phi_v dv + \dots + \phi_w dw = 0 \dots\dots\dots (2).$$

Therefore, since  $\phi$  is arbitrary,  $du, dv \dots dw$  must separately vanish, giving rise to the  $n$  equations

$$du = u_x dx + u_y dy + \dots + u_t dt = 0,$$

$$dv = v_x dx + v_y dy + \dots + v_t dt = 0,$$

$$\dots \dots \dots$$

$$dw = w_x dx + w_y dy + \dots + w_t dt = 0.$$

Solving these for the ratios, by (583), we get

$$\frac{dx}{P} = \frac{dy}{Q} = \dots = \frac{dz}{R} = \frac{dt}{S} \dots\dots\dots (3),$$

$P, Q \dots R, S$  being functions of the variables or else constants. Now,  $t$  being a function of all the rest,

$$t_x dx + t_y dy + \dots + t_z dz = dt \dots\dots\dots (4),$$

therefore, by (3) and (4), the partial differential equation required is

$$\mathbf{3385} \qquad Pt_x + Qt_y + \dots + Rt_z = S.$$

**3386** If  $u, v \dots w$  be  $n$  functions of  $n$  variables,  $x, y \dots t$ , the condition of interdependence of the functions or existence of some relation expressed by equation (1) is  $J(u, v \dots w) = 0$  (see 1606); that is, the eliminant of equations (2) must vanish.

**3387** Conversely, to integrate the partial differential equation

$$Pt_x + Qt_y + \dots + Rt_z = S \dots\dots\dots (1).$$

RULE.—Solve the system of ordinary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \&c. = \frac{dz}{R} = \frac{dt}{S} \dots\dots\dots (2),$$

and let the integrals obtained be  $u = a, v = b, \dots w = k$ ; then  $\phi(u, v, \dots w) = 0$  will be the complete primitive.

If  $P, Q \dots R, S$  are linear functions of the variables, the integrals of equations (2) can always be found by the method of (3346).

NOTE.—Suppose, in equation (1), that any coefficients  $P, Q$  vanish; then, by (2),  $dx = 0$ ,  $dy = 0$ , and therefore the corresponding integrals are  $x = a$ ,  $y = b$ . The complete primitive thus becomes

$$\phi(x, y, u, v \dots w) = 0.$$

**3389** When only one independent variable occurs in the derivatives of the partial differential equation, the equation may be integrated as though the others were constant, adding functions of the remaining variables for the constants of integration.

**3390** (Ex. 1):  $\frac{dz}{dx} = \frac{y}{\sqrt{y^2 - x^2}}$ . Integrating for  $x$  as though  $y$  were constant, the complete primitive is

$$z = y \sin^{-1} \frac{x}{y} + \phi(y).$$

Some equations are reducible to the above class by a transformation. Thus:

**3391** Ex. (2):  $z_{xy} = x^2 + y^2$ . Put  $z_x = u$ ,  
therefore  $u_y = x^2 + y^2$ , therefore  $u = z_x = x^2 y + \frac{1}{3} y^3 + \phi(x)$ ,  
therefore  $z = \frac{1}{3} x^3 y + \frac{1}{3} x y^3 + \int \phi(x) dx + \psi(y)$ ,  
or  $z = \frac{1}{3} (x^3 y + x y^3) + \chi(x) + \psi(y)$ .

**3392** Ex. (3):  $(x-a)z_x + (y-b)z_y = c-z$ .  
Solving by (3283),  $\frac{dx}{x-a} = \frac{dy}{y-b} = \frac{dz}{z-c}$ .

The integrals are

$$\left. \begin{aligned} \log(y-b) - \log(x-a) &= \log C \\ \log(z-c) - \log(x-a) &= \log C' \end{aligned} \right\}, \quad \text{or } \frac{y-b}{x-a} = C, \quad \frac{z-c}{x-a} = C',$$

therefore  $\frac{y-b}{x-a} = \phi\left(\frac{z-c}{x-a}\right)$  is the complete primitive.

For the converse process in respect of the same equation, see (3382).

**3393** Ex. (4).—To find the surface which cuts orthogonally all the spheres whose equations (varying  $a$ ) are

$$x^2 + y^2 + z^2 - 2ax = 0 \dots\dots\dots (1).$$

Let  $\phi(x, y, z) = 0$  be the surface. Then

$$(x-a)\phi_x + y\phi_y + z\phi_z = 0$$

by the condition of normals at right angles. Substitute the value of  $a$  from (1), and divide by  $\phi_x$ ; thus,

$$(x^2 - y^2 - z^2)z_x + 2xyz_y = 2xz.$$

By (3383),  $\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$ ,

$\frac{dy}{y} = \frac{dz}{z}$  gives  $\frac{y}{z} = c$  for one integral.

Substituting  $y = cz$ , we then have

$$\frac{dx}{x^2-(c^2+1)z^2} = \frac{dz}{2xz},$$

which, being a homogeneous equation in  $x$  and  $z$ , may be solved by putting  $z = vx$  (3186). The resulting integral is  $\frac{x^2+y^2+z^2}{z} = C$ . Hence the complete primitive is  $\frac{x^2+y^2+z^2}{z} = \phi\left(\frac{y}{z}\right)$  and the equation of the surface sought.

**3394** Ex. (5).—To find an integrating factor of the equation

$$(x^3y-2y^4)dx + (xy^3-2x^4)dy = 0 \dots\dots\dots (1).$$

Assuming  $z$  for that factor, the condition  $(Mz)_y = (Nz)_x$  (3087) produces the P. D. equation

$$(xy^3-2x^4)z_x + (2y^4-x^3y)z_y = 9(x^3-y^3)z \dots\dots\dots (2).$$

The system of ordinary equations (3283) is

$$\frac{dx}{xy^3-2x^4} = \frac{dy}{2y^4-x^3y} = \frac{dz}{9(x^3-y^3)z}.$$

The first of these equations is identical with (1) (and such an agreement always occurs). Its integral is  $\frac{x}{y^3} + \frac{y}{x^3} = c$ .

Also 
$$\frac{ydx + xdy}{xy^4-2x^4y+2xy^4-x^4y} = \frac{dz}{9(x^3-y^3)z},$$

which reduces to 
$$\frac{3dx}{x} + \frac{3dy}{y} + \frac{dz}{z} = 0;$$

and thus the second integral is  $x^3y^3z = c'$ .

Hence the complete primitive and integrating factor is

$$z = \frac{1}{x^3y^3} \phi\left(\frac{x}{y^3} + \frac{y}{x^3}\right).$$

Any linear P. D. equation may be written as a homogeneous equation with one additional variable; thus, equation (3387) may be written

**3395** 
$$Pu_x + Qu_y + \dots + Ru_z = Su_t.$$

#### SIMULTANEOUS LINEAR FIRST ORDER P. D. EQUATIONS.

**3396** PROP. I.—The solution of such equations may be made to depend upon a system of ordinary 1st order differential

equations having a number of variables exceeding by more than one the number of equations.

Let there be  $n$  equations reduced to the homogeneous form (3395) involving one dependent variable  $P$  and  $n+m$  independent. Select  $n$  of the latter,  $x, y \dots z$ , and let the remaining  $m$  be  $\xi, \eta \dots \zeta$ . From the  $n$  equations find  $P_x, P_y \dots P_z$  in terms of  $P_\xi, P_\eta \dots P_\zeta$ , and arrange the results as under :

$$\left. \begin{aligned} P_x + a_1 P_\xi + b_1 P_\eta \dots + k_1 P_\zeta &= 0 \\ P_y + a_2 P_\xi + b_2 P_\eta \dots + k_2 P_\zeta &= 0 \\ \dots &\dots \dots \\ P_z + a_n P_\xi + b_n P_\eta \dots + k_n P_\zeta &= 0 \end{aligned} \right\} \dots \dots \dots (1).$$

Multiply these equations by  $\lambda_1, \lambda_2, \dots \lambda_n$  respectively, and add ; thus,

$$\lambda_1 P_x + \lambda_2 P_y \dots + \lambda_n P_z + \Sigma (\lambda a) P_\xi + \Sigma (\lambda b) P_\eta \dots + \Sigma (\lambda k) P_\zeta = 0 \dots \dots (2).$$

From this, as in (3387), we have the auxiliary system

$$\frac{dx}{\lambda_1} = \frac{dy}{\lambda_2} \dots = \frac{dz}{\lambda_n} = \frac{d\xi}{\Sigma (\lambda a)} = \frac{d\eta}{\Sigma (\lambda b)} \dots = \frac{d\zeta}{\Sigma (\lambda k)} \dots \dots (3),$$

and, by eliminating  $\lambda_1, \lambda_2 \dots \lambda_n$ ,

$$\left. \begin{aligned} d\xi - a_1 dx - a_2 dy \dots - a_n dz &= 0 \\ d\eta - b_1 dx - b_2 dy \dots - b_n dz &= 0 \\ \dots &\dots \dots \\ d\zeta - k_1 dx - k_2 dy \dots - k_n dz &= 0 \end{aligned} \right\} \dots \dots \dots (4).$$

Then, if  $u = a, v = b, \&c.$  be the integrals of (4), they will be values of  $P$  satisfying the equivalent system (1), and the integral of that system will be  $F(u, v, \dots) = 0$ .

**3397** PROP. II.—*To integrate a system of linear 1st order P. D. equations.*

Let  $\Delta = ad_x + bd_y \dots + kd_z,$

so that  $\Delta P = 0$  represents a homogeneous linear P. D. equation of the 1st order.

RULE.—“Reduce the equations to the homogeneous form (1); express the result symbolically by

$$\Delta_1 P = 0, \quad \Delta_2 P = 0, \quad \dots \Delta_n P = 0,$$

and examine whether the condition

$$(\Delta_1\Delta_2 - \Delta_2\Delta_1) P = 0$$

is identically satisfied for every pair of equations of the system. If it be so, the equations of the auxiliary system (Prop. I.) will be reducible to the form of exact differential equations, and their integrals being  $u = a$ ,  $v = b$ ,  $w = c$ , ..., the complete value of  $P$  will be  $F(u, v, w, \dots)$ , the form of  $F$  being arbitrary.

"If the condition be not identically satisfied, its application will give rise to one or more new partial differential equations. Combine any one of these with the previous reduced system, and again reduce in the same way.

"With the new reduced system proceed as before, and continue this method of reduction and derivation until either a system of  $P$ . D. equations arises, between every two of which the above condition is identically satisfied, or, which is the only possible alternative, the system  $P_x = 0$ ,  $P_y = 0$ , ... appears. In the former case, the system of ordinary equations corresponding to the final system of  $P$ . D. equations, will admit of reduction to the exact form, and the general value of  $P$  will emerge from their integrals as above. In the latter case, the given system can only be satisfied by supposing  $P$  a constant."

**3398** "Ex.:  $P_x + (t + xy + xz) P_z + (y + z - 3x) P_t = 0$ ,  
 $P_y + (xzt + y - xy) P_z + (zt - y) P_t = 0$ .

Representing these in the form  $\Delta_1 P = 0$ ,  $\Delta_2 P = 0$ , it will be found that  $(\Delta_1\Delta_2 - \Delta_2\Delta_1) P = 0$  becomes, after rejecting an algebraic factor,  $xP_z + P_t = 0$ , and the three equations prepared in the manner explained in the Rule will be found to be

$$P_x + (3x^2 + t) P_z = 0, \quad P_y + yP_z = 0, \quad P_t + xP_z = 0.$$

No other equations are derivable from these. We conclude that there is but one final integral.

"To obtain it, eliminate  $P_x$ ,  $P_y$ ,  $P_z$  from the above system combined with

$$P_x dx + P_y dy + P_z dz + P_t dt = 0,$$

and equate to zero the coefficient of  $P_z$  in the result. We find

$$dz - (t + 3x^2) dx - y dy - x dt = 0,$$

the integral of which is  $z - xt - x^3 - \frac{1}{2}y^2 = c$ .

"An arbitrary function of the first member of this equation is the general value of  $P$ ."

[Boole, *Sup.*, Ch. xxv.

For Jacobi's researches in the same subject, see *Crelle's Journal*, Vol. lx.



NON-LINEAR FIRST ORDER PARTIAL DIFFERENTIAL  
EQUATIONS.

**3399** *Type*  $F(x, y, z, z_x, z_y) = 0 \dots\dots\dots (1).$

CHARPITS'S SOLUTION.—Writing  $p, q$  instead of  $z_x$  and  $z_y$ , assume the equations

$$\frac{dx}{-q_p} = dy = \frac{dz}{q - pq_p} = \frac{dp}{q_x + pq_z} \dots\dots\dots (2).$$

Find a value of  $p$  from these by integration, and the corresponding value of  $q$  from the given equation, and substitute in the equation

$$dz = p dx + q dy \dots\dots\dots (3),$$

and integrate by (3322) to obtain the final integral.

PROOF.—Since  $dz = p dx + q dy$ , we have, by the condition of integrability,  $p_y = q_x$ . Express  $p_y$  and  $q_x$  on the hypothesis that  $z$  is a function of  $x, y$ ;  $p$  a function of  $x, y, z$ ;  $q$  a function of  $x, y, z, p$ ; considering  $x$  constant when finding  $p_y$ , and  $y$  as constant when finding  $q_x$ . Equating the values of  $p_y$  and  $q_x$  so obtained, the result is the equation

$$Ap_x + Bp_y + Cp_z = D,$$

in which  $A, B, C, D$  stand for  $-q_p, 1, q - pq_p, q_x + pq_z$ .

Hence, to solve this equation, we have, by (3387), the system of ordinary equations (2).

**3400** NOTE.—More than one value of  $p$  obtained from equations (2) may give rise to more than one complete primitive.

The first two of equations (2) taken together involve equation (3).

DERIVATION OF THE GENERAL PRIMITIVE AND SINGULAR  
SOLUTION FROM THE COMPLETE PRIMITIVE.

RULE.—Let the complete primitive of a P. D. equation of the 1st order be

$$z = f(x, y, a, b) \dots\dots\dots (1).$$

**3401** The general primitive is obtained by eliminatting  $a$  between  $z = f\{x, y, a, \phi(a)\}$  and  $f_a = 0 \dots\dots\dots (2),$  the form of  $\phi$  being specified at pleasure.

**3402** The singular solution is obtained by eliminating  $a$  and  $b$  between the complete primitive and the equations

$$f_a = 0, \quad f_b = 0 \dots\dots\dots (3).$$

PROOF.—By varying  $a$  and  $b$  in (1),

$$p = f_x + f_a a_x + f_b b_x, \quad q = f_y + f_a a_y + f_b b_y.$$

Therefore, reasoning as in (3171), we must have

$$f_a a_x + f_b b_x = 0 \quad \text{and} \quad f_a a_y + f_b b_y = 0 \dots\dots\dots (3),$$

therefore either  $f_a = 0$ ,  $f_b = 0$ , leading to the singular solution; or, eliminating  $f_a, f_b$ ,

$$a_x b_y - a_y b_x = 0,$$

and therefore, by (3167),  $b = \phi(a)$ . Multiply equations (3) by  $dx, dy$  respectively, and add, thus  $f_a da + f_b db = 0$ . Substitute  $b = \phi(a)$  in this and in (1), and the equations (2) are the result.

### SINGULAR SOLUTION DERIVED FROM THE DIFFERENTIAL EQUATION.

**3403** RULE.—*Eliminate  $p$  and  $q$  from the differential equation by means of the equations*

$$z_p = 0, \quad z_q = 0.$$

PROOF.—Let the D. E. be  $z = f(x, y, p, q)$ , and the C. P.  $z = F(x, y, a, b)$ . Now  $p$  and  $q$  being implicit functions of  $a$  and  $b$ , we have, from the first equation,

$$z_a = z_p p_a + z_q q_a, \quad z_b = z_p p_b + z_q q_b.$$

Hence the conditions  $z_a = 0$ ,  $z_b = 0$  in (3) involve, and are equivalent to,  $z_p = 0$ ,  $z_q = 0$ .

**3404** All possible solutions of a P. D. equation of the 1st order are represented by the complete primitive, the general primitive, and the singular solution. [Boole, p. 343.]

**3405** To connect any given solution with the complete primitive.

Let  $z = F(x, y, a, b)$  be the complete primitive, and  $z = \phi(x, y)$  some other solution.

Determine the values of  $a$  and  $b$  which satisfy the three equations

$$F = \phi, \quad F_x = \phi_x, \quad F_y = \phi_y.$$

If these values are constant, the solution is a particular case of the complete primitive; if they are variable so that one is a function of the other, the solution is a particular case of the general primitive; if they are variable and unconnected, the solution is a singular solution.

**3406** COR.—Any two solutions springing from different complete primitives are equivalent.

**3407** Ex.:  $z = px + qy + pq \dots\dots\dots (1).$

By (3299),  $\frac{dx}{A} = dy = \frac{dz}{C} = \frac{dp}{D} \dots\dots\dots (2),$

and we have  $q = \frac{z-px}{p+y}; \quad A = -q_p = \frac{xy+z}{(p+y)^2};$

$$C = q - pq_p = \frac{2pz+yz-p^2x}{(p+y)^2}; \quad D = q_z + pq_z = \frac{-p+p}{p+y} = 0.$$

Hence (2) becomes  $\frac{(p+y)^2}{xy+z} dx = dy = \frac{(p+y)^2 dz}{2pz+yz-p^2x} = \frac{dp}{0};$

$\therefore dp = 0, p = a; \therefore q = \frac{z-ax}{a+y}.$  Substituting in  $dz = p dx + q dy,$

$$dz = a dx + \frac{z-ax}{a+y} dy \dots\dots\dots (3).$$

By (3322), making  $z$  constant,  $\frac{adx}{z-ax} + \frac{dy}{a+y} = 0,$

therefore  $-\log(z-ax) + \log(a+y) = \phi(z) \dots\dots\dots (4).$

Differentiate for  $x, y, z,$  and equate with (3), thus  $\phi'(z) = 0,$  therefore  $\phi(z) = \text{constant (say } -\log b);$  therefore, by (4),  $z = ax + by + ab,$  the C. P. of (1).

**3408** To find a singular solution by (3402), we must eliminate  $a$  and  $b$  between  $z_a = 0, z_b = 0;$  that is,  $x+b=0$  and  $y+a=0,$

therefore  $z = -xy - xy + xy = -xy$   
is the singular solution.

To find the general primitive by (3401), eliminate  $a$  between the two equations  $z = ax + (y+a)\phi(a)$  and  $x + (y+a)\phi'(a) + \phi(a) = 0.$

#### NON-LINEAR FIRST ORDER P. D. EQUATIONS WITH MORE THAN TWO INDEPENDENT VARIABLES.

**3409** PROP.—To find the complete primitive of the differential equation

$$F(x_1, x_2 \dots x_n, z, p_1, p_2 \dots p_n) = 0 \dots\dots\dots (1),$$

where  $p_1 = \frac{dz}{dx_1}, \quad p_2 = \frac{dz}{dx_2}, \quad \&c.$

**3410** RULE.—Form the linear P. D. equation in  $\Phi$  denoted by

$$\Sigma_r \left\{ \left( \frac{dF}{dx_r} + p_r \frac{dF}{dz} \right) \frac{d\Phi}{dp_r} - \frac{dF}{dp_r} \left( \frac{d\Phi}{dx_r} + p_r \frac{d\Phi}{dz} \right) \right\} = 0,$$

the summation extending from  $r=1$  to  $r=n.$  From the auxiliary system (3387)  $n-1$  integrals

$$\Phi_1 = a_1, \quad \Phi_2 = a_2, \quad \dots \quad \Phi_{n-1} = a_{n-1}$$

may be obtained. From these equations, together with (1), find  $p_1, p_2 \dots p_n$  in terms of  $x_1, x_2 \dots x_n$ , substitute the values in

$$dz = p_1 dx_1 + p_2 dx_2 \dots + p_n dx_n,$$

and the integral of this last equation will furnish the solution required in the form

$$f(x_1, x_2 \dots x_n, z, a_1, a_2 \dots a_n) = 0.$$

[Boole, *Diff. Eq.*, Ch. xiv., and *Sup.*, Ch. xxvii.]

## SECOND ORDER P. D. EQUATIONS.

**3420** Type  $F(x, y, z, z_x, z_y, z_{2x}, z_{xy}, z_{2y}) = 0$ .

The derivatives  $z_x, z_y, z_{2x}, z_{xy}, z_{2y}$  are briefly denoted by  $p, q, r, s, t$  respectively.

$z$  being a function of the two independent variables  $x$  and  $y$ , the following values are of frequent use

**3421**  $dz = p dx + q dy; \quad dp = r dx + s dy; \quad dq = s dx + t dy.$

If  $u$  be any function of  $x, y$ , and  $z$ , the complete derivatives of  $u$  are indicated by brackets, thus

**3422**  $(u_x) = u_x + p u_z, \quad (u_y) = u_y + q u_z.$

A linear 2nd order P. D. equation is of the type

**3423**  $Rr + Ss + Tt = V \dots \dots \dots (1),$

in which  $R, S, T, V$  are functions of  $x, y, z, p, q$ .

PROPOSITION.—Any P. D. equation of the 2nd order which has a first integral of the form  $u = f(v)$ , where  $u$  and  $v$  involve  $x, y, z, p, q$ , is of the form

**3424**  $Rr + Ss + Tt + U(rt - s^2) = V \dots \dots \dots (2),$

where  $R, S, T, U, V$  are functions of  $x, y, z, p, q$ , and

**3425**  $U = u_p v_q - u_q v_p \dots \dots \dots (3).$

PROOF.—Differentiate  $u = f(v)$  for  $x$  and  $y$  separately, considering  $x, y, z, p, q$  all involved in  $u$  and  $v$ , and eliminate  $f'(v)$ . The result is equation (2), with the values

**3426**  $\mu R = u_p(v_y) - (u_y)v_p, \quad \mu T = v_q(u_x) - (v_x)u_q,$   
 $\mu S = v_p(u_x) - (v_x)u_p - v_q(u_y) + (v_y)u_q,$   
 $\mu U = u_pv_q - u_qv_p, \quad \mu V = (u_y)(v_x) - (u_x)(v_y),$   
 with the notation (3422),  $\mu$  being an undetermined constant.

**3427** COR.—The condition to be fulfilled in order that equation (1) may have a first integral of the form  $u = f(v)$  is

$$u_pv_q - u_qv_p = 0.$$

#### SOLUTION BY MONGE'S METHOD OF

**3428**  $Rr + Ss + Tt = V.$

RULE.—Write the two equations

$$Rdy^2 - Sdx dy + Tdx^2 = 0 \dots\dots\dots (1),$$

$$Rdp dy - Vdx dy + Tdq dx = 0 \dots\dots\dots (2).$$

Resolve (1) into its factors, producing the two equations

$$dy - m_1 dx = 0 \quad \text{and} \quad dy - m_2 dx = 0.$$

From  $dy = m_1 dx$  and equation (2) combined, if necessary, with  $dz = p dx + q dy$ , find two 1st integrals  $u = a, v = b$ ; then  $u = f(v)$  will be one 1st integral of the given equation. Similarly from  $dy = m_2 dx$  find another 1st integral.

**3429** The final 2nd integral may be found from one of the 1st integrals by Lagrange's method (3383).

**3430** Otherwise, determine  $p$  and  $q$  in terms of  $x, y, z$  from the two 1st integrals; substitute in  $dz = p dx + q dy$ , and then integrate by (3322) to obtain the final integral.

**3431** If equation (1) is a perfect square, there will be only one 1st integral, and Lagrange's method only is applicable.

PROOF.—By (3427) we may put  $u_q = mu_p, v_q = mv_p$ ; and also  $dz = p dx + q dy$  (3321) in the complete derivatives

$$\begin{aligned} (du) &= u_x dx + u_y dy + u_z dz + u_p dp + u_q dq = 0, \quad (dv) = \&c. = 0; \\ \therefore \text{by (3422)} \quad &\left. \begin{aligned} (u_x) dx + (u_y) dy + u_p (dp + m dq) &= 0 \\ (v_x) dx + (v_y) dy + v_p (dp + m dq) &= 0 \end{aligned} \right\} \dots\dots\dots (3). \end{aligned}$$

Solving these equations for the ratios  $dx : dy : dp + m dq$ , we obtain at once

$$\frac{dx}{R} = \frac{dy + m dx}{S} = \frac{m dy}{T} = \frac{dp + m dq}{V} \dots\dots\dots (4),$$

with the values of  $R, S, T, V$  in (3426).

Equations (1) and (2) are the result of eliminating  $m$  from (4). These two equations with  $dz = p dx + q dy$  suffice to determine a first integral of (3428) when it exists in the form  $u = f(v)$ .

**3432** Ex. (i.) :  $q(1+q)r - (p+q+2pq)s + p(1+p)t = 0$ .

Solving the quadratic equation (1), we find

$$p dx + q dy = 0, \text{ or } (1+p) dx + (1+q) dy = 0 \dots\dots\dots(5).$$

First,  $dz = p dx + q dy = 0, \therefore z = A$ .

Monge's equation (2) is  $q(1+q) dp dy + p(1+p) dq dx = 0$ ,

which, by  $p dx = -q dy$ , gives  $\frac{dp}{1+p} = \frac{dq}{1+q}$ ; and, integrating,  $\frac{1+p}{1+q} = B$ .

Hence a first integral is  $\frac{1+p}{1+q} = \phi(z) \dots\dots\dots(6)$ .

Next, taking the second equation of (5) with

$$dz = p dx + q dy, \quad dx + dy + dz = 0, \quad \therefore x + y + z = C.$$

Also, by (5), equation (2) now reduces to  $q dp = p dq$ , and by integration,  $p = qD$ ; therefore the other first integral is  $p = q\psi(x+y+z)$ .

For the final integral integrate  $p - q\psi = 0$ ; i.e.,  $z_x - \psi z_p = 0$ , by (3383);

$$\therefore dx = -\frac{dy}{\psi(x+y+z)} = \frac{dz}{0}, \quad \therefore z = A, \quad \text{and} \quad dx = \frac{dx + dy + dz}{1 - \psi(x+y+z)}.$$

$$\therefore x = \int \frac{d(x+y+z)}{1 - \psi(x+y+z)} = F(x+y+z) + B.$$

Hence the second integral is  $x - f(x+y+z) = F(z)$ .

**3433** Ex. (ii.) :  $z_{2x} - a^2 z_{2y} = 0$ .

(i.) Here, in (3428),  $R = 1$ ,  $S = 0$ ,  $T = -a^2$ ,  $V = 0$ ; therefore (1) and (2) become  $dy^2 - a^2 dx^2 = 0$ ,  $dp dy - a^2 dq dx = 0$ .

From (1)  $dy + a dx = 0$ , giving  $y + ax = c$ , and converting (2) into  $dp + a dq = 0$ , which gives  $p + aq = c'$ ; therefore a first integral is

$$p + aq = \phi(y + ax) \dots\dots\dots(3).$$

Similarly, from (1),  $dy - a dx = 0$  gives rise to another first integral

$$p - aq = \psi(y - ax) \dots\dots\dots(4).$$

Eliminating  $p$  and  $q$  by means of (3) and (4) from  $dz = p dx + q dy$ , we find

$$dz = (2a)^{-1} \{ \phi(y + ax)(dy + a dx) - \psi(y - ax)(dy - a dx) \},$$

therefore, by integrating,  $z = \Phi(y + ax) + \Psi(y - ax)$ .

For the symbolic solution of the same equation, see (3566).

#### SOLUTION OF THE P. D. EQUATION.

**3434**  $Rr + Ss + Tt + U(rt - s^2) = V \dots\dots\dots(1).$

Let  $m_1, m_2$  be the roots of the quadratic equation

**3435**  $m^2 - Sm + RT + UV = 0 \dots\dots\dots(2).$



Let  $u_1 = a$ ,  $v_1 = b$ , and  $u_2 = a'$ ,  $v_2 = b'$  be respectively the solutions of the two systems of ordinary differential equations.

$$\begin{aligned} \text{3436} \quad \left. \begin{aligned} Udp &= m_2 dy - T dx \\ Udq &= m_1 dx - R dy \\ dz &= p dx + q dy \end{aligned} \right\} (3), \quad \left. \begin{aligned} Udp &= m_1 dy - T dx \\ Udq &= m_2 dx - R dy \\ dz &= p dx + q dy \end{aligned} \right\} (4). \end{aligned}$$

Then the first integrals of (1) will be

$$u_1 = f(v_1), \quad u_2 = f(v_2).$$

To obtain a second integral:

**3437** 1st.—When  $m_1, m_2$  are unequal, assign any particular forms to  $f_1$  and  $f_2$ , then substitute the values of  $p$  and  $q$ , found from these equations in terms of  $x$  and  $y$ , in  $dz = p dx + q dy$ , which integrate. Otherwise, assign the form of one only of the functions  $f_1, f_2$ , involving an arbitrary constant  $C$ , solve for  $p$  and  $q$ , and integrate  $dz = p dx + q dy$ , adding an arbitrary function of  $C$  for the constant of integration.

**3438** 2ndly.—When  $m_1, m_2$  are equal, and therefore, by (2),

$$S^2 = 4(RT + UV) \dots\dots\dots (5).$$

Equations (3) and (4) coincide, and, since  $m = \frac{1}{2}S$ , reduce to

$$\text{3439} \quad Udp = \frac{1}{2}S dy - T dx \dots\dots\dots (6),$$

$$Udq = \frac{1}{2}S dx - R dy \dots\dots\dots (7),$$

$$dz = p dx + q dy \dots\dots\dots (8).$$

Here  $p_y = q_x$ , and therefore the last equation is integrable if the values of  $p$  and  $q$ , obtained by integrating (6) and (7), be substituted in it. Let  $u = a$ ,  $v = b$  be the integrals of (6) and (7); and let  $z = \phi(x, y, a, b, c) \dots\dots\dots (9)$  be the integral obtained from (8).

The general integral is found by making the parameters  $a, b, c$  vary subject to two conditions  $b = f(a)$ ,  $c = F(a)$ ; that is, by differentiating

$$z = \phi\{x, y, a, f(a), F(a)\}$$

for  $a$ , and eliminating  $a$ .

**3440** The general integral therefore represents the envelope of the surface whose equation is (9).

PROOF.—(Boole, *Sup.*, p. 147.) Assuming a 1st integral of the form  $u = f(v)$ , eliminate  $\mu$  and  $v$  from equations (3426) by multiplying (i.) by  $(u_x) u_q$ , (ii.) by  $(u_y) u_p$ , (iv.) by  $(u_x)(u_y)$ , (v.) by  $u_p u_q$ , and adding. Again, eliminate  $\mu$  and  $v$  by multiplying (i.) by  $(u_x)^2$ , (ii.) by  $(u_y)^2$ , (iii.) by  $(u_x)(u_y)$ , (v.) by  $(u_x) u_p + (u_y) u_q$ , and adding. The two resulting equations are

$$\left. \begin{aligned} R(u_x) u_q + T(u_y) u_p - U(u_x)(u_y) + V u_p u_q &= 0 \\ R(u_x)^2 + S(u_x)(u_y) + T(u_y)^2 + V\{(u_x) u_p + (u_y) u_q\} &= 0 \end{aligned} \right\} \dots\dots(10).$$

Multiply the 2nd of these by  $m$ , divide by  $V$ , and add to the 1st equation; the result is expressible in two factors either as (11) or (12),

$$\{R(u_x) + m_1(u_y) + V u_p\} \{m_1(u_x) + T(u_y) + V u_q\} = 0 \dots\dots\dots(11),$$

$$\{R(u_x) + m_2(u_y) + V u_p\} \{m_2(u_x) + T(u_y) + V u_q\} = 0 \dots\dots\dots(12),$$

$m_1, m_2$  being the roots of the quadratic (2). By equating to zero one factor of (11) and one of (12), we have four systems of two linear 1st order P. D. equations. Taking each system in turn with the equations

$$(u_x) + r u_p + s u_q = 0,$$

$$(u_y) + s u_p + t u_q = 0,$$

and eliminating  $(u_x), (u_y), u_p, u_q$ , we have the determinant annexed for the case in which the 1st factor of (11) and the 2nd of (12) are equated to zero. In this case, and also when the 2nd factor of (11) and the 1st of (12) are chosen, transposing  $m_1, m_2$  in the determinant, the eliminant is equivalent to

$$V\{Rr + Ss + Tt + U(rt - s^2) - V\} = 0,$$

having regard to the values of  $m_1 m_2$  and  $m_1 + m_2$  from (2).

When the 1st factor of both (11) and (12) is taken, the 2nd order P. D. equation produced by the elimination is

$$Vt - R(rt - s^2) = 0,$$

and when the 2nd factor of each is taken, the elimination produces

$$Vr - T(rt - s^2) = 0.$$

Hence the hypothesis of a 1st integral of (1), of the form  $u = f(v)$ , involves the satisfying one or other of the systems of two simultaneous equations, (13) or (14), below:

$$\left. \begin{aligned} R(u_x) + m_1(u_y) + V u_p &= 0 \\ m_2(u_x) + T(u_y) + V u_q &= 0 \end{aligned} \right\} \dots(13). \quad \left. \begin{aligned} R(u_x) + m_2(u_y) + V u_p &= 0 \\ m_1(u_x) + T(u_y) + V u_q &= 0 \end{aligned} \right\} \dots(14).$$

Now multiply the 2nd equation of (13) by  $\lambda$  and add it to the 1st. In the result, collect the coefficients of  $u_x, u_y, u_p, u_q, u_z$ . The Lagrangean system of auxiliary equations (3387) will then be found to be

$$\frac{dx}{R + \lambda m_2} = \frac{dy}{m_1 + \lambda T} = \frac{dp}{V} = \frac{dq}{\lambda V} = \frac{dz}{Rp + m_1 q + \lambda(Tq + m_2 p)} = \frac{du}{0}.$$

Eliminating  $\lambda$ , equations (3) are produced. Treating equations (14) in the same way, equations (4) are produced.

## POISSON'S EQUATION.

**3441**

$$P = (rt - s^2)^n Q,$$

where  $P$  is a function of  $p, q, r, s, t$ , and homogeneous in  $r, s, t$ ;  $Q$  is a function of  $x, y, z$ , and derivatives of  $z$ , which does not become infinite when  $rt - s^2$  vanishes, and  $n$  is positive.

**RULE.**—Assume  $q = \phi(p)$  and express  $s$  and  $t$  in terms of  $q_p$  and  $r$ ; thus,  $rt - s^2$  vanishes, and the left side becomes a function of  $p, q$ , and  $q_p$ .

Solve for a 1st integral in terms of  $p$  and  $q$ , and integrate again for the final solution.

**PROOF.**—  $s = q_x = q_p p_x = q_p r$ ;  $t = q_y = q_p p_y = q_p^2 r$ ; therefore  $rt - s^2 = 0$ . Also  $P$  is of the form  $(r, s, t)^m = r^m (1, q_p, q_p^2)^m$ . Hence the equation takes the form  $(1, q_p, q_p^2)^m = 0$ .

## LAPLACE'S REDUCTION OF THE EQUATION.

**3442**

$$Rr + Ss + Tt + Pp + Qq + Zz = U \dots\dots\dots (1),$$

where  $R, S, T, P, Q, Z, U$  are functions of  $x$  and  $y$  only.

Let two integrals of Monge's equation (3428)

$$Rdy^2 - Sdx dy + Tdx^2 = 0$$

be

$$\phi(x, y) = a, \quad \psi(x, y) = b.$$

Assume

$$\xi = \phi(x, y), \quad \eta = \psi(x, y).$$

**3443** To change the variables in equation (1) to  $\xi$  and  $\eta$ , we have

$$\begin{aligned} p &= z_x = z_\xi \xi_x + z_\eta \eta_x; & q &= z_y = z_\xi \xi_y + z_\eta \eta_y; \\ r &= z_{xx} = z_{\xi\xi} \xi_x^2 + 2z_{\xi\eta} \xi_x \eta_x + z_{\eta\xi} \eta_x^2 + z_\xi \xi_{xx} + z_\eta \eta_{xx}; \\ t &= z_{yy} = z_{\xi\xi} \xi_y^2 + 2z_{\xi\eta} \xi_y \eta_y + z_{\eta\xi} \eta_y^2 + z_\xi \xi_{yy} + z_\eta \eta_{yy}; \\ s &= z_{xy} = z_{\xi\xi} \xi_x \xi_y + z_{\xi\eta} \xi_x \eta_y + z_{\eta\xi} (\xi_y \eta_x + \xi_x \eta_y) + z_\xi \xi_{xy} + z_\eta \eta_{xy}. \end{aligned} \quad (1701)$$

The transformed equation is of the form

$$z_{\xi\eta} + Lz_\xi + Mz_\eta + Nz = V \dots\dots\dots (2),$$

where  $L, M, N, V$  are functions of  $\xi$  and  $\eta$ . This equation may be written in the form

$$(d_\xi + M)(d_\eta + L)z + (N - LM - L_\xi)z = V \dots\dots\dots (3).$$

If

$$N - LM - L_\xi = 0 \dots\dots\dots (4),$$

we shall have

$$(d_\xi + M)z' = V \quad \text{with} \quad (d_\eta + L)z = z',$$

and the solution by a double application of (3210) is obtained from

$$z = e^{-\int L d\eta} \left\{ \phi(\xi) + \int z' e^{\int L d\eta} d\eta \right\},$$

$$z' = e^{-\int M d\xi} \left\{ \psi(\eta) + \int V e^{\int M d\xi} d\xi \right\}.$$

By symmetry, equation (1) is also solvable, if

$$N - LM - M_\eta = 0 \dots\dots\dots (5).$$

But if neither of these conditions is found to hold, find  $z$  in terms of  $z'$  from (3). It will be of the form

$$z = Az'_\xi + Bz'_\eta + C,$$

where  $A, B, C$  contain  $\xi$  and  $\eta$ . Substitute this for  $z$  in  $(d_\eta + L)z = z'$ , and the result is of the form

$$z'_{\xi\eta} + L'z'_\xi + M'z'_\eta + N'z' = V'.$$

The same conditions of integrability, if fulfilled for this equation, will lead to a solution of (1), and, if not fulfilled, the transformation may be repeated until one of the equations, similar to (4) or (5), is satisfied.

**3444** COR.—The solution of the equation

$$z_{\xi\eta} + az_\xi + bz_\eta + abz = V$$

is  $z = e^{-a\eta} \phi(\xi) + e^{-b\xi} \psi(\eta) + e^{-a\eta-b\xi} \iint e^{a\eta+b\xi} V d\eta d\xi.$

**3445** For the solution of equation (2), when  $L, M, V$  contain also  $z$ , see Prof. Tanner, *Proc. Lond. Math. Soc.*, Vol. viii., p. 159.

## LAW OF RECIPROCITY.

[Boole, ch. xv.]

**3446** Let a differential equation of the 1st order be

$$\phi(x, y, z, p, q) = 0 \dots\dots\dots (1).$$

Let the result of interchanging  $x$  and  $p$ ,  $y$  and  $q$ , and of changing  $z$  into  $px + qy - z$ , be

$$\phi(p, q, px + qy - z, x, y) = 0 \dots\dots\dots (2);$$

then, if  $z = \psi(x, y)$  be the solution of either (1) or (2), the

solution of the other will be obtained by eliminating  $\xi$  and  $\eta$  between the equations

$$x = d_{\xi}\psi(\xi, \eta), \quad y = d_{\eta}\psi(\xi, \eta), \quad z = \xi x + \eta y - \psi(\xi, \eta).$$

**3447** Ex.—Let  $z = pq \dots\dots (1)$ ,  $px + qy - z = xy \dots\dots (2)$ ,  
be the two reciprocal equations.

The integral of (2) is  $z = xy + xf\left(\frac{y}{x}\right)$ ,  $\therefore \psi(\xi\eta) = \xi\eta + xf\left(\frac{\eta}{\xi}\right)$ .

$\xi, \eta$  have now to be eliminated between

$$x = \eta - \frac{\eta}{\xi} f'\left(\frac{\eta}{\xi}\right) + f\left(\frac{\eta}{\xi}\right), \quad y = \xi + f'\left(\frac{\eta}{\xi}\right), \quad z = \xi\eta \dots\dots (3).$$

Each form assigned to  $f$  gives a particular integral of (1). If  $f\left(\frac{\eta}{\xi}\right) = a\frac{\eta}{\xi} + b$ , the equations (3) become  $x = \eta + b$ ,  $y = \xi + a$ ,  $z = \xi\eta$ , and the elimination produces  $z = (x-b)(y-a)$ .

**3448** In an equation of the 2nd order, the reciprocal equation is formed by making the changes in (3446), and, in addition, changing

$$r \text{ into } \frac{t}{rt-s^2}, \quad s \text{ into } \frac{-s}{rt-s^2}, \quad t \text{ into } \frac{r}{rt-s^2};$$

then, if the 2nd integral of either equation be  $z = \psi(x, y)$ , that of the other will be found by the same rule.

**3449** The above transformation makes any equation of the form  $\phi(p, q)r + \psi(p, q)s + \chi(p, q)t = 0$  dependent for solution upon one of the form

$$\chi(x, y)r - \psi(x, y)s + \phi(x, y)t = 0.$$

**3450** And, in the same way, an equation of the form

$$Rr + Ss + Tt = V(rt - s^2)$$

is dependent for solution upon one of the form

$$Rr + Ss + Tt = V.$$

See *De Morgan*, Camb. Phil. Trans., Vol. VIII.

## SYMBOLIC METHODS.

### FUNDAMENTAL FORMULÆ.

$Q$  denoting a function of  $\theta$ ,

**3470**  $(d_{\theta} - m)^{-1} Q = e^{m\theta} \int e^{-m\theta} Q d\theta.$

PROOF.—The right is the value of  $y$  in the solution of  $d_\theta y - my = Q$  by (3210). But this equation is expressed symbolically by  $(d_\theta - m)y = Q$  (see 1492), therefore  $y = (d_\theta - m)^{-1} Q$ .

Let  $x = e^\theta$ , therefore  $d_\theta = x d_x$  and  $x d\theta = dx$ . Hence (3470) may be written

$$3471 \quad (x d_x - m)^{-1} Q = x^m \int x^{-m-1} Q dx.$$

$$3472 \quad \text{COR.—} \quad (d_\theta - m)^{-1} 0 = C e^{m\theta},$$

$$3473 \quad \text{or} \quad (x d_x - m)^{-1} 0 = C x^m.$$

Let  $F(m)$  denote a rational integral function of  $m$ ; then, since  $d_\theta e^{m\theta} = m e^{m\theta}$ ,  $d_{2\theta} e^{m\theta} = m^2 e^{m\theta}$ , &c., the operation  $d_\theta$  is always replaced by the operation  $m \times$ . Hence, in all cases,

$$3474 \quad F(d_\theta) e^{m\theta} = e^{m\theta} F(m).$$

$$3475 \quad F(d_\theta) e^{m\theta} Q = e^{m\theta} F(d_\theta + m) Q.$$

Formula (2161) is a particular case of this theorem.

$$3476 \quad e^{m\theta} F(d_\theta) Q = F(d_\theta - m) e^{m\theta} Q.$$

Also, by (3474–6),

$$3477 \quad F(m) = e^{-m\theta} F(d_\theta) e^{m\theta}.$$

$$3478 \quad F(d_\theta + m) Q = e^{-m\theta} F(d_\theta) e^{m\theta} Q.$$

$$3479 \quad F(d_\theta) Q = e^{-m\theta} F(d_\theta - m) e^{m\theta} Q.$$

To the last six formulæ correspond

$$3480 \quad F(x d_x) x^m = x^m F(m).$$

$$3481 \quad F(x d_x) x^m Q = x^m F(x d_x + m) Q.$$

$$3482 \quad x^m F(x d_x) Q = F(x d_x - m) x^m Q.$$

$$3483 \quad F(m) = x^{-m} F(x d_x) x^m.$$

$$3484 \quad F(x d_x + m) Q = x^{-m} F(x d_x) x^m Q.$$

$$3485 \quad F(x d_x) Q = x^{-m} F(x d_x - m) x^m Q.$$

If  $U = a + bx + cx^2 + \&c.$ , then, by (3480),

$$3486 \quad F(x d_x) U = F(0) a + F(1) bx + F(2) cx^2 + \&c.$$



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$$3487 \quad F^{-1}(xd_x) U = F^{-1}(0) a + F^{-1}(1) bx + F^{-1}(2) cx^2 + \&c.$$

$$3488 \quad F(xd_x, yd_y, zd_z, \dots) x^m y^n z^p \dots = F(m, n, p \dots) x^m y^n z^p \dots$$


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$$3489 \quad x^n u_{nx} = d_\theta (d_\theta - 1) (d_\theta - 2) \dots (d_\theta - n + 1) u,$$

or, more succinctly, writing  $D$  for  $d_\theta$ ,

$$D(D-1) \dots (D-n+1) u \quad \text{or} \quad D_{-1}^{(n)} u \quad (2452).$$

$$3490 \quad \text{Otherwise } x^n u_{nx} = xd_x(xd_x - 1) \dots (xd_x - n + 1) u.$$

PROOF.—As in (1770). Otherwise, by Induction, differentiating again, and remembering that  $x_\theta = x$ .

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NOTE.—In the symbolic solution of differential equations, we may either employ the operator  $xd_x$  directly, or the operator  $d_\theta$  after substituting  $e^\theta$  for  $x$ . Formulæ (3480–5) or (3474–9) will be required accordingly.

$$3491 \quad \begin{aligned} & \{\phi(D) e^{r\theta}\}^n Q \\ &= e^{nr\theta} \phi(D+nr) \phi(D+\overline{n-1}.r) \dots \phi(D+r) Q \\ &= \phi(D) \phi(D-r) \phi(D-2r) \dots \phi\{D-(n-1)r\} e^{nr\theta} Q. \end{aligned}$$

PROOF.—By repeated application of (3475) or (3476).

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For ready reference, formulæ (1520, '21) are reprinted here.

$$3492 \quad f(x+h) = e^{hd_x} f(x).$$

$$3493 \quad f(x+h, y+k) = e^{hd_x + kd_y} f(x, y).$$


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Let  $a_0 + a_1 x + a_2 x^2 \dots + a_n x^n = f(x)$ ,  
then, denoting  $d_\theta$  by  $D$ ,

$$3494 \quad f(D) uv = uf(D) v + u_\theta f'(D) v + \frac{u_{2\theta}}{1.2} f''(D) v + \&c.,$$

where  $f'(D)$  means that  $D$  is to be written for  $x$  after differentiating  $f(x)$ .

PROOF.—Expand  $uv$ ,  $D.uv$ ,  $D^2.uv \dots D^n.uv$  by Leibnitz's theorem (1460); multiply the equations respectively by  $a_0, a_1, a_2 \dots a_n$ , and add the results.

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$$3495 \quad uf(D) v = f(D).uv - f'(D) u_\theta v + \frac{1}{1.2} f''(D) u_{2\theta} v - \&c.$$

PROOF.—Expand  $uv_1, uv_2, uv_3 \dots uv_n$ , by theorem (1472), and proceed as in the last.

$$3496 \quad F(d_{2x}) \frac{\sin}{\cos} mx = F(-m^2) \frac{\sin}{\cos} mx.$$

A more general theorem is

$$3497 \quad F(\pi^2)(u_{im} + u_{-im}) = F(-m^2)(u_{im} + u_{-im}),$$

where  $u$  and  $\pi$  have the meanings assigned below (3499), and  $i = \sqrt{-1}$ .

*Theorem.*—If  $\phi$  and  $\psi$  denote any algebraic functions of  $x$  and  $y$ , it may be shown, by (3474) and (3475), that

$$3498 \quad \psi(d_x + y) \phi(x) = \phi(d_y + x) \psi(y).$$

3499 Let  $u$ , or, more definitely,  $u_n = (x, y, z, \dots)^n$ , represent a homogeneous function of the  $n^{\text{th}}$  degree in severable variables, and let

$$3500 \quad \pi = xd_x + yd_y + zd_z + \&c.$$

Then, by (3480),

$$3501 \quad \pi u = nu, \quad \pi^2 u = n^2 u, \quad \pi^3 u = n^3 u, \quad \&c.$$

$$3502 \quad \text{Hence} \quad F(\pi) u = F(n) u.$$

#### REDUCTION OF $F(\pi_1)$ TO $f(\pi)$ .

3503 Let  $u$  be any implicit function of the variables, and let  $\pi = \pi_1 + \pi_2$ , where  $\pi_1$  operates only upon  $x$  as contained in  $u$ , and  $\pi_2$  only upon  $x$  as contained in  $\pi u$ , &c. after repetitions of the operation  $\pi$ . Then

$$3504 \quad \begin{array}{lll} \pi_1 u = \pi u, & \pi_1^2 u = (\pi - 1) \pi u, \\ \dots & \dots & \dots \end{array}$$

$$3506 \quad \pi_1^r u = (\pi - r + 1) \dots (\pi - 2)(\pi - 1) \pi u.$$

$$\text{PROOF.—} \quad \pi_1 u = (\pi - \pi_2) u = \pi u,$$

since  $\pi_2$  has here no subject to operate upon.

$$\pi_1^2 u = (\pi - \pi_2) \pi u = (\pi - 1) \pi u,$$

for,  $\pi u$  being of the 1st degree,  $\pi_2$  and 1 are equivalent as operators. In the next step,  $\pi_2$  and 2 are equivalent, and so on.

COR.—When  $u$  is a homogeneous function, we have, by

(3501),  $\pi^r u = n^r u$ , therefore  $\pi$  and  $n$  are equivalent operators upon  $u$ . Hence (3506) may be written

$$\text{3507} \quad \pi_1^r u = (n-r+1) \dots (n-2)(n-1)nu = n_{-1}^{(r)} u,$$

which is Euler's theorem of homogeneous functions (1625), since in that theorem the operator is confined to  $u$ .

**3508** As an illustration, let  $\pi u = (xd_x + yd_y)u = \pi_1 u$ ,

$$\text{then } \pi_1^2 u = (x^2 d_{xx} + 2xy d_{xy} + y^2 d_{yy})u, \quad \pi^2 u = (\pi_1 + \pi_2) \pi u = \pi_1^2 u + \pi_2 \pi u.$$

Here

$$\pi_2 \pi u = (xd_x + yd_y)(xd_x + yd_y)u,$$

the operation being confined to  $x$  and  $y$  in the second factor (3503), and therefore producing  $(xd_x + yd_y)u$  merely.

Hence  $\pi^2 u = (x^2 d_{xx} + 2xy d_{xy} + y^2 d_{yy} + xd_x + yd_y)u$ , which proves (3505).

If  $U = u_0 + u_1 + u_2 + \dots$ , a series of homogeneous functions of dimensions 0, 1, 2, ..., then, by (3502),

$$\text{3509} \quad F(\pi)U = F(0)u_0 + F(1)u_1 + F(2)u_2 + \dots,$$

$$\text{3510} \quad F^{-1}(\pi)U = F^{-1}(0)u_0 + F^{-1}(1)u_1 + F^{-1}(2)u_2 + \dots$$

$$\text{3511} \quad \text{Ex. 1:} \quad a^\pi U = u_0 + au_1 + a^2 u_2 + \dots,$$

$$\text{3512} \quad a^{-\pi} U = u_0 + a^{-1}u_1 + a^{-2}u_2 + \dots$$

Ex. 2: If  $u$  have the meaning in (3499),

**3513**

$$F(\pi)e^u = F(0)1 + F(n)u + F(2n)\frac{u^2}{1.2} + F(3n)\frac{u^3}{1.2.3} + \&c.,$$

and similarly for the inverse operation  $F^{-1}(\pi)$ .

PROOF.—By (3502) applied to the expansion of the subject by (150).

$$\text{3514} \quad \frac{C(n, m)u_n}{m!} = \sum \frac{x^p d_{px} y^q d_{qy} z^r d_{rz} \dots u_n}{p! q! r! \dots},$$

where  $p+q+r+\dots = m$ , and  $p! = 1.2 \dots p$ .

PROOF.—Equate coefficients of  $a^m$  in the expansion of

$$(1+a)^\pi U = (1+a)^{xd_x} (1+a)^{yd_y} (1+a)^{zd_z} \dots U,$$

reducing by (3490).

**3515** The general symbolic solution of the equation  $F(d_\theta)u = Q$  is

$$u = F^{-1}(d_\theta)Q + F^{-1}(d_\theta)0, \quad \text{by (1488-90).}$$

**3516** The solution of the equation (3238), viz.,

$$y_{nx} + a_1 y_{(n-1)x} + \dots + a_{(n-1)} y_x + a_n y = Q \dots \dots \dots (1),$$

where  $Q$  is a function of  $x$ , is most readily obtained by the symbolic method. Thus  $m_1, m_2, \dots m_n$  being the roots of the auxiliary equation in (3239), and  $A, B, C \dots N$  the numerators of the partial fractions into which  $(m^n + a_1 m^{n-1} + \dots + a_n)^{-1}$  can be resolved, the complete primitive will be

**3517**

$$y = \{A(d_x - m_1)^{-1} + B(d_x - m_2)^{-1} \dots + N(d_x - m_n)^{-1}\} (Q + 0),$$

where  $(d_x - m)^{-1} Q = e^{mx} \int e^{-mx} Q dx, \quad (3470)$

and the whole operation upon zero produces, by (3472), for the complementary term,

**3518**  $C_1 e^{m_1 x} + C_2 e^{m_2 x} \dots + C_n e^{m_n x}.$

PROOF.—Equation (1) may be written

$$(d_{nx} + a_1 d_{(n-1)x} + a_2 d_{(n-2)x} \dots + a_n) y = Q,$$

or  $(d_x - m_1)(d_x - m_2) \dots (d_x - m_n) y = Q,$

$\therefore$  by (3515),  $y = \{(d_x - m_1)(d_x - m_2) \dots (d_x - m_n)\}^{-1} (Q + 0),$

which, by partial fractions, is converted into the formula above.

If  $r$  of the roots  $m_1, m_2, \dots$  are each  $= m$ , those roots give rise in (3517) to a single term of the form

**3519**  $(A + Bd_x + Cd_{2x} \dots + Rd_{rx}) e^{mx} \int_{rx} e^{-mx} Q.$

PROOF.—By (1918), the  $r$  roots equal to  $m$  will produce

$$\{A'(d_x - m)^{-r} + B'(d_x - m)^{-r+1} \dots + R'(d_x - m)^{-1}\} Q,$$

or  $(A + Bd_x + Cd_{2x} \dots + Rd_{rx})(d_x - m)^{-r} Q.$

**3520** But, by (3470),  $(d_x - m)^{-2} Q = (d_x - m)^{-1} e^{mx} \int e^{-mx} Q dx$   
 $= e^{mx} \int \{e^{-mx} e^{mx} \int e^{-mx} Q dx\} dx = e^{mx} \int_{2x} e^{-mx} Q dx, \text{ and so on.}$

**3521** Ex. (1):  $y_{2x} - y_{1x} - 5y_x - 3 = Q.$

Here  $m^2 - m - 5m - 3 = (m - 3)(m + 1)^2,$

and  $\frac{1}{(m - 3)(m + 1)^2} = \frac{1}{16(m - 3)} - \frac{1}{16(m + 1)} - \frac{1}{4(m + 1)^2},$

therefore  $y = \frac{1}{16} (d_x - 3)^{-1} Q - \frac{1}{16} (d_x + 1)^{-1} Q - \frac{1}{4} (d_x + 1)^{-2} Q$   
 $= \frac{1}{16} e^{3x} \int e^{-3x} Q dx - \frac{1}{16} e^x \int e^{-x} Q dx - \frac{1}{4} e^{-x} \int \int e^x Q dx^2. \quad (3520)$   

3 X

**3522** Ex. (2):  $u_{2x} + a^2 u = Q,$

therefore  $u = (d_{2x} + a^2)^{-1} Q.$

Here  $(m^2 + a^2)^{-1} = (2ia)^{-1} \{ (m - ia)^{-1} - (m + ia)^{-1} \},$

therefore 
$$\begin{aligned} u &= (2ia)^{-1} \{ (d_x - ia)^{-1} Q - (d_x + ia)^{-1} Q \} \\ &= (2ia)^{-1} \{ e^{iax} \int e^{-iax} Q dx - e^{-iax} \int e^{iax} Q dx \} \quad (3470) \\ &= a^{-1} \sin ax \int \cos ax Q dx - a^{-1} \cos ax \int \sin ax Q dx, \end{aligned}$$

by the exponential values (766-7).

**3523** COR. 1.—The solution of  $u_{2x} + a^2 u = 0$  is

$$u = A \cos ax + B \sin ax.$$

**3524** COR. 2.—The solution of  $u_{2x} - a^2 u = 0$  is

$$u = Ae^{ax} + Be^{-ax}.$$

Change  $a$  into  $ia$  in the fifth line of (3522), and put  $Q = 0$ .

**3525** When  $Q$  is a function whose derivatives of the  $n^{\text{th}}$  and higher orders vanish, proceed as in the following example.

Ex. (3):  $u_{2x} + a^2 u = (1+x)^2,$

therefore 
$$\begin{aligned} u &= (d_{2x} + a^2)^{-1} (1+x)^2 + (d_{2x} + a^2)^{-1} 0 \\ &= (a^{-2} - a^{-4} d_{2x} + a^{-6} d_{4x} - \&c.) (1+2x+x^2) + (d_{2x} + a^2)^{-1} 0 \\ &= a^{-2} (1+x)^2 - 2a^{-4} + A \cos ax + B \sin ax, \end{aligned}$$

the last two terms by (3523).

### *Exceptional Case of the Inverse Process.*

**3526** Ex. (4):  $u_{2x} + a^2 u = \cos nx,$

$$\begin{aligned} \therefore u &= (d_{2x} + a^2)^{-1} (\cos nx + 0) = \frac{1}{2} (d_{2x} + a^2)^{-1} (e^{inx} + e^{-inx} + 0) \\ &= \frac{1}{2} (e^{inx} + e^{-inx}) (-n^2 + a^2)^{-1} + A \cos ax + B \sin ax \text{ by (3474) and (3523)} \\ &= \cos nx (a^2 - n^2)^{-1} + \&c. \end{aligned}$$

Now, if  $n = a$ , the first term becomes infinite. In such cases proceed as follows:—

Put  $A = A' - (a^2 - n^2)^{-1}$ , and find the value of  $\frac{\cos nx - \cos ax}{a^2 - n^2}$ , when  $n = a$ . By (1580) it is  $= \frac{x \sin ax}{2a}$ . Thus the solution is

$$u = \frac{x \sin ax}{2a} + A' \cos ax + B \sin ax.$$

The same result is obtained by making  $Q = \cos ax$  in the solution of (3522).

For another example, see (3559).

**3527** Ex. (5):  $y_{2x} - 9y_x + 20y = x^3 e^{2x},$

therefore  $y = \{(d_x - 4)(d_x - 5)\}^{-1} x^3 e^{2x} + \{(d_x - 4)(d_x - 5)\}^{-1} 0$   
 $= e^{2x} \{(d_x - 1)(d_x - 2)\}^{-1} x^3 + Ae^{4x} + Be^{5x}. \quad (3475, 3517, 3472)$

Now  $(m^3 - 3m + 2)^{-1} = \frac{1}{2} \left(1 - \frac{3m - m^3}{2}\right)^{-1}$   
 $= \frac{1}{2} \left\{1 + \frac{3m - m^3}{2} + \left(\frac{3m - m^3}{2}\right)^2 + \&c.\right\}$

Hence the solution becomes

$$y = e^{2x} \left\{ \frac{1}{2} + \frac{3}{4}d_x + \frac{1}{8}d_{2x} + \&c. \right\} x^3 + Ae^{4x} + Be^{5x}$$

$$= e^{2x} \left\{ \frac{x^3}{2} + \frac{3x^2}{2} + \frac{7x}{4} \right\} + Ae^{4x} + Be^{5x}.$$

**3528** Ex. (6):  $(d_x - a)^n u = e^{ax},$

therefore  $u = (d_x - a)^{-n} e^{ax} = e^{ax} (d_x)^{-n} 1 \quad (3476) = e^{ax} \left( \frac{x^n}{n!} + \int_{nx} 0 \right). \quad (2149)$

**3529** Ex. (7):  $(d_x + a)^2 y = \sin mx,$

therefore  $y = (d_x + a)^{-2} \sin mx + (d_x + a)^{-2} 0$   
 $= (d_x - a)^2 (d_{2x} - a^2)^{-2} \sin mx + e^{-ax} (d_x)^{-2} 0 \quad [\text{by (3478) with } Q = 0]$   
 $= (-m^2 - a^2)^{-2} (d_x - a)^2 \sin mx + e^{-ax} (Ax + B) \quad (\text{by 3496})$   
 $= (m^2 + a^2)^{-2} (-m^2 \sin mx - 2am \cos mx + a^2 \sin mx) + e^{-ax} (Ax + B).$

#### REDUCTION OF AN INTEGRAL OF THE $n^{\text{th}}$ ORDER.

**3530**  $\int_{nx} Q = \frac{1}{n-1!} \left\{ x^{n-1} \int Q dx - (n-1) x^{n-2} \int Q x dx \right.$   
 $\left. + C(n, 2) x^{n-3} \int Q x^2 dx \dots \pm \int Q x^{n-1} dx, \right.$

where  $n-1! = 1.2 \dots n.$

PROOF.—By (3489)  $d_{nx} Q = e^{-n\theta} (d_\theta - n + 1)(d_\theta - n + 2) \dots d_\theta Q \dots \dots \dots (1),$   
therefore  $d_{-nx} Q = \{(d_\theta - n + 1)(d_\theta - n + 2) \dots d_\theta\}^{-1} e^{n\theta} Q$   
 $= \frac{1}{n-1!} \{ (d_\theta - n + 1)^{-1} - (n-1)(d_\theta - n + 2)^{-1}$   
 $\quad + C(n, 2)(d_\theta - n + 3)^{-1} - \&c. \} e^{n\theta} Q \quad (3517)$   
 $= \frac{1}{n-1!} \{ e^{(n-1)\theta} (d_\theta)^{-1} e^\theta - (n-1) e^{(n-2)\theta} (d_\theta)^{-1} e^{2\theta} + \&c. \} Q.$

Then replace  $e^\theta$  by  $x.$

The equation

**3531**  $ax^m y_{mx} + bx^n y_{nx} + \&c. = A + Bx + Cx^2 + \&c. = Q$



may, by (3489), be transformed into

$$\{a(xd_x)^{(m)}_{-1} + b(xd_x)^{(n)}_{-1} + \&c.\} y = Q \quad \text{or} \quad F(xd_x) y = Q.$$

The solution is then obtained from

$$\mathbf{3532} \quad y = F^{-1}(xd_x) Q + F^{-1}(xd_x) 0.$$

The value of the 1st part is given in (3487).

**3533** If  $\alpha, \beta, \gamma, \&c.$  are the roots of  $F(m) = 0$ , the second part gives rise to the arbitrary terms

$$C_1 x^\alpha + C_2 x^\beta + \&c.$$

**3534** If a root  $\alpha$  is repeated  $r$  times, the corresponding terms are

$$x^\alpha \{C_1 (\log x)^{r-1} + C_2 (\log x)^{r-2} + \dots + C_r\}.$$

PROOF.—The partial fractions into which  $F^{-1}(xd_x) 0$  can be resolved, as in (3517), are of the type  $C(xd_x - m)^{-1} 0$ ,  $m$  being a root of  $F(x) = 0$ . But  $(xd_x - m)^{-1} 0 = Cx^m$  (3473),  $C$  being an arbitrary constant.

For a root  $m$  repeated  $r$  times, the typical fraction is  $C(xd_x - m)^{-p}$ ,  $p$  being less than  $r$ . Now

$$(xd_x - m)^p Cx^m (\log x)^{p-1} = (d_x - m)^p C e^{mx} \theta^{p-1} = e^{mx} (d_x)^p C \theta^{p-1} \quad (3475) = 0, \\ \text{therefore} \quad (xd_x - m)^{-p} 0 = Cx^m (\log x)^{p-1}.$$

The equation

$$\mathbf{3535} \quad ay_{m\theta} + by_{n\theta} + \&c. = f(e^\theta, \sin \theta, \cos \theta)$$

is reducible to the form of (3531) by  $x = e^\theta$ ; or, substituting from (768), it may be written

$$F(d_\theta) y = \Sigma (A_m e^{m\theta}),$$

and the solution will take the form

$$\mathbf{3536} \quad y = \Sigma A_m e^{m\theta} F^{-1}(m) + F^{-1}(d_\theta) 0,$$

for the last term of which the forms in (3533–4) are to be substituted with  $x$  changed to  $e^\theta$ .

$$\mathbf{3537} \quad \text{Ex. (1):} \quad x^2 y_{xx} = ax^m + bx^n$$

$$xd_x (xd_x - 1)(xd_x - 2) y = ax^m + bx^n,$$

$$\therefore y = \{xd_x (xd_x - 1)(xd_x - 2)\}^{-1} (ax^m + bx^n) + \{xd_x (xd_x - 1)(xd_x - 2)\}^{-1} 0 \\ = \frac{ax^m}{m(m-1)(m-2)} + \frac{bx^n}{n(n-1)(n-2)} + A + Bx + Cx^2,$$

by (3180) and (3533). A result evident by direct integration.

**3538** Ex. (2):  $x^2 y_{xx} + 3xy_x + y = (1-x)^{-2}$ . By (3490)

$$\{xd_x(xd_x-1)+3xd_x+1\}y = (xd_x+1)^2 y = 1+2x+3x^2+\&c.,$$

$$\begin{aligned} \therefore y &= (xd_x+1)^{-2}(1+2x+3x^2+\&c.) = (0+1)^{-2}+2(1+1)^{-2}x+3(2+1)^{-2}x^2+ \\ (3480) \quad &= 1+\frac{x}{2}+\frac{x^2}{3}+\&c.+\frac{A \log x}{x}+\frac{B}{x} = -\frac{1}{x} \log(1-x)+\&c. \end{aligned}$$

**3539** Ex. (3):  $y_{2x} + (4x-1)y_x + (4x^2-2x+2)y = 0$ .

Let  $\pi = d_x + 2x$ . Then the equation may be written

$$\pi(\pi-1)y = 0, \quad \therefore y = \{\pi(\pi-1)\}^{-1} 0 = (\pi-1)^{-1} 0 - \pi^{-1} 0.$$

Let  $(\pi-1)^{-1} 0 = u$ ,  $\therefore (\pi-1)u = 0$ , or  $u_x + (2x-1)u = 0$ ,  $\therefore u = Ae^{x^2-x}$ .

**3540** The solution of a P. D. equation of the type

$$a\pi_1^m z + b\pi_1^n z + \&c. = u_1 + u_2 + \&c.,$$

where  $u_1, u_2, \&c.$  are homogeneous functions of the 1st, 2nd degrees,  $\&c.$  in  $x, y$ , and  $\pi_1 = xd_x + yd_y$  (3503), is analogous to (3531), and is obtained from that solution by substituting  $u_1, u_2, \&c.$  for  $Bx, Cx^2, \&c.$ ; and, for such terms as  $Cx^2$ , an arbitrary homogeneous function of  $x$  and  $y$  of the same degree.

**3541** Solution of  $F(\pi)u = Q$ ,

where  $F(\pi) = \pi^n + A_1\pi^{n-1} + A_2\pi^{n-2} + \dots + A_n$ ,

and  $Q = u_0 + u_1 + u_2 + \&c.$ ,

a series of homogeneous functions of  $x, y, z, \dots$  of the respective dimensions 0, 1, 2,  $\&c.$

Here  $u = F^{-1}(\pi)Q + F^{-1}(\pi)0$ .

**3542** The value of the 1st term is given in (3510). For the general value of the last term (see Proof of 3533), let  $F'(m) = 0$  have  $r$  roots  $= m$ ; then

$$\mathbf{3543} \quad C(\pi-m)^{-p} 0 = C\{u(\log x)^{p-1} + v(\log x)^{p-2} \dots + w\},$$

where  $u, v, \dots w$  are arbitrary functions of the variables all of the degree  $m$ .

**3544** COR.—  $(\pi-m)^{-1} 0 = (x, y, \dots)^m$ ,

that is, a single homogeneous function of the variables of the degree  $m$  (1620).

**3545** Ex.:  $x^2 z_{xx} + 2xy z_{xy} + y^2 z_{yy} - a(xz_x + yz_y) + az = u_m + u_n$

$u_m, u_n$  being homogeneous functions of the  $m^{\text{th}}$  and  $n^{\text{th}}$  degrees. The equation may be written

$$(\pi_1^2 - a\pi_1 + a)z = u_m + u_n;$$

or, by (3505),  $(\pi - a)(\pi - 1)z = u_m + u_n$ ,

therefore  $z = \{(\pi - a)(\pi - 1)\}^{-1} (u_m + u_n) + \{(\pi - a)(\pi - 1)\}^{-1} 0$

$$= \frac{u_m}{(m-a)(m-1)} + \frac{u_n}{(n-a)(n-1)} + U_a + U_1.$$

The first two terms by formula (3502); the last two terms are arbitrary functions of the degrees  $a$  and  $1$  respectively, and result from formula (3543) by taking  $p = 1$  and  $m = a$  and  $1$ .

**3546** To reduce a P. D. equation, when possible, to the symbolic form

$$(\Pi^n + A_1 \Pi^{n-1} + A_2 \Pi^{n-2} \dots + A_n) u = Q \dots \dots \dots (1),$$

where  $\Pi = Md_x + Nd_y + \&c.$ ,

and  $Q, M, N, \&c.$  are any functions of the independent variables.

Consider the case of two independent variables,

$$(Md_x + Nd_y)^2 u = M^2 u_{xx} + 2MN u_{xy} + N^2 u_{yy} + (MM_x + NM_y) u_x + (MN_x + NN_y) u_y \dots (2).$$

Here the form of  $\Pi$  is obtainable from the right by considering the terms involving the highest derivatives only, for these terms are algebraically equivalent to  $(Md_x + Nd_y)^2$ .

The reduction being effected, and the equation being brought to the form of (1); then, if the auxiliary equation

$$\mathbf{3547} \quad m^n + A_1 m^{n-1} + A_2 m^{n-2} \dots + A_n = 0 \dots \dots \dots (3)$$

have its roots  $a, b, \dots$  all unequal, the solution of (1) will be of the form

$$\mathbf{3548} \quad u = (\Pi - a)^{-1} Q + (\Pi - b)^{-1} Q + \&c. \dots \dots \dots (4).$$

The terms on the right involve the solution of a series of linear first order P. D. equations, the first of which is

$$\mathbf{3549} \quad Mu_x + Nu_y + \dots - au = Q,$$

and the rest involve  $b, c, \&c.$

If equal or imaginary roots occur in the auxiliary equation, we may proceed as in the following example.

**3550** Ex. :

$$(1+x^2)^2 z_{xx} - 4xy(1+x^2) z_{xy} + 4x^2 y^2 z_{yy} + 2x(1+x^2) z_x + 2y(x^2-1) z_y + a^2 z = 0.$$

Here  $\Pi = (1+x^2) d_x - 2xy d_y$ , and the equation becomes  $(\Pi^2 + a^2) z = 0$ .

Let the variables  $x, y$  be now changed to  $\xi, \eta$ , so that  $\Pi = d_\xi$ . Therefore, since  $\Pi(\xi) = 1$ ,  $\Pi(\xi) = (1+x^2) \xi_x - 2xy \xi_y = 1$ .

Therefore, by (3383), 
$$\frac{dx}{1+x^2} = \frac{dy}{-2xy} = d\xi,$$

from which, by separating the variables and integrating, we obtain

$$x^2 y + y = A \dots\dots\dots (1),$$

and, by (1436), 
$$\xi = \tan^{-1} x + B \dots\dots\dots (2).$$

Also, since  $\Pi(\eta) = \eta_\xi = 0$ ,  $(1+x^2) \eta_x - 2xy \eta_y = 0$ .

Therefore 
$$\frac{dx}{1+x^2} = \frac{dy}{-2xy} = \frac{d\eta}{0},$$

the solution of which is equation (1). Thus

$$\xi = \tan^{-1} x \quad \text{and} \quad \eta = x^2 y + y.$$

The transformed equation is now  $(d_{2\xi}^2 + a^2) z = 0$ ,

and the solution, by (3523), is

$$z = \phi(\eta) \cos a\xi + \psi(\eta) \sin a\xi,$$

arbitrary functions of the variable, which is not explicitly involved, being substituted for the constants (3389). Therefore finally,

$$z = \phi(x^2 y + y) \cos(a \tan^{-1} x) + \psi(x^2 y + y) \sin(a \tan^{-1} x).$$

## MISCELLANEOUS EXAMPLES.

**3551**

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

Put  $d_{zy} + d_{yz} = a^2$ . Thus  $u_{zz} + a^2 u = 0$ , the solution of which, by (3523), is

$$u = \phi(y, z) \cos ax + \psi(y, z) \sin ax,$$

arbitrary functions of  $y$  and  $z$  being put for the constants  $A$  and  $B$ . Expand the sine and cosine by (764-5); replace  $a^2$  by its operative equivalent, and, in the expansion of  $\sin ax$ , put  $a\psi(y, z) = \chi(y, z)$ ; thus

$$\begin{aligned} u = \phi(y, z) - \frac{x^2}{2!} (d_{zy} + d_{yz}) \phi(y, z) + \frac{x^4}{4!} (d_{zy} + d_{yz}) \phi(y, z) - \&c. \\ + x\chi(y, z) - \frac{x^3}{3!} (d_{zy} + d_{yz}) \chi(y, z) + \frac{x^5}{5!} (d_{zy} + d_{yz}) \chi(y, z) - \&c. \end{aligned}$$

[See (3626) for another solution.]

**3552**

$$u_x + u_y + u_z = xyz.$$

Here

$$\begin{aligned} u &= (d_x + d_y + d_z)^{-1} (xyz + 0) \\ &= \{d_{-x} - d_{-xz} (d_y + d_z) + d_{-xy} (d_y + d_z)^2 - \dots\} (xyz + 0). \end{aligned}$$

Operating upon  $xyz$ , we get

$$u = \frac{1}{2} x^2 y z - \frac{1}{6} x^3 (z + y) + \frac{1}{24} x^4,$$

the rest vanishing. For symmetry, take  $\frac{1}{3}$ rd of the sum of three such expressions; thus

$$u = \frac{1}{3} \left\{ \frac{1}{12} (x^4 + y^4 + z^4) - \frac{1}{6} (x^3y + xy^3 + y^3z + yz^3 + z^3x + zx^3) + \frac{1}{3} xyz(x + y + z) \right\}.$$

Operating upon zero, we have, in the first place,  $d_{-x}0 = \phi(yz)$  instead of a constant, therefore  $d_{-2x}0 = x\phi(yz)$ , &c.

The result is

$$\{1 - x(d_y + d_z) + \frac{1}{2}x^2(d_y + d_z)^2 - \dots\} \phi(yz) = e^{-x(d_y + d_z)} \phi(yz) = \phi(y - x, z - x) \quad (3493) \text{ the complementary term.}$$

**3553** Otherwise, putting  $d_y + d_z = \mathfrak{D}$ , we have, by (3478),

$$\begin{aligned} (d_x + \mathfrak{D})^{-1} xyz &= e^{-x\mathfrak{D}} d_{-x} e^{x\mathfrak{D}} xyz = e^{-x\mathfrak{D}} d_{-x} \{x(y+x)(z+x)\}, \quad (3493) \\ &= e^{-x\mathfrak{D}} \left\{ \frac{1}{2}x^2yz + \frac{1}{3}x^3(y+z) + \frac{1}{4}x^4 \right\} \\ &= \frac{1}{4}x^2(y-x)(z-x) + \frac{1}{3}x^3(y+z-2x) + \frac{1}{4}x^4, \end{aligned}$$

which agrees with the former solution.

**3554**  $au_x + bu_y + cu_z = xyz.$

Substitute  $x = a\xi$ ,  $y = b\eta$ ,  $z = c\zeta$ , and the equation becomes

$$u_\xi + u_\eta + u_\zeta = abc\xi\eta\zeta,$$

which is solved in (3552).

The same methods furnish the solution of

**3555**  $au_x + bu_y + cu_z = x^m y^n z^p.$

**3556**  $xz_x + yz_y = 2xy\sqrt{a^2 - z^2}.$

Put

$$z = a \sin v,$$

$$\therefore \pi z = a \cos v \cdot \pi v, \quad \therefore \pi v = 2xy, \quad \therefore z = a \sin(xy + c).$$

**3557**  $axu_x + byu_y + czu_z - nu = 0.$

Put

$$x = \xi^a, \quad y = \eta^b, \quad z = \zeta^c;$$

$$\therefore \xi u_\xi + \eta u_\eta + \zeta u_\zeta - nu = 0, \quad \therefore \text{by (3544)} \quad u = (x^{\frac{1}{a}}, y^{\frac{1}{b}}, z^{\frac{1}{c}})^n.$$

**3558** The solution of any P. D. equation of the type

$$F(xd_x, yd_y, zd_z, \dots) u = \Sigma Ax^m y^n z^p \dots$$

is, by (3488) and (3557),

$$u = \Sigma \frac{Ax^m y^n z^p}{F(m, n, p, \dots)} + \frac{1}{F(xd_x, yd_y, zd_z, \dots)} 0.$$

**3559** Ex. :  $xu_x + yu_y - au = Q_m,$

where  $Q_m = (x, y)^m$  (1620).

Here  $u = (m-a)^{-1}Q_m + U_a$ . When  $a = m$ , this solution becomes indeterminate. In that case, as in (3526), assume

$$U_a = V_a - \frac{Q_a}{m-a}, \quad \therefore u = \frac{Q_m - Q_a}{m-a} + V_a.$$

Differentiate for  $a$ , by (1580), putting  $Q_a$  first in the form

$$\frac{1}{2} \left\{ x^a F\left(\frac{y}{x}\right) + y^a f\left(\frac{x}{y}\right) \right\};$$

thus  $u = \frac{1}{2}Q_m(\log x + \log y) + V_m.$

Similarly, the solution of

**3560**  $xu_x + yu_y + zu_z - mu = Q_m$

is  $u = \frac{1}{3}Q_m(\log x + \log y + \log z) + V_m.$

**3561**  $xu_x + yu_y + zu_z = c.$

The solution, by (3560), is

$$u = \frac{1}{3}c(\log x + \log y + \log z) + V_0.$$

**3562**  $z_{2x} - 2az_{xy} + a^2z_{2y} = 0$  or  $(d_x - ad_y)^2 z = 0.$

$$z = (d_x - ad_y)^{-1}0 = (d_x - ad_y)^{-1} \phi(y) e^{axd_y} \quad (3472)$$

by putting  $ad_y$  for  $m$  and  $\phi(y)$  for  $C$ . The second operation produces, by (3476),

$$z = e^{axd_y} \{ x\phi(y) + \psi(y) \} = x\phi(y+ax) + \psi(y+ax). \quad (3492)$$

**3563**  $x^2z_{2x} - y^2z_{2y} + xz_x - yz_y = 0.$

This reduces to  $(xd_x + yd_y)(xd_x - yd_y)z = 0.$

Here  $\pi = xd_x + yd_y$ , and  $m = 0$  in (3544),

therefore  $z = (x, y)^0 + \left(x, \frac{1}{y}\right)^0,$

the second term being obtained by substituting  $y^{-1} = y'$ , and so converting the second factor into  $(xd_x + y'd_y)$ . The above may also be written

$$z = F\left(\frac{x}{y}\right) + f(xy),$$

$F$  and  $f$  being integral algebraic functions.

**3564**  $z_{2x} - a^2z_{2y} + 2abz_x + 2a^2bz_y = 0.$

Putting  $y = a\eta$ , this equation is equivalent to

$$(d_x - d_\eta + 2ab)(d_x + d_\eta)z = 0;$$

3 Y



putting  $x = \log x'$  and  $\eta = \log \eta'$ , this gives, by (3544),

$$z = (e^x, e^{-\eta})^{-2ab} + (e^x, e^\eta)^0 = e^{-2abx} \phi(e^{x+\eta}) + \psi(e^{x-\eta}) \\ = e^{-2abx} F(y+ax) + f(y-ax),$$

the functions being algebraic and integral.

**3565**

$$u_{2x} - a^2 u_{2y} = \phi(x, y).$$

$$\therefore u = (d_{2x} - a^2 d_{2y})^{-1} \phi(x, y) \quad (3515)$$

$$= (2ad_y)^{-1} \{ (d_x - ad_y)^{-1} - (d_x + ad_y)^{-1} \} \phi(x, y) \quad (3470)$$

$$= (2ad_y)^{-1} \left\{ e^{axd_y} \int e^{-axd_y} \phi(x, y) dx - e^{-axd_y} \int e^{axd_y} \phi(x, y) dx \right\} \quad (3470)$$

$$= (2a)^{-1} \int \{ \Phi_1(x, y+ax) - \Phi_2(x, y-ax) \} dy,$$

since

$$e^{axd_y} \phi(x, y) = \phi(x, y+ax) \quad (3492).$$

Here

$$\Phi_1(x, y) = \int \phi(x, y-ax) dx + \psi(y),$$

$$\Phi_2(x, y) = \int \phi(x, y+ax) dx + \chi(y).$$

**3566** If  $\phi(x, y) = 0$ , the solution therefore becomes

$$u = \psi_1(y+ax) + \chi_1(y-ax) \quad [\text{Boole, ch. 16.}]$$

For the solution in this case by Monge's method, see (3433).

**3567**

$$z_x - az_y = e^{mx} \cos ny.$$

$$z = (d_x - ad_y)^{-1} e^{mx} \cos ny = e^{axd_y} \int e^{-axd_y} e^{mx} \cos ny dx \quad (3470)$$

$$= e^{axd_y} \int e^{mx} \cos n(y-ax) dx \quad (3492), \text{ and this by Parts, or by (1999), is}$$

$$= e^{axd_y} e^{mx} \left\{ m \cos n(y-ax) - an \sin n(y-ax) \right\} (m^2 + a^2 n^2)^{-1} + e^{axd_y} \phi(y)$$

$$\therefore z = e^{mx} \left\{ m \cos ny - an \sin ny \right\} (m^2 + a^2 n^2)^{-1} + \phi(y+ax), \text{ by (3492).}$$

**3568**

$$z_t - az_{2x} = 0.$$

$$z = (d_t - ad_{2x})^{-1} 0 = e^{atd_{2x}} \phi(\phi), \text{ by (3472),}$$

$\phi(x)$  taking the place of the constant  $C$ .

$$\text{Therefore} \quad z = \phi(x) + at\phi_{2x} + \frac{1}{2}a^2 t^2 \phi_{4x} + \&c. \quad (3492)$$

Otherwise, to obtain  $z$  in powers of  $x$ , we have, putting  $b^2 = a^{-1}$ ,

$$z_{2x} - b^2 z_t = 0,$$

$$\therefore z = \{ (d_x + bd_t)(d_x + bd_t) \}^{-1} 0 = e^{bx d_t} \phi(t) + e^{-bx d_t} \psi(t) \quad (3518);$$

then expand by (150).

**3569**

$$z_{2x} + z_{2y} = \cos nx \cos my.$$

$$z = (d_{2x} + d_{2y})^{-1} \cos nx \cos my.$$

Treating  $d_{2y}$  and  $\cos my$  as constants, we have, by (3526), putting  $d_{2y}$  for  $a^2$ ,

$$\begin{aligned} z &= \cos nx (d_{2y} - n^2)^{-1} \cos my + A \cos ax + B \sin ax, \text{ or by (3496),} \\ &= \cos nx \cos my (-m^2 - n^2)^{-1} + \phi(y) \cos(xd_y) + \psi(y) \sin(xd_y), \end{aligned}$$

$A$  and  $B$  becoming  $\phi(y)$  and  $\psi(y)$ .

$$\mathbf{3570} \quad z_{2\phi} + 2z_{\phi\psi} + z_{2\psi} + a^2 z = \cos(m\phi + n\psi).$$

Therefore  $(\pi + ia)(\pi - ia)z = \frac{1}{2} \{e^{i(m\phi + n\psi)} + e^{-i(m\phi + n\psi)}\}$ ,

where  $\pi = d_\phi + d_\psi$ . Therefore, by (3510), with  $x = e^\phi$ ,  $y = e^\psi$ ,

$$z = \frac{1}{2} \left\{ \frac{e^{i(m\phi + n\psi)}}{a^2 - (m+n)^2} + \frac{e^{-i(m\phi + n\psi)}}{a^2 - (m+n)^2} \right\} + (e^\phi, e^\psi)^{-ia} + (e^\phi, e^\psi)^{ia},$$

or 
$$z = \frac{\cos(m\phi + n\psi)}{a^2 - (m+n)^2} + (e^\phi, e^\psi)^{-ia} + (e^\phi, e^\psi)^{ia}.$$

**3571** PROP. I.—To transform a linear differential equation of the form

$$(a + bx + cx^2 \dots) u_{nx} + (a' + b'x + c'x^2 \dots) u_{(n-1)x} + \&c. = Q \dots (1)$$

into the symbolical form

$$f_0(D)u + f_1(D)e^\theta u + f_2(D)e^{2\theta}u + \&c. = T \dots (2),$$

where  $Q$  is a function of  $x$ ,  $T$  a function of  $\theta$ ,  $x = e^\theta$  and  $D = d_\theta$ .

Multiply the equation by  $x^n$ ; then the 1st term on the left becomes, by (3489),

$$(a + be^\theta + ce^{2\theta} + \dots) D(D-1) \dots (D-n+1)u.$$

This reduces, by the repeated application of formula (3476) with the notation of (2451), to

$$\mathbf{3572} \quad aD^{(n)}u + b(D-1)^{(n)}e^\theta u + c(D-2)^{(n)}e^{2\theta}u + \&c.$$

The other terms admit of similar reductions.

**3573** Conversely, to bring back an equation from the symbolic form (2) to the ordinary form (1), employ formula (3475) so as to transfer  $e^{m\theta}$  to the left of the operative symbol.

$$\begin{aligned} \mathbf{3574} \quad \text{Ex.: } x^2(x^2u_{2x} + 7xu_x + 5u) &= e^{2\theta} \{D(D-1) + 7D + 5\}u \\ &= e^{2\theta} (D^2 + 6D + 5)u = e^{2\theta} (D+1)(D+5)u \\ &= (D-1)(D+3)e^{2\theta}u \quad (3476). \end{aligned}$$

For the converse reduction, the steps must be retraced, employing (3475). See also example (3578).

**3575** PROP. II.—To solve the equation

$$u + a_1 \phi(D) e^\theta u + a_2 \phi(D) \phi(D-1) e^{2\theta} u \dots \\ + a_n \phi(D) \phi(D-1) \dots \phi(D-n+1) e^{n\theta} u = U,$$

where  $U$  is a function of  $\theta$ .

By (3491)

$$\phi(D) \phi(D-1) \dots \phi(D-n+1) e^{n\theta} u = \{\phi(D) e^\theta\}^n u.$$

Putting  $\rho^n u$  for this, the equation becomes

$$(1 + a_1 \rho + a_2 \rho^2 \dots + a_n \rho^n) u = U.$$

**3576** Therefore

$$u = \{A_1 (1 - q_1 \rho)^{-1} + A_2 (1 - q_2 \rho)^{-1} \dots + A_n (1 - q_n \rho)^{-1}\} U,$$

where  $q_1, q_2 \dots q_n$  are the roots of the equation

$$q^n + a_1 q^{n-1} + a_2 q^{n-2} \dots + a_n = 0,$$

and

$$A_r = \frac{q_r^{n-1}}{(q_r - q_1)(q_r - q_2) \dots (q_r - q_n)}.$$

The solution will then be expressed by

$$u = A_1 u_1 + A_2 u_2 \dots + A_n u_n,$$

where  $u_r$  is given by the solution of the equation

**3577** 
$$u_r - q_r \phi(D) e^\theta u_r = U.$$

**3578** Ex.:

$$(x^2 + 5x^3 + 6x^4) u_{xx} + (4x + 25x^2 + 36x^3) u_x + (2 + 20x + 36x^2) u = 20x^3.$$

Putting  $x = e^\theta$ , and transforming by (3489),

$$(1 + 5e^\theta + 6e^{2\theta}) D(D-1) u + (4 + 25e^\theta + 36e^{2\theta}) Du + (2 + 20e^\theta + 36e^{2\theta}) u = 20e^{3\theta}.$$

The first term  $= D(D-1) u + 5(D-1)(D-2) e^\theta u + 6(D-2)(D-3) e^{2\theta} u$

by applying (3476). The other terms similarly; thus, after rearrangement,

$$(D+1)(D+2) u + 5(D+1)^2 e^\theta u + 6D(D+1) e^{2\theta} u = 20e^{3\theta}.$$

Operating upon this with  $\{(D+1)(D+2)\}^{-1}$ , we get

$$u + 5 \frac{D+1}{D+2} e^\theta u + 6 \frac{D}{D+2} e^{2\theta} u = \frac{20e^{3\theta}}{(3+1)(3+2)} = e^{3\theta}, \text{ by (3474);}$$

or  $(1 + 5\rho + 6\rho^2) u = e^{3\theta}$ , if  $\rho = (D+1)(D+2)^{-1} e^\theta$ ;

therefore  $u = \{3(1+3\rho)^{-1} - 2(1+2\rho)^{-1}\} e^{3\theta} = 3y - 2z$ ,

if  $y = (1+3\rho)^{-1} e^{3\theta}$  and  $z = (1+2\rho)^{-1} e^{3\theta}$ .

Hence  $(1+3\rho)y = e^{3\rho}$  or  $y+3(D+1)(D+2)^{-1}e^{\rho}y = e^{3\rho}$ ;  
 therefore  $(D+2)y+3(D+1)e^{\rho}y = e^{3\rho}(3+2)$ , by (3474),  
 or  $(D+2)y+3e^{\rho}(D+2)y = 5e^{3\rho}$ , by (3475);  
 that is,  $(x+3x^2)y_x+2(1+3x)\dot{y} = 5x^3$ .  
 Similarly  $(x+2x^2)z_x+2(1+2x)z = 5x^3$ .  
 Solve these by (3210), and substitute in  $u = 3y-2z$ .

**3579** PROP. III.—To transform the equation

$$u + \phi(D)e^{r\rho}u = U \quad \text{into} \quad v + \phi(D+n)e^{r\rho}v = V,$$

put  $u = e^{n\rho}v$  and  $U = e^{n\rho}V$ .

PROOF.—By (3474), because  $\phi(D)e^{(n+r)\rho}v = e^{n\rho}\phi(D+n)e^{r\rho}v$ .

**3580** PROP. IV.—To transform the equation

$$u + \phi(D)e^{r\rho}u = U \quad \text{into} \quad v + \psi(D)e^{r\rho}v = V,$$

put  $u = P_r \frac{\phi(D)}{\psi(D)} v$  and  $U = P_r \frac{\phi(D)}{\psi(D)} V$ ,

**3581** where  $P_r \frac{\phi(D)}{\psi(D)} = \frac{\phi(D)\phi(D-r)\phi(D-2r)\dots}{\psi(D)\psi(D-r)\psi(D-2r)\dots}$ .

PROOF.—Put  $u = f(D)v$  in the 1st equation, and  $e^{r\rho}f(D)v = f(D-r)e^{r\rho}v$  (3476). After operating with  $f^{-1}(D)$  it becomes

$$v + \phi(D)f(D-r)f^{-1}(D)e^{r\rho}v = f^{-1}(D)U,$$

therefore  $\phi(D)f(D-r)f^{-1}(D) = \psi(D)$  by hypothesis;

therefore  $f(D) = \frac{\phi(D)}{\psi(D)}f(D-r) = \frac{\phi(D)\phi(D-r)}{\psi(D)\psi(D-r)}f(D-2r)$ ,

and so in inf. Also  $U = f(D)V$ .

**3582** To make any elementary factor  $\chi(D)$  of  $\phi(D)$  become, in the transformed equation,  $\chi(D \pm nr)$ , where  $r$  is an integer; take  $\psi(D) = \chi(D \pm nr)\chi_1(D)$ . See example (3589).

**3583** To make any factor of  $\phi(D)$  of the form  $\frac{\chi(D)}{\chi(D \pm nr)}$  disappear in the transformed equation, take  $\psi(D) = \chi_1(D)$ , where  $\chi_1(D)$ , in each case, denotes the remaining factors of  $\phi(D)$ . See example (3591).

**3584** In the application of Proposition IV., differentiation or integration will be the last operation according as  $P_r \frac{\phi(D)}{\psi(D)}$  (3581) has its factors, after reduction, in the numerator or denominator, and therefore according as  $\psi(D)$  is formed by algebraically diminishing or increasing the several factors of  $\phi(D)$ . However, by first employing Proposition III., the given equation may frequently be so prepared that the final operation with Prop. IV. shall be differentiation only. See example (1).

For further investigation, see *Boole's Diff. Eq.*, Ch. 17, and *Supplement*, p. 187.

**3585** To reduce an equation of the homogeneous class (3531) to a binomial equation of the same order of the form

$$y_{nx} + qy = X.$$

The general theory of such solutions is as follows. Let the given equation be

$$u + q \{ (D + a_1)(D + a_2) \dots (D + a_n) \}^{-1} e^{a_1 x} u = U \dots (1),$$

$a_1, a_2, \dots, a_n$  being in descending order of magnitude. Putting  $u = e^{-a_1 x} v$ , by Prop. III.,

$$v + q \{ D(D - \overline{a_1 - a_2}) \dots (D - \overline{a_1 - a_n}) \}^{-1} e^{a_1 x} v = e^{a_1 x} U \dots (2).$$

To transform these factors, regarded as  $\phi(D)$ , by Prop. IV. into  $\psi(D) = D(D-1) \dots (D-n+1)$ , we convert  $D$  into  $D+rn$  (3582),  $r$  being an integer.

Hence for the  $p^{\text{th}}$  factor we must have

$$D + rn - a_1 + a_p = D - p + 1,$$

**3586** and therefore  $a_1 - a_p = rn + p - 1 \dots \dots \dots (3).$

If this relation holds for each of the constants  $a_1 \dots a_n$ , equation (1) is reducible to the form

$$\mathbf{3587} \quad y + q \{ D(D-1) \dots (D-n+1) \}^{-1} e^{a_1 x} y = Y \dots \dots (4),$$

which, by (3489), is equivalent to  $y_{nx} + qy = Y_{nx} = X$ .

$y$  being found in terms of  $x$  from the last equation, and,  $v$  being  $= P_n \frac{\phi(D)}{\psi(D)} y$  (3580), the solution will result from

$$\mathbf{3588} \quad u = e^{-a_1 x} P_n \frac{(D-1)(D-2) \dots (D-n+1)}{(D-a_1+a^2) \dots (D-a_1+a_n)} y;$$

while  $U$  and  $Y$  are connected by the same relation as  $u$  and  $y$ .

**3589** Ex. 1: Given  $x^3 u_{3x} + 18x^2 u_{2x} + 84xu_x + 96u + 3x^3 u = 0$ .

Putting  $x = e^z$  and employing (3489), this becomes

$$\{D(D-1)(D-2) + 18D(D-1) + 84D + 96\} u + 3e^{3z} u = 0,$$

or  $(D+8)(D+4)(D+3)u + 3e^{3z}u = 0,$

therefore  $u + 3\{(D+8)(D+4)(D+3)\}^{-1}e^{3z}u = 0 \dots\dots\dots(1).$

Employing Prop. III., put  $u = e^{-3z}v,$

therefore (3476)  $v + 3\{D(D-4)(D-5)\}^{-1}e^{3z}v = 0 \dots\dots\dots(2).$

To transform this by Prop. IV. into

$$y + 3\{D(D-1)(D-2)\}^{-1}e^{3z}y = 0 \dots\dots\dots(3),$$

we have

$$P_3 \frac{\phi(D)}{\psi(D)} = \frac{D(D-1)(D-2)(D-3)(D-4)(D-5) \dots}{D(D-4)(D-5)(D-3)(D-7)(D-8) \dots} = (D-1)(D-2),$$

$$\therefore v = (D-1)(D-2)y, \quad \therefore u = e^{-3z}(D-1)(D-2)y \dots\dots\dots(4),$$

and the solution is obtained by differentiation only, performed on the value of  $y$  as obtained by the solution of (3), that equation being equivalent to

$$D(D-1)(D-2)y + 3e^{3z}y = 0, \text{ or, by (3489), } y_{3z} + 3y = 0.$$

If, however, Prop. IV. were used to pass directly from (1) to (3), we should have

$$\begin{aligned} P_3 \frac{\phi(D)}{\psi(D)} &= \frac{D(D-1)(D-2)(D-3)(D-4)(D-5) \dots}{(D+8)(D+4)(D+3)(D+5)(D+1)D \dots} \\ &= \frac{1}{(D+8)(D+5)(D+4)(D+3)(D+2)(D+1)}, \end{aligned}$$

and equation (4) would involve integrations of  $y$  as high as  $D^{-6}y$ .

**3590** NOTE.—By the literal application of Rule IV., the right side of equation (3) ought to be  $V = \{(D-1)(D-2)\}^{-1}0$ ; but no such term is required when the original and transformed equations are of the same order, for in such cases the arbitrary constants introduced by the operation upon zero disappear with the terms containing them in the final differentiation. The result is the same as if the operation upon zero had not been performed.

In the following example,  $V$  has to be retained.

**3591** Ex. 2:  $(x-x^3)u_{2x} + (2-12x^3)u_x - 30xu = 0 \dots\dots\dots(1).$

Multiply by  $x$ , transform by (3489), and remove  $e^{2z}$  to the right of each function of  $D$  by (3476), thus

$$u - \frac{(D+4)(D+3)}{D(D+1)}e^{2z}u = 0 \dots\dots\dots(2).$$

Transform this by Prop. IV. into

$$v - \frac{D+3}{D+1}e^{2z}v = V \dots\dots\dots(3).$$

We have  $u = P_2 \frac{D+4}{D}v = (D+4)(D+2)v,$

$$V = \{(D+4)(D+2)\}^{-1}0 = Ae^{-2z} + Be^{-4z} \quad (3518).$$

The operation upon zero is required in this example (see 3590), because (3)



is of a lower order than (2); but only one term of the result need be retained, because only one additional constant is wanted. Hence (3) becomes

$$(D+1)v - (D+3)e^{2x}v = (D+1)Ae^{-2x} = -Ae^{-2x}.$$

Changing again to  $x$ , this equation becomes

$$(x^3 - x^5)v_x - 4x^3v + A = 0.$$

The value of  $v$  obtained from this by (3210) will contain two arbitrary constants. The solution of (1) will then be given by

$$u = (D+4)(D+2)v.$$

**3592** Ex. 3:  $u_{2x} - n(n+1)x^{-2}u - q^2u = 0$ , [Boole, p. 424.  
 $n$  being an integer.

Multiplying by  $x^2$  and employing (3489), this becomes

$$u - q^2\{(D+n)(D-n-1)\}^{-1}e^{2x}u = 0.$$

This is changed by Prop. III. into

$$v - q^2\{D(D-2n-1)\}^{-1}e^{2x}v = 0, \text{ with } u = e^{-2x}v,$$

and this, by Prop. IV., into

$$y - q^2\{D(D-1)\}^{-1}e^{2x}y = 0 \text{ or } y_{2x} - q^2y = 0 \quad (3489).$$

$y$  being found from this by (3524), we then have

$$u = e^{-2x}P, \frac{D-1}{D-2n-1}y = e^{-2x}(D-1)^{(n)}_{-2}y = x^{-n}(xd_x-1)^{(n)}_{-2}y.$$

But, by (3484),  $F(xd_x-m) = x^m F(xd_x)x^{-m}$ ,

$$\therefore u = x^{-n}x(xd_x)x^{-1} \cdot x^3(xd_x)x^{-3} \dots x^{2n-1}(xd_x)x^{-2n+1}y,$$

or

$$u = x^{-(n+1)}(x^3d_x)^n x^{-2n+1}y \\ = x^{-n-1}(x^3d_x)^n x^{-2n+1}(Ae^{qx} + Be^{-qx}) \quad (3525).$$

This may be evaluated by substituting  $z = x^{-2}$ . (See *Educ. Times Reprint*, Vol. XVII., p. 77.)

**3593** Ex. 4:  $u_{2x} - a^2u_{2y} - n(n+1)x^{-2}u = 0$ .

The solution is derived from that of Example (2), by putting  $q = ad_y$ , and arbitrary functions of  $y$  after the exponentials instead of  $A$  and  $B$ ; thus

$$u = x^{-n-1}(x^3d_x)^n x^{-2n+1}\{e^{axd_y}\phi(y) + e^{-axd_y}\psi(y)\} \\ = x^{-n-1}(x^3d_x)^n x^{-2n+1}\{\phi(y+ax) + \psi(y+ax)\}, \text{ by (3492).}$$

[Boole, p. 425.]

**3594**  $(1+ax^2)u_{2x} + axu_x \pm n^2u = 0$ .

To solve this equation or its symbolical equivalent obtained by (3489), viz.,

**3595**  $u + \frac{a(D-2)^2 \pm n^2}{D(D-1)}e^{2x}u = 0$ .

Substitute  $t = \int \frac{dx}{\sqrt{1+ax^3}}$  in the solution of  $u_{2t} \pm n^2u = 0$ , by (3523-4).

**3596** Similarly, to solve the equation

$$(x^3 + a) x^2 u_{xx} + (2x^3 + a) x u_x \pm n^2 u = 0,$$

or, the same in its symbolical form,

**3597** 
$$u + \frac{(D-1)(D-2)}{aD^2 \pm n^2} e^{2\theta} u = 0.$$

Substitute  $t = \int \frac{dx}{x\sqrt{(x^3+a)}}$  in the solution of  $u_{tt} \pm n^2 u = 0$ .

(3596) is obtainable from (3593) by changing  $\theta$  into  $-\theta$ .

**3598** Pfaff's equation,

$$(a + bx^n) x^2 u_{xx} + (c + ex^n) x u_x + (f + gx^n) u = Q.$$

When  $Q = 0$ , the symbolical form becomes

$$u + \frac{b(D-n)(D-n-1) + c(D-n) + g}{aD(D-1) + cD + f} e^{n\theta} u = 0 \dots \dots \dots (1).$$

If  $n$  be not  $= 2$ , substitute  $2\theta' = n\theta$ , and therefore  $2d_\theta = nd_{\theta'}$ .

**3599** Thus 
$$u + \frac{b(D-a_1)(D-a_2)}{a(D-\beta_1)(D-\beta_2)} e^{2\theta'} u = 0 \dots \dots \dots (2).$$

where  $a_1, a_2$  are the roots of the equation

$$b(\tfrac{1}{2}na - n)(\tfrac{1}{2}na - n - 1) + c(\tfrac{1}{2}na - n) + g = 0 \dots \dots \dots (3),$$

and  $\beta_1, \beta_2$  are the roots of

$$a \tfrac{1}{2}n\beta(\tfrac{1}{2}n\beta - 1) + c \tfrac{1}{2}n\beta + f = 0.$$

Four cases occur—

**3600** I.—If  $a_1 - a_2$  and  $\beta_1 - \beta_2$  are odd integers, (2) can be reduced by

Prop. IV. (3581) to the form  $v + \frac{b(D-a_1)(D-a_1-1)}{a(D-\beta_1)(D-\beta_1-1)} e^{2\theta'} v = 0$ ,

and then resolved into two equations of the first order.

**3601** II.—If any one of the four quantities  $a_1 - \beta_1, a_1 - \beta_2, a_2 - \beta_1, a_2 - \beta_2$  is an even integer, (2) can be reduced by Prop. IV. to an equation of the first order.

**3602** III.—If  $\beta_1 - \beta_2$  and  $a_1 + a_2 - \beta_1 - \beta_2$  are both odd integers, then, by Props. III. and IV., (2) is reducible to (3595).

**3603** IV.—If  $a_1 - a_2$  and  $a_1 + a_2 - \beta_1 - \beta_2$  are both odd integers, (2) is reducible in like manner to (3597). [Boole, p. 428.]

NOTE.—The integers may be either positive or negative, and when even may be zero.

## SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS BY SERIES.

**3604** CASE I.—Solution of the linear differential equation

$$f_0(D)u - f_1(D)e^{rx}u = 0 \quad \text{or} \quad f_0(xd_x)u - f_1(xd_x)x^r u = 0,$$

in which  $f_0(D)$ ,  $f_1(D)$  are polynomial expressions of the form  $a_0 + a_1D + a_2D^2 \dots + a_nD^n$  and  $f_0(D) = (D-a)(D-b)(D-c) \dots$

**3605** Let  $\phi(D) = f_1(D) \div f_0(D)$ , and let

$$\begin{aligned} \mathbf{3606} \quad \Phi(a) = & 1 + \phi(a+r)x^r + \phi(a+2r)\phi(a+r)x^{2r} \\ & + \phi(a+3r)\phi(a+2r)\phi(a+r)x^{3r} + \&c. \end{aligned}$$

Then the solution will be

$$\mathbf{3607} \quad u = Ax^a\Phi(a) + Bx^b\Phi(b) + Cx^c\Phi(c) + \&c.$$

PROOF.—Operating with  $f_0^{-1}(D)$  and writing  $\rho$  for  $\phi(D)e^{rx}$ ,

$$u - \rho u = f_0^{-1}(D)0 = Ae^{ax} + Be^{bx} + \&c. \quad (3518)$$

Therefore, by (3515),  $u = (1 - \rho)^{-1}(Ae^{ax} + Be^{bx} + \dots)$

$$= (1 + \rho + \rho^2 + \dots)Ae^{ax} + (1 + \rho + \rho^2 + \dots)Be^{bx} + \&c.$$

Now in each term substitute for  $\rho^n$  the value in (3491), and remove  $D$  by formula (3474).

CASE II.—Solution of

$$\mathbf{3608} \quad f_0(D)u + f_1(D)e^a u + f_2(D)e^{2a} u \dots + f_n(D)e^{na} u = 0 \quad \dots\dots(1),$$

where  $f_0(D) = (D-a)(D-b)(D-c) \dots$

$$\text{Let } \Psi(a) = 1 + F_1(a+1)x + F_2(a+2)x^2 + \&c.,$$

where the coefficients  $F_1(a+1)$  or  $v_1$ ,  $F_2(a+2)$  or  $v_2$ , &c. are determined in succession by the formula

$$\mathbf{3609} \quad f_0(m)v_m + f_1(m)v_{m-1} \dots + f_n(m)v_{m-n} = 0 \quad \text{and} \quad v_0 = 1 \quad \dots\dots(2).$$

The solution will then be expressed by

$$\mathbf{3610} \quad u = Ax^a\Psi(a) + Bx^b\Psi(b) + Cx^c\Psi(c) + \&c.$$

PROOF.—From (1)

$$u = \{1 + \phi_1(D)e^a \dots + \phi_n(D)e^{na}\}^{-1} f_0^{-1}(D) \cdot 0 \dots\dots\dots(3),$$

where

$$\phi_r(D) = f_r(D) \div f_0(D).$$

Here  $f_0^{-1}(D)0 = \{(D-a)(D-b) \dots\}^{-1}0 = Ae^{ax} + Be^{bx} + \dots \quad (3518);$

and  $\{1 + \phi_1(D)e^a \dots + \phi_n(D)e^{na}\}^{-1} = 1 + F_1(D)e^a + F_2(D)e^{2a} + \dots$

To determine  $F_1, F_2, \&c.$ , operate upon each side with  $\{1 + \phi(D)e^a + \&c.\}$ , and equate coefficients of powers of  $e$ ; thus formula (2) is obtained. (3) now becomes

$$u = \{1 + F_1(D)e^a + F_2(D)e^{2a} + \dots\} (Ae^{a^0} + Be^{b^0} + \dots) \dots\dots\dots (4).$$

Multiply out; apply (3474), and put  $x$  for  $e^a$ .

**3611** Ex.:  $x^2 u_{xx} - (a+b-1) x u_x + abu - qx^2 u = 0,$

or, by (3489),  $(D-a)(D-b)u - qe^{2a}u = 0.$

Here  $f_0(D) = (D-a)(D-b), f_1(D) = 0, f_2(D) = -q.$

Therefore (2) becomes  $(m-a)(m-b)v_m = qv_{m-2},$

therefore  $F_1, F_2, \&c.$  vanish, and  $F_0(a) = 1,$

$$F_2(a+2) = \frac{qF_0(a)}{(a+2-a)(a+2-b)} = \frac{q}{2(a+2-b)},$$

$$F_4(a+4) = \frac{qF_2(a+2)}{(a+4-a)(a+4-b)} = \frac{q^2}{4 \cdot 2(a+4-b)(a+2-b)}.$$

Therefore  $\Psi(a) = 1 + \frac{qx^2}{2(a+b-2)} + \frac{q^2x^4}{4 \cdot 2(a-b+4)(a-b+2)} + \dots$

Similarly we find  $F_2(b+2), F_4(b+2), \&c.$ , and thence  $\Psi(b)$ ; and, substituting in (3610), we have

$$u = Ax^a + \frac{Aqx^{a+2}}{2(a+b-2)} + \frac{Aq^2x^{a+4}}{4 \cdot 2(a-b+4)(a-b+2)} + \dots$$

$$+ Bx^b + \frac{Bqx^{b+2}}{2(b-a+2)} + \frac{Bq^2x^{b+4}}{4 \cdot 2(b-a+4)(b-a+2)} + \dots$$

**3612** The solution is arrived at more quickly by formula (3607). We have

$$\phi(D) = \frac{q}{(D-a)(D-b)},$$

$$\therefore \phi(a+2) = \frac{q}{2(a+2-b)}, \quad \phi(a+4) = \frac{q}{4(a+4-b)}, \quad \&c.,$$

producing the same series by the value of  $\Phi(a)$ . Similarly with  $\Phi(b)$ .

**3613** When  $f_0(D)$  has  $r$  factors each  $= D-a$ , the corresponding part of the value of  $u$  in equation (4) will produce

**3614**  $A_0 + A_1 \log x + A_2 (\log x)^2 \dots + A_{r-1} (\log x)^{r-1},$

where the coefficients  $A_0, A_1, \dots$  are each of the form

$$C_0x^a + C_1x^{a+1} + C_2x^{a+2} + \dots$$

**3615** But if any one of the quantities  $F_r(a+r) = 0$  (3608), then  $C_r = 0$  also.

PROOF.— $f_0^{-1}(D)$  now contains a term of the form

$$e^{a\theta}(c_0 + c_1\theta + \dots + c_r\theta^{r-1}) = e^{a\theta}v, \text{ say.}$$

The corresponding part of  $u$  in (4) is

$$\begin{aligned} & \{1 + F_1(D)e^\theta + \dots\} e^{a\theta}v \\ &= \{e^{a\theta} + e^{(a+1)\theta}F_1(D+a+1) + e^{(a+2)\theta}F_2(D+a+2) + \dots\} v \text{ by (3475).} \end{aligned}$$

Expand each function  $F$  by Taylor's theorem in powers of  $D$ , operate upon  $v$ , and arrange the result according to powers of  $\theta$ .

In practice, proceed as in the following example.

**3616** Ex. :  $xu_{xx} + u_x + q^2xu = 0.$

Multiplying by  $x$  and changing by (3489), this becomes

$$D^2u + q^2e^{2\theta}u = 0. \quad D^2u = 0 \text{ gives } u = A + B\theta.$$

Substitute this value and operate with  $D$ , considering  $A$  and  $B$  as variables, and equate to zero the coefficients of the powers of  $\theta$ ; thus

$$D^2A + q^2e^{2\theta}A + 2DB = 0, \quad D^2B + q^2e^{2\theta}B = 0.$$

Then change  $D$  into  $m$ , and  $e^{r\theta}A$  into  $a_{m-r}$ , to obtain the relations

$$m^2a_m + q^2a_{m-2} + 2mb_m = 0; \quad m^2b_m + q^2b_{m-2} = 0,$$

which determine the constants successively in terms of  $a_0$  and  $b_0$  (which are arbitrary) in the equation

$$u = a_0 + a_2x^2 + a_4x^4 + \dots + \log x (b_0 + b_2x^2 + b_4x^4 + \dots),$$

which thus becomes the solution sought.

[Boole, *Diff. Eq.*, p. 439.]

## SOLUTION BY DEFINITE INTEGRALS.\*

**3617** *Laplace's method.*—The solution of the equation

$$x\phi(d_x)u + \psi(d_x)u = 0 \dots\dots\dots (1)$$

is 
$$u = C \int \{e^{xt + \int \frac{\psi t}{\phi t} dt} (\phi t)^{-1}\} dt \dots\dots\dots (2),$$

the limits being determined by

$$e^{xt + \int \frac{\psi t}{\phi t} dt} = 0 \dots\dots\dots (3).$$

PROOF.—Assume  $u = e \int e^{xt} T dt$ , and substitute in (1), putting  $\phi(d_x)e^{xt} = \phi(t)e^{xt}$  (3474), thus

$$\int x e^{xt} \phi(t) T dt + \int e^{xt} \psi(t) T dt = 0.$$

\* This method of solution is merely indicated here, and the reader is referred to Boole's *Diff. Eq.*, Ch. xviii., for a complete investigation.

Integrating the first term by parts, this becomes

$$e^{xt} \phi(t) T - \int e^{xt} [d_t \{ \phi(t) T \} - \psi(t) T] dt = 0 \dots\dots\dots (4),$$

an equation which is satisfied by equating each term to zero. The second term thus produces a value of  $T$  by integration by (3209), and this value substituted in the first term, and in the value of  $u$ , gives the results (3) and (2).

**3618** Ex. (1):  $xu_{xx} + au_x - q^2xu = 0^*$  ..... (4).

Here  $\phi(d_x) = d_{xx} - q^2$ ,  $\psi(d_x) = ad_x$ ,  $\phi(t) = t^2 - q^2$ ,  $\psi(t) = at$ . Hence (2) and (3) become

$$u = C \int e^{xt} (t^2 - q^2)^{\frac{a}{2}-1} dt; \quad e^{xt} (t^2 - q^2)^{\frac{a}{2}} = 0;$$

$a$  being positive, the limits are  $t = \pm q$ , and, putting  $t = q \cos \theta$ , we find

$$u = C \int_0^\pi e^{qx \cos \theta} \sin^{a-1} \theta d\theta \dots\dots\dots (5).$$

**3619** The solution in series by (3608) is as follows. Equations (1) and (2) of that article are in this case

$$D(D+a-1)u - q^2e^{2x}u = 0 \quad \text{and} \quad m(m+a-1)v_m - q^2v_{m-2} = 0.$$

Thus,  $a$  in (3608) = 0, and  $b = 1 - a$ . Therefore (3610) becomes

$$u = A \left\{ 1 + \frac{q^2x^2}{2(a+1)} + \frac{q^4x^4}{2.4(a+1)(a+3)} + \&c. \right\} \\ + Bx^{1-a} \left\{ 1 + \frac{q^2x^2}{2(3-a)} + \frac{q^4x^4}{2.4(3-a)(5-a)} + \&c. \right\} \dots\dots (6).$$

Both series are convergent by (239 ii.).

The results deduced by Boole are these—

**3620** (5) is equivalent to the particular integral represented by the first series of (6).

**3621** A second particular integral, by assuming  $u = e^{(1-a)x}v$ , is found to be, when  $2-a$  is positive,

$$u = C_1 x^{1-a} \int_0^\pi e^{qx \cos \theta} \sin^{1-a} \theta d\theta \dots\dots\dots (7).$$

**3622** When  $a$  lies between 0 and 2, the complete integral is

$$u = C_1 \int_0^\pi e^{qx \cos \theta} \sin^{a-1} \theta d\theta + C_2 x^{1-a} \int_0^\pi e^{qx \cos \theta} \sin^{1-a} \theta d\theta \dots\dots\dots (8).$$

\* The method by definite integrals is elucidated by Boole chiefly in the solution of this important equation.



**3623** But, if  $a = 1$ , the solution becomes

$$u = \int_0^\pi e^{qx \cos \theta} \{ A + B \log (x \sin^2 \theta) \} d\theta \dots\dots\dots (9).$$

**3624** If  $a$  does not lie between 0 and 2, then, if  $a$  be negative, put  $a = a' - 2n$ , and replace the first term of (8) by

$$C_1 (xd_x + a' - 1)(xd_x + a' - 3) \dots (xd_x + a' - 2n + 1) \int_0^\pi e^{qx \cos \theta} \sin^{a'-1} d\theta \dots (10),$$

the transformation being effected by (3580).

**3625** And if  $a$  be positive and  $> 2$ , put  $u = e^{(1-a)s} v = x^{1-a} v$ . This converts  $a$  into  $2 - a$ , a negative quantity, and the case is reduced to the last one.

**3626** Ex. (2).—To solve by this method the P. D. equation

$$u_{xx} + u_{yy} + u_{zz} = 0 \dots\dots\dots (1)$$

when  $r = \sqrt{(x^2 + y^2)}$ .

Eliminating  $x$  and  $y$ , (1) becomes

$$ru_{xx} + u_r + ru_{zz} = 0 \dots\dots\dots (2).$$

Now the solution of this equation is number (9) of Example (1), if we change  $x$  into  $r$ ,  $q$  into  $id_z$ , and  $A$  and  $B$  into arbitrary functions of  $z$ . We thus obtain

$$u = \int_0^\pi e^{r \cos \theta id_z} \{ \phi(z) + \psi(z) \log (r \sin^2 \theta) \} d\theta \dots\dots\dots (3),$$

or, by (3492),

$$u = \int_0^\pi \phi \{ z + ir \cos \theta \} d\theta + \int_0^\pi \psi (z + ir \cos \theta) \log (r \sin^2 \theta) d\theta \dots\dots (4).$$

See (3551) for another solution.

**3627** If  $u$  be the potential of an attracting mass at an external point, and if  $u = F(z)$  when  $r = 0$ ; then, since  $\log r = \infty$ ,  $\psi(z)$  must vanish;

therefore 
$$F(z) = \int_0^\pi \phi(z) d\theta = \pi \phi(z).$$

Hence (4) reduces to 
$$u = \frac{1}{\pi} \int_0^\pi F \{ z + ir \cos \theta \} d\theta.$$

### *Parseval's Theorem.*

**3628** If, for all values of  $u$ ,

$$A + Bu + Cu^2 + \dots = \phi(u)$$

and

$$A' + B'u^{-1} + C'u^{-2} + \dots = \psi(u) \dots\dots\dots (1),$$

then

$$AA' + BB' + \dots = \frac{1}{2\pi} \int_0^\pi \left[ \phi(e^{i\theta}) \psi(e^{i\theta}) + \phi(e^{-i\theta}) \psi(e^{-i\theta}) \right] d\theta.$$

PROOF.—Form the product of equations (1), and in it put  $u = e^{i\theta}$  and  $e^{-i\theta}$  separately, and add the results. Multiply by  $d\theta$ , integrate from 0 to  $\pi$ , and divide by  $2\pi$ .

## P. D. EQUATIONS WITH MORE THAN TWO INDEPENDENT VARIABLES.

**3629** By means of Fourier's theorem (2742), the solution of the equation

$$u_{2t} - h^2 (u_{2x} + u_{2y} + u_{2z}) = 0$$

may be deduced by a general method in the form

$$u = (1 + d_t) \iiint \iiint e^{i(A+Bht)} \psi(a, b, c) da db dc d\lambda d\mu d\nu,$$

the limits of each integration being  $-\infty$  to  $\infty$ , and the function  $\psi$  being arbitrary and different in the two terms arising from the operator  $(1 + d_t)$ .

Boole, Ch. xviii., and more fully in Cauchy's *Exercice d'Analyse Mathématique*, Tom. I., pp. 53 et 178.

**3630** Poisson's solution of the same equation in the form of a double integral is

$$u = (1 + dt) \int_0^\pi \int_0^{2\pi} t \sin \xi \psi(x + ht \sin \xi \sin \eta, y + ht \sin \xi \cos \eta, z + ht \cos \xi) d\xi d\eta$$

with the same latitude in the interpretation of  $\psi$ .

[Gregory's *Examples*, p. 504.]

## DIFFERENTIAL RESOLVENTS OF ALGEBRAIC EQUATIONS.

**3631** THEOREM I. (Boole).—"If  $y_1, y_2 \dots y_n$  are the  $n$  roots of the equation

$$y^n - ay^{n-1} + 1 = 0 \dots\dots\dots (1),$$

and if the  $m^{\text{th}}$  power of any one of these roots be represented

by  $u$ , and if  $a = e^\theta$ , then  $u$  as a function of  $\theta$  satisfies the differential equation

$$u - \left( \frac{n-1}{n} D + \frac{m}{n} - 1 \right)^{(n-1)} \left( \frac{D}{n} - \frac{m}{n} - 1 \right) [D^{(n)}]^{-1} e^{n\theta} u = 0,$$

and the complete integral of the same will be

$$u = C_1 y_1^m + C_2 y_2^m + \dots + C_n y_n^m.$$

**3632** “COR. I.—If  $m = -1$  and if  $n$  be  $> 2$ , the differential equation

$$D^{(n-1)} u - \frac{1}{n} \left( \frac{n-1}{n} D - \frac{1}{n} - 1 \right)^{(n-1)} e^{n\theta} u = 0$$

has for its general integral

$$u = C_1 y_1^{-1} + C_2 y_2^{-1} \dots + C_{n-1} y_{n-1}^{-1},$$

$y, y \dots y_{n-1}$  being any  $n-1$  roots of (1).

“If  $\theta$  be changed into  $-\theta$ , and therefore  $D$  into  $-D$ , the above results are modified as follows:—

**3633** “COR. II.—The differential equation

$$u - (D-1)^{(n)} \left[ \left( \frac{n-1}{n} D - \frac{m}{n} \right)^{(n-1)} \left( \frac{D}{n} + \frac{m}{n} \right) \right]^{-1} e^{n\theta} u = 0$$

has for its complete integral

$$u = C_1 y_1^m + C_2 y_2^m \dots + C_n y_n^m,$$

$y_1, y_2 \dots y_n$  being the roots of the equation

$$ay^n - y^{n-1} + a = 0 \dots\dots\dots (2).$$

**3634** “COR. III.—The differential equation

$$u - n (D-2)^{(n-1)} \left[ \left( \frac{n-1}{n} D + \frac{1}{n} \right)^{(n-1)} \right]^{-1} e^{n\theta} u = 0,$$

supposing  $n > 2$  has for its complete integral

$$u = C_1 y_1^{-1} + C_2 y_2^{-1} \dots + C_{n-1} y_{n-1}^{-1},$$

$y_1, y_2 \dots y_{n-1}$  being any  $n-1$  roots of (2).

“ *Theorem II. (Harley).*—

**3635** “ The differential equation

$$a^r (x d_x)^{(n)} u - \left( \frac{n-r}{n} x d_x + \frac{m}{n} - 1 \right)^{(n-r)} \left( \frac{r}{n} x d_x - \frac{m}{n} - 1 \right)^{(r)} x^r u = 0$$

is satisfied by the  $m^{\text{th}}$  power of any root of the equation

$$y^n - x y^{n-r} + a = 0,$$

$u$  being considered as a function of  $x$ .

**3636** “ *COR.*—The differential equation

$$n^n \left( \frac{n-r}{r} x d_x - \frac{m}{r} \right)^{(n-r)} (x d_x)^{(r)} u - (n-1)^r \left( \frac{n}{r} x d_x - \frac{m}{r} - 1 \right)^{(n)} x^r u = 0$$

is satisfied by the  $m^{\text{th}}$  power of any root of the equation

$$y^n - n y^{n-r} + (n-1) x = 0."$$

[*Boole, Diff. Eq., Sup.* 191—199.]

**3637** See also *Boole, Phil. Trans.*, 1864; *Harley, Proc. of the Lit. and Phil. Soc. of Manchester*, Vol. II.; *Rawson, Proc. of the Lond. Math. Soc.*, Vol. 9.

# CALCULUS OF FINITE DIFFERENCES.

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## INTRODUCTION.

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**3701** In this branch of pure mathematics a function  $\phi(x)$  is denoted by  $u_x$ , and  $\phi(x+h)$  consequently by  $u_{x+h}$ . The increment  $h$  is commonly unity. If  $\Delta x$  denotes the increment  $h$ , and  $\Delta u_x$  the consequent increase in the value of  $u_x$ , we have

**3702** 
$$\Delta u_x = u_{x+\Delta x} - u_x.$$

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**3703** When  $\Delta x$  diminishes without limit, the value of

$$\frac{\Delta u_x}{\Delta x} \quad \text{or} \quad \frac{u_{x+\Delta x} - u_x}{\Delta x} \quad \text{is} \quad \frac{du_x}{dx}.$$

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**3704** The repetition of the operation  $\Delta$  is indicated as follows:

$$\Delta \Delta u_x = \Delta^2 u_x, \quad \Delta \Delta^2 u_x = \Delta^3 u_x, \quad \text{and so on.}$$

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**3705** Ex.: Let  $u_x = x^2$ ,

$x =$	1	2	3	4	5	...
$x^2 =$	1	4	9	16	25	...
$\Delta x^2 =$	3	5	7	9	...	...
$\Delta^2 x^2 =$	2	2	2	...	...	...

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## FORMULÆ FOR FIRST AND $n^{\text{th}}$ DIFFERENCES.

If  $u_x = ax^n + bx^{n-1} + cx^{n-2} + \&c.,$

**3706** 
$$\Delta^r u_x = an(n-1) \dots (n-r+1) x^{n-r} + c_1 x^{n-r-1} + \&c.,$$

**3707** 
$$\Delta^n u_x = an(n-1) \dots 3.2.1.$$

**3708** Hence the  $n$ th difference of a rational integral function of the  $n$ th degree is constant.

**3709** So also  $\Delta^n x^n = 1.2.3 \dots n.$

**3710** NOTATION.—Factorial terms are denoted as follows:—

$$u_x u_{x-1} \dots u_{x-m+1} \equiv u_x^{(m)},$$

**3711** 
$$\frac{1}{u_x u_{x+1} \dots u_{x+m-1}} \equiv u_x^{(-m)}.$$

**3712** Thus  $x(x-1) \dots (x-m+1) \equiv x^{(m)},$

**3713** 
$$\frac{1}{x(x+1) \dots (x+m-1)} \equiv x^{(-m)}.$$

Hence  $\underline{m}$ ,  $m!$ , and  $m^{(m)}$  are equivalent symbols.

**3714** According to (2452),  $x^{(m)}$  would here be denoted by  $x_{-1}^{(m)}$ . The suffix, however, being omitted, it may be understood that the common difference of the factors is always  $-1$ .

**3715**  $\Delta x^{(m)} = m x^{(m-1)}, \Delta^n x^{(m)} = m^{(n)} x^{(m-n)}, \Delta^m x^{(m)} = m^{(m)},$   
and, if  $m < n$ ,  $\Delta^n x^{(m)} = 0$ , since  $\Delta c = 0$  if  $c = \text{constant}$ .

**3718**  $\Delta x^{(-m)} = -m x^{(-m-1)}, \Delta^n x^{(-m)} = (-m)^{(n)} x^{(-m-n)}.$

**3720** 
$$\Delta u_x^{(m)} = (u_{x+1} - u_{x-m+1}) u_x^{(m-1)},$$
  
$$\Delta u_x^{(-m)} = (u_x - u_{x+m}) u_x^{(-m-1)}.$$

**3722** Ex.:

$$\Delta (ax+b)^{(m)} = am (ax+b)^{(m-1)}, \quad \Delta (ax+b)^{(-m)} = -am (ax+b)^{(-m-1)}.$$

**3724**  $\Delta \log u_x = \log \left\{ 1 + \frac{\Delta u_x}{u_x} \right\}, \quad \Delta \log u_x^{(m-1)} = \log \frac{u_{x+1}}{u_{x-m+1}}.$

**3726**  $\Delta a^x = (a-1) a^x, \quad \Delta^n a^{mx} = (a^m - 1)^n a^{mx}.$

**3728**  $\Delta^n \frac{\sin}{\cos} (ax+b) = \left( 2 \sin \frac{a}{2} \right)^n \frac{\sin}{\cos} \left\{ ax+b + \frac{n(a+\pi)}{2} \right\}.$

PROOF.— 
$$\Delta \sin (ax+b) = \sin (ax+a+b) - \sin (ax+b)$$
  
$$= 2 \sin \frac{a}{2} \sin \left( ax+b + \frac{a+\pi}{2} \right).$$

That is,  $\Delta$  is equivalent to adding  $\frac{a+\pi}{2}$  to the angle and multiplying the sine by  $2 \sin \frac{a}{2}$ .



**3729** Conversely, the same formula holds if the sign of  $n$  be changed throughout.

#### EXPANSION BY FACTORIALS.

**3730** If  $\Delta^n \phi(0)$  denote the value of  $\Delta^n \phi(x)$  when  $x=0$ , then  $\phi(x) = \phi(0) + \Delta \phi(0)x + \frac{\Delta^2 \phi(0)}{2} x^{(2)} + \frac{\Delta^3 \phi(0)}{2.3} x^{(3)} + \&c.$

**3731** If  $\Delta x = h$  instead of unity, the same expansion holds good if for  $\Delta^n \phi(0)$  we write  $(\Delta^n \phi(x) \div h^n)_{x=0}$ ; that is, making  $x=0$  after reduction.

PROOF.—Assume  $\phi(x) = a_0 + a_1 x + a_2 x^{(2)} + a_3 x^{(3)} + \&c.$   
Compute  $\Delta \phi(x)$ ,  $\Delta^2 \phi(x)$ , &c., and put  $x=0$  to determine  $a_0, a_1, a_2$ , &c.

#### GENERATING FUNCTIONS.

**3732** If  $u_x t^x$  be the general term in the expansion of  $\phi(t)$ , then  $\phi(t)$  is called the *generating function* of  $u_x$  or  $\phi(t) = Gu_x$ .

Ex.:  $(1-t)^{-2} = G(x+1)$ , for  $x+1$  is the coefficient of  $t^x$  in the expansion.

$$\mathbf{3733} \quad Gu_x = \phi(t), \quad Gu_{x+1} = \frac{\phi(t)}{t}, \quad \dots \quad Gu_{x+n} = \frac{\phi(t)}{t^n}.$$

$$\mathbf{3734} \quad G\Delta u_x = \left(\frac{1}{t} - 1\right)\phi(t), \quad \dots \quad G\Delta^n u_x = \left(\frac{1}{t} - 1\right)^n \phi(t).$$

PROOF.—  $G\Delta u_x = Gu_{x+1} - Gu_x$  &c.

#### THE OPERATIONS $E$ , $\Delta$ , AND $d_x$ .

**3735**  $E$  denotes the operation of increasing  $x$  by unity,

$$Eu_x = u_{x+1} = u_x + \Delta u_x = (1 + \Delta) u_x.$$

The symbols  $E$  and  $\Delta$  both follow the laws of *distribution*, *commutation*, and *repetition* (1488–90).

$$\mathbf{3736} \quad E = 1 + \Delta = e^{d_x} \quad \text{or} \quad e^D.*$$

$$\text{PROOF.—} \quad Eu_x = u_{x+1} = u_x + d_x u_x + \frac{1}{2} d_{2x} u_x + \frac{1}{2.3} d_{3x} u_x + \&c.$$

$$= (1 + d_x + \frac{1}{2} d_{2x} + \frac{1}{2.3} d_{3x} + \&c.) u_x = e^{d_x} u_x.$$

By (1520),  $\Delta x$  being unity.

\* The letter  $d$  is reserved as a symbol of differentiation only, and the suffix attached to it indicates the independent variable. See (1487).

**3737** Hence  $\Delta = e^D - 1$  and  $D = \log E$ .

**3739** Consistently with (3735)  $E^{-1}$  denotes the diminishing  $x$  by unity; thus  $E^{-1}u_x = u_{x-1}$ .

For  $E u_{x-1} = u_x$ ,  $\therefore u_{x-1} = E^{-1}u_x$ .

$u_{x+n}$  in terms of  $u_x$  and successive differences.

**3740**  $u_{x+n} = u_x + n\Delta u_x + C(n, 2)\Delta^2 u_x + C(n, 3)\Delta^3 u_x + \&c.$

PROOF.—(i.) By induction, or (ii.) by generating functions, or (iii.) by the symbolic law:

$$(ii.) \quad G u_{x+n} = \left(\frac{1}{t}\right)^n \phi(t) = \left\{1 + \left(\frac{1}{t} - 1\right)\right\}^n \phi(t).$$

Expand by the Binomial theorem, and apply (3734).

$$(iii.) \quad u_{x+n} = E^n u_x = (1 + \Delta)^n u_x.$$

Apply the laws in (3735) by expanding the binomial and distributing the operation upon  $u_x$ .

Conversely to express  $\Delta^n u_x$  in terms of  $u_x, u_{x+1}, u_{x+2}, \&c.$

**3741**  $\Delta^n u_x = u_{x+n} - n u_{x+n-1} + C(n, 2) u_{x+n-2} \dots (-1)^n u_x.$

PROOF.—  $\Delta^n u_x = (E - 1)^n u_x$  (3736).

Expand, and apply (3735) as before. Putting  $x = 0$ , we also have

**3742**  $\Delta^n u_0 = u_n - n u_{n-1} + C_{n,2} u_{n-2} \dots (-1)^n u_0.$

**3743**  $\Delta^n x^m = (x+n)^m - n(x+n-1)^m + C(n, 2)(x+n-2)^m - \&c.$

**3744**  $\Delta^n 0^m = n^m - n(n-1)^m + C(n, 2)(n-2)^m - C(n, 3)(n-3)^m + \&c.$

**3745** Ex.: By (3717)  $\Delta^n 0^n = n!$  Hence a proof of theorem (285) is obtained.

**3746**  $\Delta^n u_x v_x = (EE' - 1)^n u_x v_x,$

where  $E$  operates only upon  $u_x$  and  $E'$  only upon  $v_x$ .

PROOF.  $\Delta u_x v_x = u_{x+1} v_{x+1} - u_x v_x = E u_x \cdot E' v_x - u_x v_x = (EE' - 1) u_x v_x.$

*Applications of (3746).*

**3747** Ex. (1):  $\Delta^n u_x v_x = (-1)^n (1 - EE')^n u_x v_x.$

Expand the binomial, and operate upon the subjects  $u_x, v_x$ ; thus

**3748**  $\Delta^n u_x v_x = (-1)^n \{ u_x v_x - n u_{x+1} v_{x+1} + C(n, 2) u_{x+2} v_{x+2} - \&c. \}.$

**3749** Ex. (2): To expand  $a^x \sin x$  by successive differences of  $\sin x$ .

$$\begin{aligned}\Delta^n a^x \sin x &= \{E(1+\Delta')-1\}^n a^x \sin x = \{\Delta + E\Delta'\}^n a^x \sin x \\ &= \{\Delta^n + n\Delta^{n-1}E\Delta' + C(n, 2)\Delta^{n-2}E^2\Delta'^2 + \&c.\} a^x \sin x \\ &= \Delta^n a^x \sin x + n\Delta^{n-1}a^{x+1}\Delta \sin x + C(n, 2)\Delta^{n-2}a^{x+2}\Delta^2 \sin x + \&c. \\ &= a^x \{(a-1)^n \sin x + n(a-1)^{n-1}a\Delta \sin x + C(n, 2)(a-1)^{n-2}a^2\Delta^2 \sin x + \&c.\},\end{aligned}$$

by (3727), while  $\Delta^r \sin x$  is known from (3728).

**3750** Ex. (3): To expand  $\Delta^n u_x v_x$  in differences of  $u_x$  and  $v_x$  alone: put  $E = 1 + \Delta$ ,  $E' = 1 + \Delta'$  in (3746), thus

$$\Delta^n u_x v_x = (\Delta + \Delta' + \Delta\Delta')^n u_x v_x,$$

which must be expanded.

$\Delta^n u_x$  in differential coefficients of  $u_x$ .

**3751**  $\Delta^n u_x = d_x^n u_x + A_1 d_x^{n+1} + A_2 d_x^{n+2} u_x + \&c.$

PROOF.—  $\Delta^n u_x = (e^{d_x} - 1)^n u_x$  (3737).

Expand by (150) and (125) as if  $d_x$  were a quantitative symbol. See also (3761).

$\frac{d^n u}{dx^n}$  in successive differences of  $u$ .

**3752**  $\frac{d^n u}{dx^n} = \{\log(1+\Delta)\}^n u.$

The expansion by (155) and (125) will present a series of ascending differences of  $u$ .

PROOF.—  $e^{d_x} = 1 + \Delta$ ,  $\therefore d_x = \log(1 + \Delta).$

**3753** Ex.: If  $n = 1$ ,  $\frac{du}{dx} = \Delta u - \frac{\Delta^2 u}{2} + \frac{\Delta^3 u}{3} - \frac{\Delta^4 u}{4} + \&c.$

If  $C$  be a constant,

**3754**  $\phi(D) C = \phi(\Delta) C = \phi(0) C$  and  $\phi(E) C = \phi(1) C$ ,

Since every term of  $\phi(D)$ , or of  $\phi(\Delta) C$ , operating upon  $C$ , produces 0; and every term of  $\phi(E)$  operating upon  $C$  produces  $C$ .

HERSCHEL'S THEOREM.

**3757**  $\phi(e^t) = \phi(E) e^{0 \cdot t}$

**3758**  $= \phi(1) + \phi(E) 0 \cdot t + \phi(E) 0^2 \cdot \frac{t^2}{1 \cdot 2} + \&c.$

PROOF.—Let  $\phi(e^t) = A_0 + A_1 e^t \dots + A_n e^{nt}$

$$= A_0 e^{0 \cdot t} + A_1 E e^{0 \cdot t} \dots + A_n E^n e^{0 \cdot t} = (A_0 + A_1 E \dots + A_n E^n) e^{0 \cdot t}$$

$$= \phi(E) e^{0 \cdot t} = \phi(E) \left\{ 1 + 0 \cdot t + \frac{0^2 t^2}{1 \cdot 2} + \&c. \right\},$$

and  $\phi(E) 1 = \phi(1)$  by (3756).

A THEOREM CONJUGATE TO MACLAURIN'S (1507).

**3759**  $\phi(t) = \phi(D) e^{0 \cdot t}$

**3760**  $= \phi(0) + \phi(d_0) 0 \cdot t + \phi(d_0) 0^2 \cdot \frac{t^2}{1 \cdot 2} + \&c.$

PROOF.—  $\phi(t) = \phi(\log e^t) = \phi(\log E) e^{0 \cdot t}$  (3757)

$$= \phi(D) e^{0 \cdot t} \text{ (3738)} = \phi(D) \left\{ 1 + 0 \cdot t + \dots + \&c. \right\},$$

and

$$\phi(D) 1 = \phi(0) \text{ (3754).}$$

$n$  being a positive integer,

**3761**

$$\Delta^n u = \frac{d^n u}{dx^n} + \frac{\Delta^n 0^{n+1}}{1 \cdot 2 \dots (n+1)} \frac{d^{n+1} u}{dx^{n+1}} + \frac{\Delta^n 0^{n+2}}{1 \cdot 2 \dots (n+2)} \frac{d^{n+2} u}{dx^{n+2}} + \dots$$

PROOF.—By (3758), putting  $\phi(e^t) \equiv (e^t - 1)^n$ ,

$$(e^t - 1)^n = (E - 1)^n 0 \cdot t + (E - 1)^n 0^2 \cdot \frac{t^2}{1 \cdot 2} + \&c. = \Delta^n 0 \cdot t + \Delta^n 0^2 \frac{t^2}{1 \cdot 2} + \&c.$$

Put  $t = d_x$ , and employ (3736) and (3737).

## INTERPOLATION.

*Approximate value of  $u_x$  in terms of  $n$  particular equidistant values.*

**3762** If  $u_x$  is an integral algebraic function of the degree  $n-1$ ,  $\Delta^n u_x$  vanishes, and therefore by making  $x=0$ , and writing  $x$  for  $n$  in (3740),

**3763**  $u_x = u_0 + x \Delta u_0 + C_{x,2} \Delta^2 u_0 \dots + C_{x,n-1} \Delta^{n-1} u_0.$

This is formula (265). The given values are  $u_0, \Delta u_0, \Delta^2 u_0, \&c.$ , corresponding to  $a, b, c, \dots$

**3764** For an application of the formula to the problem of interpolation, see (267), in which example  $x = 1 \cdot 54$  and  $u_x = \log 72 \cdot 54$ .

**3765** When the term to be interpolated is one of a set of equidistant terms, employ (3741).  $\Delta^n u_x = 0$ , as in (3762); therefore

$$\mathbf{3766} \quad u_n - nu_{n-1} + C_{n,2}u_{n-2} - C_{n,3}u_{n-3} \dots + (-1)^n u_0 = 0.$$

**3767** Ex.: From  $\sin 0$ ,  $\sin 30^\circ$ ,  $\sin 45^\circ$ , and  $\sin 60^\circ$ , to deduce the value of  $\sin 15^\circ$ .

The formula gives  $\sin 0 - 4 \sin 15^\circ + 6 \sin 30^\circ - 4 \sin 45^\circ + \sin 60^\circ = 0$ ,  
or  $-4 \sin 15^\circ + 3 - 2\sqrt{2} + \frac{1}{2}\sqrt{3} = 0$ ,  
from which  $\sin 15^\circ = \frac{1}{8}(6 - 4\sqrt{2} + \sqrt{3}) = .2594$ .  
The true value is .2588; the error .0006.

#### LAGRANGE'S INTERPOLATION FORMULA.

**3768** Let  $a, b, c, \dots k$  be  $n$  values of  $x$ , not equidistant, for which the values of  $u_x$  are known; then generally

**3769**

$$u_x = u_a \frac{(x-b)(x-c) \dots (x-k)}{(a-b)(a-c) \dots (a-k)} + u_b \frac{(x-a)(x-c) \dots (x-k)}{(b-a)(b-c) \dots (b-k)} \\ \dots \dots \dots + u_k \frac{(x-a)(x-b)(x-c) \dots}{(k-a)(k-b)(k-c) \dots}.$$

PROOF.—Assume  $u_x = A(x-b)(x-c) \dots (x-k)$   
 $+ B(x-a)(x-c) \dots (x-k) + C(x-a)(x-b)(x-d) \dots (x-k) + \&c.$ ,  
and determine  $A, B, C, \&c.$  by making  $x = a, b, c, \&c.$ , in turn.

If the values of  $a, b, c, \dots k$  are  $0, 1, 2, \dots n-1$ , (3769) reduces to

$$\mathbf{3770} \quad u_x = u_{n-1} \frac{x(x-1) \dots (x-n+2)}{1.2.3 \dots (n-1)} \\ - u_{n-2} \frac{x(x-1) \dots (x-n+3)(x-n+1)}{1.1.2.3 \dots (n-2)} \\ + u_{n-3} \frac{x(x-1) \dots (x-n+4)(x-n+2)(x-n+1)}{2.1.1.2.3 \dots (n-3)} - \&c., \text{ or}$$

**3771**

$$u_x = \frac{x^{(n)}}{(n-1)!} \left\{ \frac{u_{n-1}}{x-n+1} - \frac{C_{n-1,1}u_{n-2}}{x-n+2} + \frac{C_{n-1,2}u_{n-3}}{x-n+3} - \&c. \right\}.$$

## MECHANICAL QUADRATURE.

The area of a curve whose equation is  $y = u_x$  in terms of  $n+1$  equidistant ordinates  $u, u_1, \dots u_n$ , is approximately

$$\begin{aligned} 3772 \quad nu + \frac{n^2}{2} \Delta u + \left( \frac{n^3}{3} - \frac{n^2}{2} \right) \frac{\Delta^2 u}{1.2} + \left( \frac{n^4}{4} - n^3 + n^2 \right) \frac{\Delta^3 u}{3!} \\ + \left( \frac{n^5}{5} - \frac{3n^4}{2} + \frac{11n^3}{3} - 3n^2 \right) \frac{\Delta^4 u}{4!} \\ + \left( \frac{n^6}{6} - 2n^5 + \frac{35n^4}{4} - \frac{50n^3}{3} + 12n^2 \right) \frac{\Delta^5 u}{5!} \\ + \left( \frac{n^7}{7} - \frac{15n^6}{6} + 17n^5 - \frac{225n^4}{4} + \frac{274n^3}{3} - 60n^2 \right) \frac{\Delta^6 u}{6!}. \end{aligned}$$

PROOF.—The area is  $= \int_0^n u_x dx$ . Take the value of  $u_x$  in terms of  $u_0, u_1 \dots u_{n-1}$  from (3763) and integrate.

$$3773 \quad \text{When } n=2, \quad \int_0^2 u_x dx = \frac{u+4u_1+u_2}{3}.$$

$$3774 \quad n=3, \quad \int_0^3 u_x dx = \frac{3}{8} (u+3u_1+3u_2+u_3),$$

$$3775 \quad n=4, \quad \int_0^4 u_x dx = \frac{14(u+u_4)+64(u_1+u_3)+24u_2}{45}.$$

$$3776 \quad n=6, \quad \int_0^6 u_x dx = \frac{3}{10} \{u+u_2+u_4+u_6+5(u_1+u_5)+6u_3\}.$$

In the last formula, which is due to Mr. Weddle, the coefficient of  $\Delta^6 u$  is taken as  $\frac{3}{10}$  instead of  $\frac{41}{140}$ , its true value. These results are obtained from (3772) by substituting for each  $\Delta$  its value from (3742).

## COTES'S AND GAUSS'S FORMULÆ.

3777 These give the area of the curve directly in terms of fixed abscissæ.

They are obtained by integrating Lagrange's value of  $u_x$  (3769–71), and are fully discussed in articles (2995–7).



## LAPLACE'S FORMULA.

$$\begin{aligned}
 3778 \quad \int_0^n u_x dx &= \frac{u_0}{2} + u_1 + u_2 \dots + \frac{u_n}{2} \\
 &- \frac{1}{12} (\Delta u_n - \Delta u_0) + \frac{1}{24} (\Delta^2 u_n - \Delta^2 u_0) - \&c.,
 \end{aligned}$$

the coefficients being those in the expansion of  
 $t \{ \log (1+t) \}^{-1}.$

$$\begin{aligned}
 \text{PROOF.} \quad \Delta w_x &= d_x \left\{ \frac{\Delta}{\log (1+\Delta)} \right\} w_x, \text{ by (3736),} \\
 &= d_x \left\{ 1 + \frac{\Delta}{2} - \frac{\Delta^2}{12} + \frac{\Delta^3}{24} - \frac{19}{720} \Delta^4 + \&c. \right\} w_x.
 \end{aligned}$$

Hence, putting  $u_x = \Delta w_x$

$$\begin{aligned}
 \int_0^1 u_x dx &= \frac{u_0 + u_1}{2} - \frac{\Delta^2 u_0}{12} + \frac{\Delta^3 u_0}{24} - \&c., \\
 \int_1^2 u_x dx &= \frac{u_1 + u_2}{2} - \frac{\Delta^2 u_1}{12} + \frac{\Delta^3 u_1}{24} - \&c.,
 \end{aligned}$$

and so on; then add together the  $n$  equations.

**3779** Formula (3778) contains  $\Delta u_n$ ,  $\Delta^2 u_n$ , &c., which cannot be found from  $u_0, u_1 \dots u_n$ .

The following formula does not involve differences higher than  $\Delta u_{n-1}$ .

$$\begin{aligned}
 3780 \quad \int_0^n u_x dx &= \frac{u_0}{2} + u_1 + u_2 \dots + \frac{u_n}{2} \\
 &- \frac{1}{12} (\Delta u_{n-1} - \Delta u_0) - \frac{1}{24} (\Delta^2 u_{n-1} - \Delta^2 u_0) - \&c.
 \end{aligned}$$

PROOF.—In the proof of (3778), change  $\frac{\Delta}{\log (1+\Delta)}$  into  $E \frac{-\Delta E^{-1}}{\log (1-\Delta E^{-1})}$ , and put  $E^{-1} w_x = w_{x-1}$  (3739) after expansion, and proceed as before.

## SUMMATION OF SERIES.

**3781** Definition:  $\Sigma u_x = u_a + u_{a+1} + u_{a+2} \dots + u_{x-1}.$

**3782** Theorem:  $\Sigma u_x = \Delta^{-1} u_x + C,$

where  $C$  is constant for all the assigned values of  $x$ .

PROOF.—Let  $\phi(x)$  be such that  $\Delta\phi(x) = u_x$ , then  $\phi(x) = \Delta^{-1}u_x$ , therefore  $u_a = \phi(a+1) - \phi(a)$ . Write thus, and add together the values of  $u_a, u_{a+1}, \dots, u_{x-1}$ . Therefore, by (3781),  $\Sigma u_x = \phi(x) - \phi(a) = \Delta^{-1}u_x - \phi(a)$ , and  $\phi(a)$  is constant with respect to  $x$ .

Taken between the limits  $x=a, x=b-1$ , we have the notation,

$$3783 \quad \sum_{x=a}^{x=b-1} u_x \quad \text{or} \quad \Sigma_a^{b-1} u_x = \Sigma u_b - \Sigma u_a = \Delta^{-1}u_b - \Delta^{-1}u_a.$$

Functions integrable in finite terms :

$$3784 \quad \text{Class I.} \quad \Sigma x^{(m)} = \frac{x^{(m+1)}}{m+1} + C.$$

$$3785 \quad \Sigma (ax+b)^{(m)} = \frac{(ax+b)^{(m+1)}}{a(m+1)} + C.$$

$$3786 \quad \text{Class II.} \quad \Sigma x^{(-m)} = \frac{x^{(-m+1)}}{-m+1} + C. \quad \begin{array}{l} \text{By (3718),} \\ \text{and notation (3711).} \end{array}$$

$$3787 \quad \Sigma (ax+b)^{(-m)} = C - \frac{(ax+b)^{(-m+1)}}{a(m-1)}.$$

Formulae (3785) and (3786) are equivalent to the rules (269) and (271). They are the direct results of theorem (3782).

$$3788 \quad \Sigma a^x = \frac{a^x}{a-1}. \quad [\text{By (3726).}]$$

Class III.—If  $u_x$  be a rational integral function,

$$3789 \quad \Sigma_a^{a+x-1} u_x = \{x + C_{x,2}\Delta + C_{x,3}\Delta^2 + \dots\} u_a.$$

PROOF.—By (3735) and (3736),

$$u_a + u_{a+1} \dots + u_{a+x-1} = (1 + E + E^2 \dots + E^{x-1}) v_a = \frac{E^x - 1}{E - 1} u_a = \frac{(1 + \Delta)^x - 1}{\Delta} u_a \\ = \text{the expansion above.}$$

3790 The formula has been given at (266) and an example of its application. The series there summed is  $1 + 5 + 15 + 35 + 70 + 126 + \dots$  to 100 terms. The function  $u_x$  which gives rise to these terms is found by (3763) to be

$$u_x = (x^4 + 10x^3 + 35x^2 + 50x + 24) \div 24.$$

**3791** If this function be presented as  $u_x$ , and  $\Sigma_0^{x-1} u_x$  be required, we first find  $u_0=1, u_1=5, u_2=15, \&c.$ ; then the differences  $\Delta u_0, \Delta^2 u_0, \dots \Delta^4 u_0 = 1, 4, 6, 4, 1$ , and then, by (3789), the required sum, as in the example referred to.

**3792** For another example, let  $\Sigma_1^n x^3 = 1 + 2^3 + \dots + n^3$  be required.

Here  $\Delta x^3 = 3x^2 + 3x + 1, \Delta^2 x^3 = 6x + 6, \Delta^3 x^3 = 6,$   
therefore  $\Delta 0^3 = 1, \Delta^2 0^3 = 6, \Delta^3 0^3 = 6 \dots \dots \dots (1).$

$x^3$  may now be expressed in factorials, and the summation may then be effected by (3784). First, by (3730),

$$x^3 = x + 3x(x-1) + x(x-1)(x-2);$$

therefore, by (3784),  $\Sigma_1^n x^3 = \Sigma (n+1)^3$  (3783),

$$\Sigma_1^n x^3 = \frac{n(n+1)}{2} + \frac{3(n+1)n(n-1)}{3} + \frac{(n+1)n(n-1)(n-2)}{4} = \frac{n^2(n+1)^2}{4}.$$

**3793** Otherwise, by (3789), taking  $a=0$ , we have

$$u_x = x^3, u_0 = 0, \Delta u_0 = 1, \Delta^2 u_0 = 6, \Delta^3 u_0 = 6, \text{ as above.}$$

Therefore

$$\Sigma_0^{n-1} u_x = \frac{n(n-1)}{2} + \frac{6n(n-1)(n-2)}{1.2.3} + \frac{6n(n-1)(n-2)(n-3)}{1.2.3.4} = \frac{n^2(n-1)^2}{4},$$

therefore, changing  $n$  into  $n+1$ ,  $\Sigma_0^n u_x = \frac{(n+1)^2 n^2}{4}.$

**3794** *Class IV.*—When the general term of a series is a rational fraction of the form

$$\frac{A + Bx + Cx^2 + \dots}{u_x u_{x+1} \dots u_{x+m}}, \text{ where } u_x = ax + b,$$

and the degree of the numerator is not higher than  $x+m-2$ ; resolve the numerator into

$$A' + B'u_x + C'u_x u_{x+1} + \dots + D'u_x u_{x+1} u_{x+2} \dots u_{x+m-2},$$

by (3730). The fraction then separates into a series of fractions with constant numerators which can be summed by (3787).

**3795** If the factors  $u_x \dots u_{x+m}$  are not consecutive, introduce the missing ones in the denominator and numerator, and then resolve the fraction as in the foregoing rule.

**3796** Ex.: To sum the series  $\frac{1}{1.4} + \frac{1}{2.5} + \frac{1}{3.6} + \dots$  to  $n$  terms.

$$\begin{aligned} \text{The } n^{\text{th}} \text{ term is } \frac{1}{n(n+3)} &= \frac{(n+1)(n+2)}{n(n+1)(n+2)(n+3)} \\ &= \frac{n(n+1)+2n+2}{n(n+1)(n+2)(n+3)} = \frac{1}{(n+2)(n+3)} + \frac{2}{(n+1)(n+2)(n+3)} \\ &\quad + \frac{2}{n(n+1)(n+2)(n+3)}. \end{aligned}$$

The sum of  $n$  terms is, therefore, by the rule (271),

$$\begin{aligned} \left(\frac{1}{3} - \frac{1}{n+3}\right) + \frac{1}{2} \left(\frac{2}{2 \cdot 3} - \frac{2}{(n+2)(n+3)}\right) + \frac{1}{3} \left(\frac{2}{2 \cdot 3} - \frac{2}{(n+1)(n+2)(n+3)}\right) \\ = \frac{11}{18} - \frac{3n^2+12n+11}{3(n+1)(n+2)(n+3)}. \end{aligned}$$

If the form in (3787) is used, the total constant part  $C$  is determined finally by making  $n = 0$ , which gives  $C = \frac{11}{18}$ .

**3797** *Theorem.*  $f(E) a^x \phi(x) = a^x f(aE) \phi(x)$ ,  
 $f$  being an algebraic function.

PROOF.—Let  $a = e^m$ , then the left  
 $= f(E) e^{mx} \phi(x) = f(e^D) e^{mx} \phi(x)$  (3736)  $= e^{mx} f(e^{D \cdot m}) \phi(x)$  (3475)  
 $= a^x f(aE) \phi(x)$ .

*Class V.*—If  $\phi(x)$  be a rational integral function,

**3798**

$$\Sigma a^x \phi(x) = \frac{a^x}{a-1} \left\{ \phi(x) - \frac{a}{a-1} \Delta \phi(x) + \frac{a^2}{(a-1)^2} \Delta^2 \phi(x) - \&c. \right\}.$$

The upper limit is understood to be  $x-1$ , and a constant is to be added, (3781-2).

PROOF.— $\Sigma a^x \phi(x) = \Delta^{-1} a^x \phi(x) = (E-1)^{-1} a^x \phi(x) = a^x (aE-1)^{-1} \phi(x)$   
 (3797)  $= a^x \left\{ a(1+\Delta) - 1 \right\}^{-1} \phi(x) = \frac{a^x}{a-1} \left( 1 + \frac{a\Delta}{a-1} \right)^{-1} \phi(x)$ .

Then expand the binomial.

$\Sigma a^x \phi(x)$  in successive derivatives of  $\phi(x)$ .

**3799**  $\Sigma a^x \phi(x) = \frac{a^x}{a-1} \left\{ 1 + A_1 \phi'(x) + \frac{A_2}{1 \cdot 2} \phi''(x) + \&c. \right\},$

where  $A_n = \left( \frac{aE-1}{a-1} \right)^{-1} 0^n = \left( 1 + \frac{a\Delta}{a-1} \right)^{-1} 0^n$ .

PROOF.—By (3757),  $\psi(e^D) = \psi(E) e^{0 \cdot D}$ ; therefore (see last proof)  
 $a^x (aE-1)^{-1} \phi(x)$  (putting  $E = e^D$ )  $= a^x (aE-1)^{-1} e^{0 \cdot D} \phi(x)$   
 $= \frac{a^x}{a-1} \left( 1 + \frac{a\Delta}{a-1} \right)^{-1} \left\{ 1 + 0 \cdot D + \frac{0^2}{1 \cdot 2} D^2 + \&c. \right\} \phi(x)$ .

**3800** Ex.: To sum the series  $2.1 + 4.8 + 8.27 + 16.64 + \dots$  to  $n$  terms.

We require  $2^x x^3 + \Sigma 2^x x^3 = 2^x x^3 + 2^x (x^3 - 2\Delta x^3 + 2^2 \Delta^2 x^3 - 2^3 \Delta^3 x^3)$

$$= 2^x x^3 + 2^x \{ x^3 - 2(3x^2 + 3x + 1) + 4(6x + 6) - 8.6 \}$$

$$= 2^x \{ 2x^3 - 6x^2 + 18x - 26 \}.$$

**3801** If  $\Delta^{-n} u_x$  be known for all integral values of  $n$ , and if  $v_x$  be rational and integral,

$$\Sigma u_x v_x = \Sigma^1 u_x \cdot v_{x-1} - \Sigma^2 u_x \cdot \Delta v_{x-2} + \Sigma^3 u_x \cdot \Delta^2 v_{x-3} - \&c.$$

PROOF.  $\Sigma u_x v_x = (EE' - 1)^{-1} u_x v_x = (\Delta E' + \Delta')^{-1} u_x v_x$  (3746) and (3736). Expand the binomial operator, observing (3738).

**3802**

$$\Sigma^n u_x v_x = u_x \Sigma^n v_x - n \Delta u_x \Sigma^{n+1} v_{x+1} + C_{n,2} \Delta^2 u_x \Sigma^{n+2} v_{x+2} - \&c.$$

PROOF.  $\Sigma^n u_x v_x = (\Delta' + \Delta E')^{-n} u_x v_x$ , as in (3801),  
 $= \Delta^{-n} v_x u_x - n (\Delta^{-n-1} E) v_x \Delta u_x + C_{n,2} (\Delta^{-n-2} E^2) v_x \Delta^2 u_x - \&c.,$   
 producing the above by (3735) and (3782).

Observe that, in (3801) and (3802), two forms are obtainable in each case by expanding the binomial operator from either end of the series.

**3803** Ex.: To sum the series  $\sin a + 2^1 \sin 2a + 3^1 \sin 3a + \dots$  to  $x$  terms.

The sum is  $= x^3 \sin ax + \Sigma x^2 \sin ax$ . Taking  $u_x = \sin ax$  and  $v_x = x^2$ , we know  $\Delta^{-n} \sin ax$ , by (3729); therefore (3801) gives

$$\Sigma x^2 \sin ax = (2 \sin \tfrac{1}{2}a)^{-1} \sin \left\{ ax - \tfrac{1}{2}(a + \pi) \right\} (x-1)^2$$

$$- (2 \sin \tfrac{1}{2}a)^{-2} \sin \left\{ ax - (a + \pi) \right\} (2x-3) + (2 \sin \tfrac{1}{2}a)^{-3} \sin \left\{ ax - \tfrac{3}{2}(a + \pi) \right\} 2.$$

## APPROXIMATE SUMMATION.

**3820** The most useful formula is the following

$$\Sigma u_x = C + \int u_x dx + \frac{u_x}{2} + \frac{B_2}{2!} \frac{du_x}{dx} - \frac{B_4}{4!} \frac{d^3 u_x}{dx^3} + \&c.$$

$$= C + \int u_x dx + \frac{u_x}{2} + \frac{1}{12} \frac{du_x}{dx} - \frac{1}{720} \frac{d^3 u_x}{dx^3} + \frac{1}{30240} \frac{d^5 u_x}{dx^5}$$

$$- \frac{1}{1209600} \frac{d^7 u_x}{dx^7} + \frac{1}{47900160} \frac{d^9 u_x}{dx^9} - \&c.$$

PROOF.—  $\Sigma u_x = (e^D - 1)^{-1} u_x$ . Expand by (1539) with  $D$  in the place of  $x$ .

Ex. 1: The value of  $\Sigma x^p$  at (2939) is given at once by the formula.

**3821** Ex. 2: To sum the series  $1 + \frac{1}{2} + \frac{1}{3} \dots + \frac{1}{x}$  approximately,

$$\frac{1}{x} + \Sigma \frac{1}{x} = \frac{1}{x} + C + \log x - \frac{1}{2x} - \frac{1}{12x^3} + \frac{1}{120x^5} - \&c.$$

Put  $x=10$  to determine the constant; thus

$$1 + \frac{1}{2} + \frac{1}{3} \dots + \frac{1}{10} = C + \log 10 + \frac{1}{20} - \frac{1}{1200} + \&c.,$$

from which  $C = .577215$ , and the required sum is

$$.577215 + \log x + \frac{1}{2x} - \frac{1}{12x^3} + \frac{1}{120x^5} - \&c.$$

**3822** Ex. 3:  $1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \&c.,$

$$\Sigma \frac{1}{x^3} = C - \frac{1}{2x^3} - \frac{1}{2x^3} - \frac{3B_3}{2x^5} + \frac{5B_4}{2x^6} - \frac{7B_5}{2x^7} + \frac{9B_6}{2x^{10}},$$

$$\therefore \Sigma_1 \frac{1}{x^3} = \frac{1}{2} + \frac{1}{2} + \frac{1}{4} - \frac{1}{12} + \frac{1}{12} - \frac{3}{20} + \frac{5}{12}.$$

The convergent part of this series, consisting of the first five terms, is an approximation to the sum of all the terms.

**3823** A much nearer approximation is obtained in this and analogous cases by starting with the summation formula at a more advanced term.

$$\begin{aligned} \text{E.g.:} \quad & 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \Sigma_5 \frac{1}{x^3} \\ &= \frac{2035}{1728} + \frac{1}{2.5^3} + \frac{1}{2.5^3} + \frac{3B_3}{2.5^5} - \frac{5B_4}{2.5^6} + \&c. \\ &= \frac{2035}{1728} + \frac{1}{50} + \frac{1}{250} + \frac{1}{2500} - \frac{1}{187500} + \&c. \end{aligned}$$

The converging part now consists of a far greater number of terms than before, and the convergence at first is much more rapid.

**3824** Ex. 4: The series for  $\log \Gamma(x+1)$  at (2773) can be obtained by the above formula when  $x$  is an integer. For, in that case,

$$\log \Gamma(x+1) = \log 1 + \log 2 + \log 3 \dots + \log x = \log x + \Sigma \log x,$$

and (3820) gives the expansion in question, the constant being determined by making  $x$  infinite.

**3825** Formula (3820) may also be used to find  $\int u_x dx$  by the process of summation, and thus answers the purpose of Laplace's formula (3778).



$\Sigma^n u_x$  in a series of derivatives of  $u_x$ .

**3826** Lemma.—

$$n-1! (e^t-1)^{-n} = (-1)^{n-1} (d_t+n-1)(d_t+n-2) \dots (d_t+1) \{e^t-1\}^{-1}.$$

PROOF.—Put  $v_n$  for  $n-1! (e^t-1)^{-n}$ . Then

$$v_{n+1} = -(d_t+n) v_n = (d_t+n)(d_t+n-1) v_{n-1}.$$

$\Sigma^n u_x$  may now be developed.

**3827** Ex.—To develop  $\Sigma^3 u_x$ , (Boole, p. 97)

$$\begin{aligned} 2(e^t-1)^{-3} &= (d_t+2)(d_t+1) \{e^t-1\}^{-1} \\ &= (d_t+3d_t+2) \left\{ \frac{1}{t} - \frac{1}{2} + A_1 t + A_2 t^2 + \&c. \right\} \end{aligned}$$

with  $A_{2r} = 0$ , and

$$\begin{aligned} A_{2r+1} &= (-1)^r B_{2r+1} \div (2r+2)! = \frac{2}{t^3} - \frac{3}{t^2} + \frac{2}{t} + (2A_3+3A_1-1) \\ &\quad + \Sigma_1 \left\{ (r+2)(r+1) A_{r+2} + 3(r+1) A_{r+1} + 2A_r \right\} t^r. \end{aligned}$$

Therefore, changing  $t$  into  $d_x$ , we get

$$\Sigma^3 u_x = \iiint u_x dx - \frac{3}{2} \iint u_x dx + \int u_x dx - \frac{3}{8} u_x + \frac{19}{240} \frac{du_x}{dx} - \&c.$$

**3828**  $\Sigma^n u_x$  in a series of derivatives of  $u_{x-\frac{n}{2}}$ .

Let  $x^n \operatorname{cosec}^n x = 1 - C_2 x^2 + C_4 x^4 - \&c.$ , then

$$\Sigma^n u_x = D^{-n} \left\{ 1 + C_2 \left( \frac{D}{2} \right)^2 + C_4 \left( \frac{D}{2} \right)^4 + \&c. \right\} u_{x-\frac{n}{2}}.$$

[Boole, p. 98.]

**3829**

$$\phi(0) - \phi(1) + \phi(2) - \&c. = \frac{1}{2} \left\{ 1 - \frac{\Delta}{2} + \frac{\Delta^2}{4} - \&c. \right\} \phi(0).$$

By this formula, a series of the given type may often be transformed into one much more convergent.

$$\text{PROOF.—The left} = \frac{1}{1+E} \phi(0) = \frac{1}{2+\Delta} \phi(0) = \frac{1}{2} \frac{1}{1+\frac{1}{2}\Delta} \phi(0),$$

the expansion of which is the series on the right.

**3830** Ex.—To sum  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \&c.$  Summing the first six terms,

it becomes  $\frac{37}{60} + \frac{1}{7} - \frac{1}{8} + \&c.$  Taking  $\phi(0) = (0+7)^{-1}$ ,

$$\frac{1}{7} - \frac{1}{8} + \&c. = \frac{1}{2} \left\{ \frac{1}{7} + \frac{1}{2 \cdot 7 \cdot 8} + \frac{2}{4 \cdot 7 \cdot 8 \cdot 9} + \frac{2 \cdot 3}{8 \cdot 7 \cdot 8 \cdot 9 \cdot 10} + \&c. \right\}.$$

The sum after six terms converges rapidly by this formula, and more rapidly than if the formula had been applied to the series from its commencement.

# PLANE COORDINATE GEOMETRY.

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## SYSTEMS OF COORDINATES.

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### CARTESIAN COORDINATES.

**4001** In this system (Fig. 1)\* the position of a point  $P$  in a plane is determined by its distances from two fixed straight lines  $OX, OY$ , called axes of coordinates. These distances are measured parallel to the axes. They are the abscissa  $PM$  or  $ON$  denoted by  $x$ , and the ordinate  $PN$  denoted by  $y$ . The axes may be rectangular or oblique. The abscissa  $x$  is reckoned positive or negative according to the position of  $P$  to the right or left of the  $y$  axis, and the ordinate  $y$  is positive or negative according as  $P$  lies above or below the  $x$  axis conformably to the rules (607, '8).

**4002** These coordinates are called *rectangular* or *oblique* according as the axes of reference are or are not at right angles.

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### POLAR COORDINATES.

**4003** The polar coordinates of  $P$  (Fig. 1) are  $r$ , the radius vector, and  $\theta$ , the inclination of  $r$  to  $OX$ , the initial line, measured as in Plane Trigonometry (609).

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**4004** To change rectangular into polar coordinates, employ the equations  $x = r \cos \theta$ ,  $y = r \sin \theta$ .

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**4005** To change polar into rectangular coordinates, employ

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

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\* See the end of the volume.

## TRILINEAR COORDINATES.

**4006** The trilinear coordinates of a point  $P$  (Fig. 2) are  $\alpha, \beta, \gamma$ , its perpendicular distances from three fixed lines which form the *triangle of reference*,  $ABC$ , hereafter called the *trigon*. These coordinates are always connected by the relation

$$\mathbf{4007} \quad \alpha\alpha + \beta\beta + \gamma\gamma = \Sigma,$$

$$\mathbf{4008} \quad \text{or} \quad \alpha \sin A + \beta \sin B + \gamma \sin C = \text{constant},$$

where  $\alpha, \beta, \gamma$  are the sides of the trigon, and  $\Sigma$  is twice its area.

**4009** If  $x, y$  are the Cartesian coordinates of the point  $\alpha\beta\gamma$ , the equations connecting them with the trilinear coordinates are, by (4094),

$$\begin{aligned} \alpha &= x \cos \alpha + y \sin \alpha - p_1, \\ \beta &= x \cos \beta + y \sin \beta - p_2, \\ \gamma &= x \cos \gamma + y \sin \gamma - p_3. \end{aligned}$$

**4010** Here  $\alpha$  has two significations. On the left, it is the length of the perpendicular from the point in question upon the side  $AB$  of the trigon. On the right, it is the inclination of that perpendicular to the  $x$  axis of Cartesian coordinates. Similarly  $\beta$  and  $\gamma$ .

**4011** The angles  $\alpha, \beta, \gamma$  are connected with the angles  $A, B, C$  by the equations

$$\gamma - \beta = \pi - A, \quad \alpha - \gamma = \pi - B, \quad \alpha - \beta = \pi + C,$$

only two of which are independent.

**4012**  $p_1, p_2, p_3$  are the perpendiculars from the origin upon the sides of the triangle  $ABC$ .

## AREAL COORDINATES.

If  $A, B, C$  (Fig. 2) be the trigon as before, the areal coordinates  $\alpha', \beta', \gamma'$  of the point  $P$  are

$$\mathbf{4013} \quad \alpha' = \frac{\alpha}{p_1} = \frac{PBC}{ABC}, \quad \beta' = \frac{\beta}{p_2} = \frac{PCA}{ABC}, \quad \gamma' = \frac{\gamma}{p_3} = \frac{PAB}{ABC}.$$

The equation connecting the coordinates is now

$$4014 \quad \alpha' + \beta' + \gamma' = 1.$$

**4015** To convert any homogeneous trilinear equation into the corresponding areal equation.

**4016** Substitute  $\alpha\alpha' = \Sigma\alpha'$ ,  $\beta\beta' = \Sigma\beta'$ ,  $\gamma\gamma' = \Sigma\gamma'$ .

Also any relation between the coefficients  $l, m, n$  in the equation of a right line in trilinears will be adapted to areals by substituting  $la, mb, nc$  for  $l, m, n$ . Similarly for  $a, b, c, f, g, h$ , in the general equation of a conic (4656), substitute  $a\alpha^2, bb^2, cc^2, fbc, gca, hab$ .

In either the trilinear or areal systems, a point is determined if the ratios only of the coordinates are known.

Thus, if  $\alpha : \beta : \gamma = P : Q : R$ , then, with trilinear coordinates,

$$4017 \quad \alpha = \frac{P\Sigma}{aP+bQ+cR}; \text{ and, with areal, } \alpha = \frac{P}{P+Q+R}.$$

#### TANGENTIAL COORDINATES.

**4019** In this system the position of a straight line is determined by coordinates, and the position of a point by an equation. If  $la + m\beta + n\gamma = 0$  be the trilinear equation of a straight line  $EDF'$  (Fig. 3); then, making  $\alpha, \beta, \gamma$  constant, and  $l, m, n$  variable, the equation becomes the tangential equation of the point  $O(\alpha, \beta, \gamma)$ ; whilst  $l, m, n$  are the coordinates of some right line passing through that point.

Let  $\lambda, \mu, \nu$  (Fig. 3) be the perpendiculars from  $A, B, C$  upon  $EDF'$ , and let  $p_1, p_2, p_3$  be the perpendiculars from  $A, B, C$  upon the opposite sides of the trigon; then, by (4624), we have

$$4020 \quad R\lambda = lp_1, \quad R\mu = mp_2, \quad R\nu = np_3,$$

where  $R = \sqrt{l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C}$ .

Hence the equation of the point  $O$  becomes

$$4021 \quad \lambda \frac{\alpha}{p_1} + \mu \frac{\beta}{p_2} + \nu \frac{\gamma}{p_3} = 0 \quad \text{or} \quad \lambda \frac{\sin \theta_1}{\rho_1} + \mu \frac{\sin \theta_2}{\rho_2} + \nu \frac{\sin \theta_3}{\rho_3} = 0,$$

where  $\rho_1 = OA$ ,  $\theta_1 = \angle BOC$ , &c., and  $2\Delta BOC = \rho_2 \rho_3 \sin \theta$ .

Formula (4021) shows that, when the perpendiculars  $\lambda, \mu, \nu$  are taken for the coordinates of the line, the coefficients become the areal coordinates of the point referred to the same trigon.

**4023** Any homogeneous equation in  $l, m, n$  as tangential coordinates is expressed in terms of  $\lambda, \mu, \nu$  by substituting for  $l, m, n, \frac{\lambda}{p_1}, \frac{\mu}{p_2}, \frac{\nu}{p_3}$  respectively. By (4020).

**4024** An equation in  $\lambda, \mu, \nu$  of a degree higher than the first represents a curve such that  $\lambda, \mu, \nu$  are always the perpendiculars upon the tangent. The curve must therefore be the envelope of the line  $(\lambda, \mu, \nu)$ .

#### TWO-POINT INTERCEPT COORDINATES.

Let  $\lambda = AD, \mu = BE$  (Fig. 4) be variable distances from two fixed points  $A, B$  measured along two fixed parallel lines, then

**4025** 
$$a\lambda + b\mu + c = 0$$

is the equation of a fixed point  $O$  through which the line  $DE$  always passes. This may easily be proved directly, but we shall show that it is a particular case of the system of three-point tangential coordinates.

Let one of the vertices ( $C$ ) of the trigon in that system be at infinity (Fig. 3). Then equation (4022) becomes

$$\frac{\lambda \sin \theta_1}{\rho_1} + \frac{\mu \sin \theta_2}{\rho_2} + \sin COE \sin \theta_3 = 0.$$

For  $\nu : \rho_3 = \sin COE$  always. Divide by  $\sin COE$ ; then  $\lambda \div \sin COE = AD$ , &c., and the equation becomes

$$\frac{\sin \theta_1}{\rho_1} AD + \frac{\sin \theta_2}{\rho_2} AE + \sin \theta_3 = 0.$$

The only variables are  $AD$  and  $AE$ . Calling these  $\lambda$  and  $\mu$ , the equation may be written

$$a\lambda + b\mu + c = 0,$$

the form taken by  $a\lambda + b\mu + c'\nu = 0$  when  $\nu = \infty$  and  $c'$  vanishes.

#### ONE-POINT INTERCEPT COORDINATES.

**4026** Let  $a, b$  be the Cartesian coordinates of the point  $O$  (Fig. 5); and let the reciprocals of the intercepts on the axes

of any line  $DOE$  passing through  $O$  be  $\xi = \frac{1}{AE}$ ,  $\eta = \frac{1}{AD}$ .  
Then, by (4053),

$$4027 \quad a\xi + b\eta = 1$$

is the equation of the point  $O$ , the variables being  $\xi, \eta$ .

This is a case of the system of three-point tangential coordinates in which two of the vertices ( $B, C$ ) of the trigon are at infinity. Equation (4022)

now becomes  $\frac{\lambda \sin \theta_1}{\rho_1} + \sin BOD \sin \theta_2 + \sin COE \sin \theta_3 = 0$ ,

or  $\frac{\sin \theta_1}{\rho_1} + \frac{\sin \theta_2}{AD} + \frac{\sin \theta_3}{AE} = 0$ ,

which is of the form  $a\xi + b\eta = 1$ .

#### TANGENTIAL RECTANGULAR COORDINATES.

**4028** This name has been given to the system last described when the two fixed lines are at right angles (Fig. 6).

The coordinates  $\xi, \eta$ , which are defined as the reciprocals of the intercepts of the line they determine, have now also the following values.

**4029** Let  $x, y$  be the rectangular coordinates of the pole of the line in question with respect to a circle whose centre is the origin and whose radius is  $k$ ; then

$$\xi = \frac{x}{k^2} \quad \text{and} \quad \eta = \frac{y}{k^2},$$

since  $x.OM = y.ON = k^2$ ; for  $M, N$  are the poles of  $y = 0$ ,  $x = 0$ .

**4030** The equation of a point  $P$  on  $NM$  whose rectangular coordinates are  $OR = a$ ,  $OS = b$ , is

$$a\xi + b\eta = 1, \text{ by (4053),}$$

this equation being satisfied by the coordinates of all lines passing through that point.

**4031** In all these systems an equation of a higher degree in  $\xi, \eta$  represents a curve the coordinates of whose tangents satisfy the equation.



# ANALYTICAL CONICS

## IN

## CARTESIAN COORDINATES.

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### LENGTHS AND AREAS.

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Coordinates of the point dividing in the ratio  $n : n'$  the right line which joins the two points  $xy, x'y'$ .

**4032** 
$$\xi = \frac{nx' + n'x}{n + n'}, \quad \eta = \frac{ny' + n'y}{n + n'}.$$

PROOF.—(Fig. 7.)  $\xi = x + AC = x + \frac{n}{n+n'}(x'-x)$ . Similarly for  $\eta$ .

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**4033** If  $n = n'$ , 
$$\xi = \frac{x+x'}{2}, \quad \eta = \frac{y+y'}{2}.$$

---

**4034** Length of the line joining the points  $xy, x'y'$

$$= \sqrt{(x-x')^2 + (y-y')^2}.$$

The same with oblique axes

**4035** 
$$\sqrt{(x-x')^2 + (y-y')^2 + 2(x-x')(y-y') \cos \omega}.$$

PROOF.—By (Fig. 7), Euc. I. 47, and (702).

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Area  $A$  of a triangle in terms of the coordinates of its angular points  $x_1y_1, x_2y_2, x_3y_3$ .

**4036** 
$$A = \frac{1}{2} \{x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3\}.$$

PROOF.—(Fig. 8.) By considering the three trapezoids formed by  $y_1, y_2, y_3$  and the sides of the triangle, we have

$$A = \frac{1}{2} (y_1 + y_2)(x_2 - x_1) + \frac{1}{2} (y_2 + y_3)(x_3 - x_2) - \frac{1}{2} (y_3 + y_1)(x_3 - x_1).$$

Area of the triangle contained by the  $x$  axis and the lines

$$y = m_1x + c_1, \quad y = m_2x + c_2, \quad (4052)$$

$$4037 \quad A = \frac{(c_1 - c_2)^2}{2(m_1 - m_2)} = \frac{(B_1C_2 - B_2C_1)^2}{2B_1B_2(A_1B_2 - A_2B_1)}. \quad (4056)$$

PROOF.—(Fig. 9.) Area =  $\frac{1}{2}(c_1 - c_2)p$ , and  $p$  is found from  
 $pm_2 - pm_1 = c_1 - c_2$ . The sign of the area is not regarded.

COR.—Area of the triangle contained by the lines

$$y = m_1x + c_1, \quad y = m_2x + c_2, \quad y = m_3x + c_3,$$

$$4038 \quad A = \frac{1}{2} \left\{ \frac{(c_1 - c_2)^2}{m_1 - m_2} + \frac{(c_2 - c_3)^2}{m_2 - m_3} + \frac{(c_3 - c_1)^2}{m_3 - m_1} \right\}.$$

$$4039 \quad = \frac{\{c_1(m_2 - m_3) + c_2(m_3 - m_1) + c_3(m_1 - m_2)\}^2}{2(m_1 - m_2)(m_2 - m_3)(m_3 - m_1)}.$$

$$4040 \quad = \frac{(B_1C_2 - B_2C_1)^2}{2B_1B_2(A_1B_2 - A_2B_1)} + \&c.$$

$$4041 \quad = \frac{\text{Square of Determinant } (A_1B_2C_3)}{2(A_1B_2 - A_2B_3)(A_2B_3 - A_3B_1)(A_3B_1 - A_1B_3)}.$$

PROOF.—(Fig. 10.)  $ABC = AEF + CDE - BED$ . Employ (4037).  
F

### Area of Polygon of $n$ sides.

First in terms of the coordinates of the angular points  
 $x_1y_1, x_2y_2, \dots, x_ny_n$ .

$$4042 \quad 2A = (x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \dots + (x_ny_1 - x_1y_n) \\ = x_1(y_2 - y_n) + x_2(y_3 - y_1) + \dots + x_n(y_1 - y_{n-1}).$$

Secondly, when the equations to the sides are given, as in (4037).

$$4043 \quad 2A = \frac{(c_1 - c_2)^2}{m_1 - m_2} + \frac{(c_2 - c_3)^2}{m_2 - m_3} + \dots + \frac{(c_n - c_1)^2}{m_n - m_1}.$$

$$4044 \quad \text{Also three values similar to (4039, '40, '41).}$$

PROOF.—By (4367), adding the component triangles.

4047 Each expression for the area of a triangle or polygon will be adapted to oblique axes by multiplying by  $\sin \omega$ .

## TRANSFORMATION OF COORDINATES.

**4048** To transform the origin to the point  $hk$ .

Put  $x = x' + h, \quad y = y' + k.$

To transform to rectangular axes inclined at an angle  $\theta$  to the original axes.

**4049** Put

$$x = x' \cos \theta - y' \sin \theta, \quad y = y' \cos \theta + x' \sin \theta. \quad (\text{Fig. 11.})$$

PROOF.—Consider  $x'$  as  $\cos \phi$  and  $y'$  as  $\sin \phi$ . Then  $x = \cos(\phi + \theta)$  and  $y = \sin(\phi + \theta)$  (627, '9).

Generally (Fig. 12), let  $\omega$  be the angle between the original axes; and let the new axes of  $x$  and  $y$  make angles  $\alpha$  and  $\beta$  respectively with the old axis of  $x$ .

**4050** Put  $x \sin \omega = x' \sin(\omega - \alpha) + y' \sin(\omega - \beta)$

and  $y \sin \omega = x' \sin \alpha + y' \sin \beta.$

PROOF.—(Fig. 12.) The coordinates of  $P$  referred to the old axes being  $OC = x$ ,  $PC = y$ , and referred to the new axes,  $OM = x'$ ,  $PM = y'$ , we have, by projecting  $OCP$  and  $OMP$  at right angles first to  $CP$  and then to  $OC$ ,

$$CD = MF - ME, \quad PN = ML + PK,$$

which are equivalent to the above equations.

To change Rectangular coordinates into Polar,  $hk$  being the pole  $O$ ,  $\alpha$  the inclination of the initial line to the  $x$  axis (Fig. 13), and  $xy$  the point  $P$ .

**4051** Put  $x = h + r \cos(\theta + \alpha), \quad y = k + r \sin(\theta + \alpha).$

## THE RIGHT LINE.

## EQUATIONS OF THE RIGHT LINE.

**4052**  $y = mx + c \dots \dots \dots (1),$

**4053**  $\frac{x}{a} + \frac{y}{b} = 1 \dots \dots \dots (2),$

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4054  $x \cos \alpha + y \sin \alpha = p \dots\dots\dots (3).$

4055  $Ax + By + C = 0 \dots\dots\dots (4).$

PROOF.—(Fig. 14.) Let  $AB$  be the line. Take any point  $P$  upon it, coordinates  $ON = x$ ,  $PN = y$ . Then, in (1);  $m = \tan \theta$ , where  $\theta = BAX$ , the inclination to the  $X$  axis; therefore  $mx = -OC$ , and  $c$  is the intercept  $OB$ . In (2),  $a, b$  are the intercepts  $OA, OB$ . In (3),  $p = OS$ , the perpendicular from  $O$  upon the line;  $\alpha = \angle AOS$ .

$$p = OR + LP = x \cos \alpha + y \sin \alpha.$$

(4) is the general equation.

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4056  $m = \tan \theta = -\frac{A}{B} = -\frac{b}{a} = -\cot \alpha.$

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4060  $\sin \theta = \frac{A}{\sqrt{A^2 + B^2}}, \quad \cos \theta = -\frac{B}{\sqrt{A^2 + B^2}}.$

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4062  $p = c \sin \alpha = \frac{c}{\sqrt{1+m^2}} = -\frac{C}{\sqrt{A^2 + B^2}}.$

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#### *Oblique Axes.*

Equations (4052, '53, '55) hold for oblique axes, but (4054) must be written

4065  $x \cos \alpha + y \cos \beta = p. \quad (\text{Fig. 14})$

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4066  $\tan \theta = \frac{m \sin \omega}{1 + m \cos \omega} = \frac{A \sin \omega}{A \cos \omega - B},$

$\omega$  being the angle between the axes.

PROOF.—From  $m = \sin \theta \div \sin (\omega - \theta).$

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4068  $p = \frac{c \sin \omega}{\sqrt{1 + 2m \cos \omega + m^2}} = \frac{C \sin \omega}{\sqrt{A^2 + B^2 - 2AB \cos \omega}}.$

PROOF.—From  $p = c \sin (\omega - \theta)$  and (4066).

---

The equations of two lines being given in the forms (4052) or (4055), the angle,  $\phi$ , between them is given by

4070  $\tan \phi = \frac{m - m'}{1 + mm'} \quad \text{or} \quad \frac{AB' - A'B}{AA' + BB'}.$

PROOF.—(Fig. 15)  $\tan \phi = \tan (\theta - \theta').$  Expand by (632).

To oblique axes :

$$4072 \quad \tan \phi = \frac{(m-m') \sin \omega}{1+(m+m') \cos \omega + mm'}.$$

PROOF.—As in the last, employing (4066).

Equation of a line passing through  $x'y'$ :

$$4073 \quad y-y' = m(x-x'), \quad (\text{Fig. 8})$$

$$4074 \quad \text{or} \quad y-mx = y'-mx',$$

$$4075 \quad \text{or} \quad Ax+By = Ax'+By'.$$

PROOF.—From Figure (13),  $m$  being  $= \tan \theta$ .

Condition of parallelism of two lines :

$$4076 \quad m = m', \quad \text{or} \quad AB' = A'B.$$

Hence the equations differ by a constant.

Condition of perpendicularity :

$$4078 \quad mm' = -1 \quad \text{or} \quad AA' + BB' = 0. \quad (4070)$$

The same to oblique axes :

$$4080 \quad 1+(m+m') \cos \omega + mm' = 0. \quad (4072)$$

$$4081 \quad \text{or} \quad AA' + BB' = (AB' + A'B) \cos \omega,$$

$$4082 \quad \text{or} \quad m' = -\frac{1+m \cos \omega}{m + \cos \omega}.$$

A line passing through the points  $x_1 y_1, x_2 y_2$  :

$$4083 \quad \frac{y-y_1}{x-x_1} = \frac{y_1-y_2}{x_1-x_2} = m.$$

PROOF.—(Fig. 16.) By the similar right-angled triangles  $PCA, ADB$ .

$$4084 \quad \text{Or} \quad y = mx + \frac{x_1 y_2 - x_2 y_1}{x_1 - x_2},$$

$$4085 \quad \text{or} \quad (x-x_1)(y-y_2) = (x-x_2)(y-y_1).$$

PROOF.—This equation represents a straight line because it is of the first degree; and the coordinates of each of the given points satisfy the equation.

A line passing through  $x'y'$  and perpendicular to a given line ( $m$ ):

$$4086 \quad y - y' = -\frac{1}{m}(x - x'). \quad (4073, '78)$$

$$4087 \quad \text{or} \quad Bx - Ay = Bx' - Ay'.$$

The two lines passing through  $x'y'$  and making an angle  $\beta (= \tan^{-1} m_2)$  with a given line ( $m_1$ ):

$$4088 \quad \frac{y - y'}{x - x'} = \frac{m_1 - m_2}{1 + m_1 m_2} \quad \text{and} \quad \frac{m_1 + m_2}{1 - m_1 m_2}. \quad (4073, '70)$$

A line passing through  $hk$  and dividing the line which joins  $x_1 y_1$  and  $x_2 y_2$  in the ratio  $n_1 : n_2$ :

$$4089 \quad \frac{y - k}{x - h} = \frac{n_1 (y_2 - k) + n_2 (y_1 - k)}{n_1 (x_2 - h) + n_2 (x_1 - h)}. \quad (4073, '32)$$

Coordinates of the point of intersection of two lines:

$$4090 \quad x = \frac{c_1 - c_2}{m_2 - m_1} = \frac{B_1 C_2 - B_2 C_1}{A_1 B_2 - A_2 B_1}.$$

$$4092 \quad y = \frac{c_1 m_2 - c_2 m_1}{m_2 - m_1} = -\frac{A_1 C_2 - A_2 C_1}{A_1 B_2 - A_2 B_1}. \quad (4116)$$

Length of the perpendicular from a point  $x'y'$  upon a given line

$$4094 \quad = x' \cos a + y' \sin a - p.$$

PROOF.—Let  $AB$  (Fig. 14) be the line, and  $Q$  the point  $x'y'$ . Then, by (4054),  $x' \cos a + y' \sin a = OT$ , the perpendicular from  $O$  upon a parallel line through  $Q$ , and  $p = OS$ .

Otherwise, the same perpendicular

$$4095 \quad = \frac{Ax' + By' + C}{\sqrt{A^2 + B^2}}. \quad (4060, '61, '94)$$

The same with oblique axes

$$4096 \quad = \frac{(Ax' + By' + C) \sin \omega}{\sqrt{(A^2 + B^2 - 2AB \cos \omega)}},$$

obtained in a similar way from (4065–69).



Condition of three lines intersecting in one point :

$$4097 \quad c_1 m_2 - c_2 m_1 + c_2 m_3 - c_3 m_2 + c_3 m_1 - c_1 m_3 = 0.$$

The area in 1009 must vanish.

**4098** *Otherwise.*—If certain values of the constants  $l, m, n$  make the expression

$$l(A_1x + B_1y + C_1) + m(A_2x + B_2y + C_2) + n(A_3x + B_3y + C_3)$$

vanish identically, the three lines indicated intersect in one point.

PROOF.—By (4099), for then values of  $x$  and  $y$  which make (1) and (2) vanish also make (3) vanish.

A line passing through the point of intersection of the lines  $Ax + By + C = 0$  and  $A'x + B'y + C' = 0$  is

$$4099 \quad Ax + By + C = k(A'x + B'y + C'),$$

$$4100 \quad \text{or } l(Ax + By + C) - m(A'x + B'y + C') = 0,$$

$k, l$ , and  $m$  being any constants.

**4101** RULE.—If the equation of a right line contains a third variable  $k$  in the first degree, the line always passes through a fixed point.

PROOF.—For the values of  $x$  and  $y$ , which satisfy simultaneously the given equations, also satisfy (4099), whatever  $k$  may be. See (4604).

**4102** If in the equation of a line  $Ax + By + C = 0$ , the coefficients  $A, B, C$  involve  $x', y'$ , the coordinates of a point which moves along a fixed right line, then the first line passes through some fixed point.

PROOF.—By means of the equation of the fixed line,  $y'$  may be eliminated, and  $x'$  then remains a third variable in the first degree (4101).

**4103** To find the point in which the line  $Ax + By + C$  intersects the line joining the points  $xy, x'y'$ ; substitute

$$Ax + By + C \text{ for } n, \text{ and } Ax' + By' + C \text{ for } n' \text{ in (4032).}$$

PROOF.—By (4095), since the segments intercepted are in the ratio of the perpendiculars from  $xy, x'y'$  upon the line  $Ax + By + C$ .

Equations of the line with  $l, m$  for direction-ratios,  $hk$  a fixed point on the line, and  $r$  the distance of the variable point  $xy$  from  $hk$ .

$$4104 \quad r = \frac{x-h}{l} = \frac{y-k}{m},$$

$$4105 \quad \text{where } l = \frac{\sin(\omega - \theta)}{\sin \omega}, \quad m = \frac{\sin \theta}{\sin \omega}, \quad (\text{Oblique})$$

$$4106 \quad \text{or } l = \cos \theta, \quad m = \sin \theta. \quad (\text{Rectangular})$$

*Polar Equation of a Straight Line.*

$$4107 \quad r \cos(\theta - \alpha) = p.$$

(Fig. 17.) Here  $p$  is the perpendicular to the line from the pole  $O$ , and  $\alpha$  is the inclination of  $p$  to the initial line  $OA$ .

When the line passes through the pole, the equation is

$$4108 \quad \theta = \text{constant}.$$

A line passing through the two points  $r_1\theta_1, r_2\theta_2$ .

$$4109 \quad rr_1 \sin(\theta - \theta_1) + r_1r_2 \sin(\theta_1 - \theta_2) + r_2r \sin(\theta_2 - \theta) = 0.$$

PROOF.—(Fig. 18.)  $\Delta POA + AOB - POB = 0$ . Then by (707).

EQUATIONS OF TWO OR MORE RIGHT LINES.

The homogeneous equation of the  $n^{\text{th}}$  degree,

$$4110 \quad x^n + p_1x^{n-1}y + p_2x^{n-2}y^2 + \dots + p_ny^n = 0,$$

represents  $n$  right lines, real or imaginary, passing through the origin.

For it is resolvable into  $n$  factors of the form  $(x - ay)$ , by (405).

For the case of two right lines represented by the general equation of the second degree, see (4469).

Equation of two right lines through the origin :

$$4111 \quad ax^2 + 2hxy + by^2 = 0.$$

If  $\phi$  be the angle between the lines,

$$4112 \quad \tan \phi = \frac{\sqrt{(h^2 - ab)}}{a + b} \quad \text{or} \quad \frac{2 \sin \omega \sqrt{(h^2 - ab)}}{a + b - 2h \cos \omega},$$

according as the axes are rectangular or oblique.

PROOF.—Assume  $(y - m_1 x)(y - m_2 x) = 0$ , and apply (4088).

Equation of the bisectors of the angle  $\phi$  :

$$4113 \quad hx^2 - (a - b)xy - hy^2 = 0.$$

PROOF.—Let  $y = \mu x$  be a bisector ( $\mu = \tan \psi$ ); then, since  $2\psi = \theta_1 + \theta_2$ ,

$$\frac{2\mu}{1 - \mu^2} = \frac{m_1 + m_2}{1 - m_1 m_2} = \frac{2h}{a - b}, \text{ by (4111); and } \mu = \frac{y}{x}.$$

The roots of this equation are always real.

## GENERAL METHODS.

APPLICABLE TO ALL EQUATIONS OF PLANE CURVES.

4114 Let  $F(x, y) = 0 \dots (i.)$  and  $f(x, y) = 0 \dots (ii.)$  be the equations of two curves of any degree.

4115 To find the intercepts on the  $x$  and  $y$  axes.

Put  $y = 0$  in (i.), then  $x$  becomes the intercept on the  $x$  axis. Similarly, put  $x = 0$  for the intercept on the  $y$  axis.

4116 To find the points of intersection of (i.) and (ii.).

Solve as simultaneous equations. Each pair of values of  $x$  and  $y$  so obtained gives a point of intersection. Imaginary values give an imaginary point.

4117 To determine equation (i.) so that the line may pass through certain fixed points,  $x_1 y_1$ ,  $x_2 y_2$ , &c.

Substitute  $x_1 y_1$ ,  $x_2 y_2$ , &c. for  $xy$  successively, so forming as many equations as there are points. From these equations the constants in (i.) must be determined in terms of  $x_1, y_1$ ,  $x_2, y_2$ , &c.

4118 The number of arbitrary points cannot exceed the number of constants in the equation.

**4119** Condition that (i.) and (ii.) may touch.

*At a point of contact two or more points of intersection must coincide, and therefore the equation for  $x$  or  $y$ , obtained as in (4116), must have two or more equal roots for each point of contact. The contact is said to be of the second order when there are three coincident points; of the third order when there are four, and so on.*

**4120** To find the equation of the tangent at a point  $x'y'$  on the curve  $f(x, y) = 0$ .

*Form the equation to the secant through two adjacent points  $x_1y_1, x_2y_2$  (4083), and determine the limiting value of  $\frac{y_1 - y_2}{x_1 - x_2}$  when the points coincide by means of the equations  $f(x_1, y_1) = 0$ ,  $f(x_2, y_2) = 0$ .*

**4121** Otherwise  $m = \frac{dy}{dx}$ , by (5101).

**4122** For the equation of the normal, change  $m$  of the tangent into  $-\frac{1}{m}$  (4086).

**4123** To express the equation of the tangent, or normal, in terms of  $m$  and the constants of the curve.

*From the equation of the tangent or normal, the equation to the curve, and the equation furnished by the value of  $m$ , eliminate  $x' y'$ , the coordinates of the point of contact of the tangent.*

#### THEORY OF POLES AND POLARS.

**4124** Let  $F(x', y', x, y) = 0$  represent the equation to the tangent of a curve at the point  $x'y'$ .

Then  $F(x, y, x', y') = 0$ , the equation obtained by interchanging the constants  $x', y'$  with the variables  $x, y$ , represents the polar of any fixed point  $x'y'$  not on the curve.

Let  $x_1y_1, x_2y_2$  (Fig. 19) be points  $A, B$  on the curve, and let the tangents at those points intersect in  $x'y'$ . Consider the equations

$$F(x_1, y_1, x, y) = 0 \dots (1), \quad F(x_2, y_2, x, y) = 0 \dots (2), \quad F(x, y, x', y') = 0 \dots (3).$$

Here (1), (2) are the tangents, and (3) is some straight line or curve according to the dimensions of  $x$  and  $y$ . Also (3) passes through the points

of contact  $x_1y_1$ ,  $x_2y_2$ , and may therefore be called the *curve of contact*; or, if a right line, the *chord of contact* of tangents drawn from  $x'y'$ , i.e., the polar.

**4125** Hence the coordinates of the points of contact of tangents from an external point  $x'y'$  will be determined by solving (3) and the equation of the curve simultaneously.

**4126** Again, let  $x'y'$  (Figs. 20 and 21) be any point  $P$  not on the curve. Then, from the equations

$$F(x', y', x, y) = 0 \dots (4), \quad F(x, y, x_3, y_3) = 0 \dots (5), \quad F(x, y, x_4, y_4) = 0 \dots (6),$$

we see that (4) is some straight line; that, if  $x_3y_3$  and  $x_4y_4$  are any two points upon it, (5) and (6) are the curves of contact of tangents from those points; and that these curves of contact pass through the point  $x'y'$ .

**4127** If the points  $x_3y_3$ ,  $x_4y_4$  are taken at  $A$ ,  $B$ , where (4) intersects the curve, (5) and (6) then become curves touching the given curve at  $A$  and  $B$ , and passing through  $x'y'$ . We may call these lines the *curve tangents* from  $x'y'$ .

**4128** Lastly, let  $x'y'$  in (3) be a point within the given curve (Fig. 22), then the equations

$$F(x, y, x', y') = 0 \dots (7), \quad F(x_3, y_3, x, y) = 0 \dots (8), \quad F(x_4, y_4, x, y) = 0 \dots (9)$$

show that (7) is the locus of a point, the curve tangents from which have their chord of contact always passing through a fixed point. When  $x'y'$  is without the curve, as in Fig. (19), the same definition applies to every part of the locus (3) from which tangents can be drawn.

**4129** If the given curve be of a degree higher than the second, the line of contact of the tangents from a point is a curve, and the line of contact of the curve tangents from a point is a straight line (Figs. 19 and 20). A similar converse relation is exhibited in Figures (21) and (22).

If the curve be of the second degree, equations (3) and (4) become identical. The line of contact or the polar is always in this case a straight line, and so is the locus (7).

Figures (19) and (20) now become identical, as also (21) and (22).

**4130** The polar of the point of intersection of two right lines with regard to a conic passes through their poles.

PROOF.—As in (4124). Let (1) and (2) be the two lines,  $(x_1y_1)$ ,  $(x_2y_2)$  their poles, and  $x'y'$  their point of intersection.

**4131** To find the ratio in which the line joining two given points  $xy, x'y'$  is cut by the curve  $f(x, y) = 0$ .

*Substitute for  $x$  and  $y$ , the supposed coordinates of the point of intersection, the values*

$$\frac{nx' + n'x}{n + n'}, \quad \frac{ny' + n'y}{n + n'}, \quad \text{by (4032)}$$

*and determine the ratio  $n : n'$  from the resulting equation. The real roots of this equation correspond to the real points of intersection.*

**4132** To form the equation of all the tangents that can be drawn to the curve from a point  $x'y'$ .

*Express the condition for equal roots of the equation in (4131), and consider  $xy$  a variable point.*

**4133** To form the equation of the lines drawn from  $x'y'$  to all the points of intersection of two curves.

*Substitute  $nx' + n'x, ny' + n'y$  for  $x$  and  $y$  in both curves, and eliminate the ratio  $n : n'$ .*

**PROOF.**—Take any other point  $xy$  on the line through  $x'y'$  and a point of intersection. The ratio  $n : n'$  (4131) is the same for each curve, and therefore may be eliminated.

**4134** To find the length,  $r = AP$  or  $AP'$  (Fig. 23), of the segment intercepted between the point  $A$  or  $x'y'$  and the curve  $f(x, y) = 0$  on a straight line drawn from  $A$  at an inclination  $\theta$  to the  $X$  axis. That is, to form the polar equation with  $x'y'$  for the pole and the initial line parallel to the  $x$  axis.

*Substitute for  $x$  and  $y$ , the assumed coordinates of the point of intersection, the values  $x = ON$  or  $ON'$ ,  $y = PN$  or  $P'N'$ , that is,*

$$x = x' + r \cos \theta, \quad y = y' + r \sin \theta,$$

*and determine  $r$  from the resulting equation. That is, put  $a = 0$  in (4051).*

*The real values of  $r$  are the distances of the points of intersection from  $x'y'$ .*

**4135** When an equation has been obtained for determining  $x$  the length of a line, important results may frequently be arrived at by applying theorem (406) respecting the sum and product of the roots.



## THE CIRCLE.

Equation with the centre for origin.

$$4136 \quad x^2 + y^2 = r^2. \quad (\text{Fig. 24.})$$

Equations of the tangent at the point  $P$  or  $x'y'$ .

$$4137 \quad y - y' = -\frac{x'}{y'}(x - x'). \quad (4120)$$

$$4138 \quad xx' + yy' = r^2.$$

Also, by (4124), the polar of  $x'y'$ , any point not on the curve.

$$4139 \quad y = mx + r\sqrt{1+m^2}; \quad m = -\frac{x'}{y'}. \quad (4123)$$

$$4140 \quad x \cos \alpha + y \sin \alpha = r,$$

 $\alpha$  being the inclination to the  $x$  axis of the radius to the point  $x'y'$ .Equation of the circle with  $a, b$  for the coordinates of the centre  $Q$ . (Fig. 24.)

$$4141 \quad (x-a)^2 + (y-b)^2 = r^2.$$

Tangent at  $x'y'$ , or Polar,

$$4142 \quad (x-a)(x'-a) + (y-b)(y'-b) = r^2, \quad (4138)$$

$$4143 \quad \text{or} \quad (x-a) \cos \alpha + (y-b) \sin \alpha = r,$$

 $\alpha$  being the inclination of the radius to the point  $x'y'$ .

General equation of the circle :

$$4144 \quad x^2 + y^2 + 2gx + 2fy + c = 0.$$

$$4145 \quad \text{Centre } (-g, -f). \quad \text{Radius } \sqrt{g^2 + f^2 - c}.$$

PROOF.—By equating coefficients with (4141).

Equation of the circle with oblique axes : (Fig. 25.)

$$4146 \quad (x-a)^2 + (y-b)^2 + 2(x-a)(y-b) \cos \omega = r^2, \quad (702)$$

$$4147 \quad \text{or} \quad x^2 + 2xy \cos \omega + y^2 - 2(a + b \cos \omega)x - 2(b + a \cos \omega)y + a^2 + 2ab \cos \omega + b^2 = r^2.$$

*General Equation.*

$$4148 \quad x^2 + 2xy \cos \omega + y^2 + 2gx + 2fy + c = 0.$$

The coordinates of the centre are

$$4149 \quad a = \frac{f \cos \omega - g}{\sin^2 \omega}, \quad b = \frac{g \cos \omega - f}{\sin^2 \omega}.$$

$$4150 \quad \text{Radius} = \frac{\sqrt{\{g^2 - 2fg \cos \omega + f^2 - c \sin^2 \omega\}}}{\sin \omega}.$$

PROOF.—By equating coefficients with (4147).

*Polar Equation.*

$$4151 \quad r^2 + l^2 - 2rl \cos(\theta - \alpha) = c^2, \quad (\text{Fig. 26})$$

$$4152 \quad \text{or } r^2 - 2l \cos \alpha r \cos \theta - 2l \sin \alpha r \sin \theta + l^2 - c^2 = 0.$$

PROOF.—By (702), the coordinates of  $P$  being  $r$  and  $\theta$ .

General form of the polar equation:—

$$4153 \quad r^2 + 2gr \cos \theta + 2fr \sin \theta + c = 0.$$

$$4154 \quad \tan \alpha = \frac{f}{g}, \quad l = \sqrt{g^2 + f^2}.$$

PROOF.—By equating coefficients with (4152).

4156 Equation of the circle passing through the three points  $x_1y_1$ ,  $x_2y_2$ ,  $x_3y_3$ .

$$(x^2 + y^2) \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} - (x_1^2 + y_1^2) \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x & y & 1 \end{vmatrix} + (x_2^2 + y_2^2) \begin{vmatrix} x_3 & y_3 & 1 \\ x & y & 1 \\ x_1 & y_1 & 1 \end{vmatrix} - (x_3^2 + y_3^2) \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

PROOF.—Eliminate  $g$ ,  $f$ , and  $c$  from (4144) by (4117).

Equation of the chord joining  $x_1y_1$ ,  $x_2y_2$ , two points on the circle  $x^2 + y^2 = r^2$ :

$$4157 \quad x(x_1 + x_2) + y(y_1 + y_2) = x_1x_2 + y_1y_2 + r^2, \quad (4083, 4136)$$

$$4158 \quad \text{or } x \cos \frac{1}{2}(\theta_1 + \theta_2) + y \sin \frac{1}{2}(\theta_1 + \theta_2) = r \cos \frac{1}{2}(\theta_1 - \theta_2),$$

where  $r \cos \theta_1 = x_1$ ,  $r \sin \theta_1 = y_1$ , &c.

**4159** NOTE.—The coordinates  $x, y$  of a point on the circle  $x^2 + y^2 = r^2$  may often be expressed advantageously in this way in terms of  $\theta$ , a single variable.

**4160** Let  $S = (x-a)^2 + (y-b)^2 - r^2 = 0$

be any circle (Fig. 27). Then, if  $xy$  be a point  $P$  outside the circle,  $S$  becomes the square of the tangent from  $P$ . If  $xy$  be a point  $P'$  within the circle,  $S$  becomes minus the square of the ordinate drawn through  $P'$  at right angles to the radius through  $P'$ .

#### CO-AXAL CIRCLES.

(See also 984 and 1021.)

**4161** If  $S = x^2 + y^2 + 2gx + 2fy + c = 0,$   
 $S' = x^2 + y^2 + 2g'x + 2f'y + c' = 0$

be two circles, the equation to the radical axis is

$$S - S' = 0.$$

If  $x = 0$  be taken for the radical axis, the equation to any circle (radius  $r$ ) of the system of coaxal circles (1021) is

**4162**  $x^2 + y^2 - 2kx \pm \delta^2 = 0$  and  $k^2 - r^2 = \pm \delta^2,$

+ in Figure (1), - in Figure (2). Here  $\delta = IR$  a constant, and  $k = IO$  a variable.

**4164** The polar of  $x'y'$  for any circle of the system passes through the intersection of

$$xx' + yy' \pm \delta^2 = 0 \quad \text{and} \quad x + x' = 0.$$

PROOF.—Its equation is  $xx' + yy' - k(x + x') \pm \delta^2 = 0$  (4121). Then by (4099).

**4165** When  $k = \delta$ , then  $k = ID = ID'$ .  $D$  and  $D'$  are Poncelet's limiting points.

**4166** The polar of  $D$  with respect to any of the circles passes through  $D'$ , and *vice versa*, by (4164).

**4167** Tangents from any point on the radical axis to all circles of the system are equal (4160, '61).

**4168** The radical axes of three circles,  $S_1, S_2, S_3$ , meet in a point called their radical centre.

**4169** The reciprocals with respect to the origin  $D$  or  $D'$  of the system of co-axial circles are all confocal conics (4558).

The equation of the circle, centre  $Q$ , cutting the system of circles orthogonally is, putting  $IQ = h$ ,

$$\mathbf{4170} \quad x^2 + y^2 - 2hy - \delta^2 = 0. \quad (1230, 1236)$$

This circle passes through  $D$  and  $D'$ .

The common tangents to the two circles

$$(x-a)^2 + (y-b)^2 = r^2 \quad \text{and} \quad (x-a')^2 + (y-b')^2 = r'^2.$$

(See also 1037.)

The equation for  $a$  in (4143) is

$$\mathbf{4171} \quad (a-a') \cos \alpha + (b-b') \sin \alpha + r \mp r' = 0.$$

PROOF.—Assume (4143) in  $a, b, r, \alpha$ , and also in  $a', b', r', \alpha'$  as coinciding lines. Then  $\tan \alpha = \tan \alpha'$ ; therefore  $\alpha' = \alpha$  or  $\pi + \alpha$ . Take the difference of the two equations.

The chords of contact are

$$\mathbf{4172} \quad (a-a')(x-a) + (b-b')(y-b) + r(r \mp r') = 0,$$

$$\mathbf{4173} \quad (a-a')(x-a') + (b-b')(y-b') + r(r \mp r') = 0,$$

with  $-$  for exterior tangents,  $+$  for transverse.

PROOF.—For these are straight lines, and they pass through the points of contact of each pair of tangents respectively, by (4171).

The centres of similitude  $O, Q$  are the intersections of the external and transverse tangents respectively.

$$\mathbf{4174} \quad \text{Coordinates of } O, \quad \frac{a'r - ar'}{r - r'}, \quad \frac{b'r - br'}{r - r'}.$$

$$\mathbf{4175} \quad \text{Coordinates of } Q, \quad \frac{a'r + ar'}{r + r'}, \quad \frac{b'r + br'}{r + r'}.$$

PROOF.—By equating coefficients in (4172) and (4142), the polar of  $O$  or  $Q$ .

**4176** The six centres of similitude of three circles lie on

four straight lines called axes of similitude. See the figure of (1046).

PROOF.—The coordinates of the three centres of the forms (4174, '75) will in each case satisfy equation (4083).

**4177** The equation of the external axis of similitude is, in determinant notation (554),

$$(1r_2b_3)x - (1r_2a_3)y + (r_1b_2a_3) = 0.$$

PROOF.—By forming the equation of the right line passing through two of the centres of similitude whose coordinates are as in (4174).

**4178** The remaining three axes are found by changing in turn the signs of  $r_1, r_2$ ;  $r_2, r_3$ ;  $r_3, r_4$ .

**4179** If one of the circles touches the other two, one axis of similitude passes through the points of contact.

**4180** The angle  $\theta$ , at which the circle  $F(x, y) = 0$ , radius  $r$  (Fig. 29), intersects the circle whose centre is  $hk$ , and radius  $R$  is given by the equation

$$R^2 - 2Rr \cos \theta = F(h, k).$$

PROOF:  $\theta = OQP$ ,  $R^2 - 2Rr \cos \theta + r^2 = PT^2 + r^2 = F(h, k) + r^2$ , (702) and (4160).

**4181** COR. 1.—If the circles are given by the equations

$$x^2 + y^2 + 2g'x + 2f'y + c' = 0, \quad x^2 + y^2 + 2gx + 2fy + c = 0,$$

the equation for  $\cos \theta$  becomes, since  $h = -g$ ,  $k = -f$ ,

$$2Rr' \cos \theta = 2gg' + 2ff' - c - c'. \quad (4145)$$

**4182** COR. 2.—The condition that the two circles may cut orthogonally is

$$2gg' + 2ff' - c - c' = 0.$$

**4183** COR. 3.—By solving three such equations, we can find the circle cutting three given circles orthogonally (4186).

**4184** COR. 4.—The condition that four circles may have a common orthogonal circle is the determinant equation

$$\begin{vmatrix} c & g & f & 1 \\ c_1 & g_1 & f_1 & 1 \\ c_2 & g_2 & f_2 & 1 \\ c_3 & g_3 & f_3 & 1 \end{vmatrix} = 0.$$

**4185** Cor. 5.—If the circle  $x^2 + y^2 + 2Gx + 2Fy + C = 0$  cuts three other circles at the same angle  $\theta$ , we have, by (4081), three equations to determine  $G, F, C$ . The resulting determinant equation may be written

$$\begin{vmatrix} x^2 + y^2 & -x & -y & 1 \\ c_1 & g_1 & f_1 & 1 \\ c_2 & g_2 & f_2 & 1 \\ c_3 & g_3 & f_3 & 1 \end{vmatrix} + 2R \cos \theta \begin{vmatrix} 0 & -x & -y & 1 \\ r_1 & g_1 & f_1 & 1 \\ r_2 & g_2 & f_2 & 1 \\ r_3 & g_3 & f_3 & 1 \end{vmatrix} = 0.$$

**4186** The first determinant, put  $= 0$ , is the orthogonal circle (4183), and the second, expanded, is the axis of similitude.

**4187** The locus of the centre of a circle cutting three given circles at equal angles is a perpendicular from their radical centre on any of the four axes of similitude.

PROOF.—By eliminating  $R$  and  $\cos a$  between three equations, like (4180).

**4188** Each of these four perpendiculars contains the centres of two circles touching the three given circles.

PROOF.—Consider  $a = 0$  or  $180^\circ$ , in (4180).

To draw the eight circles which touch three given circles, see (946) and (1049).

**4189** The equation of the fourth degree of two of the touching circles is

$$\overline{23} \sqrt{S_1} \pm \overline{31} \sqrt{S_2} \pm \overline{12} \sqrt{S_3} = 0,$$

where  $\overline{23}$  signifies the length of the common tangent of the second and third circles, &c.

PROOF.—By first showing that, if four circles are all touched by another circle, the relation

**4190**  $\overline{12}.\overline{34} \pm \overline{14}.\overline{23} \pm \overline{31}.\overline{24} = 0$

will subsist, and then supposing the fourth circle to reduce to a point.

## THE PARABOLA.

**4200** DEF.—A conic is the locus of a point which moves in one plane so that its distance from a fixed point  $S$ , the *focus*,



is in a constant ratio ( $e$ ) to its distance from a fixed right line  $XM$  (the *directrix*).

When  $e = \text{unity}$ , the curve is a parabola. (See also p. 248, *et seq.*)

Equation of the Parabola with origin of coordinates at the vertex  $A$ .

$$4201 \quad y^2 = 4ax.$$

Here  $a = AS$ ,  $x = AN$ ,  
 $y = PN$ .

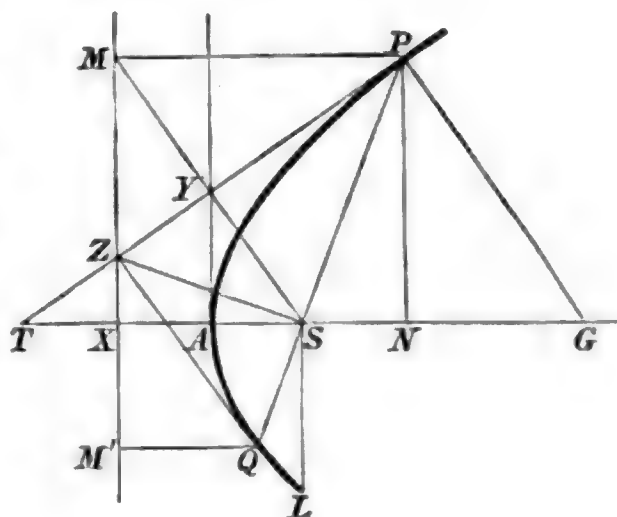
PROOF.—Geometrically, at (1229).  
Analytically, from

$$PS^2 = y^2 + (x-a)^2 = PM^2 = (x+a)^2.$$

The equations with the origin at  $S$  and  $X$  respectively are

$$4202 \quad y^2 = 4a(x+a),$$

$$y^2 = 4a(x-a). \quad (4048)$$



Equations of the tangent at  $x'y'$  :

$$4204 \quad y - y' = \frac{2a}{y'}(x - x'). \quad (4120)$$

$$4205 \quad yy' = 2a(x + x').$$

$$4206 \quad y = mx + \frac{a}{m}, \quad m = \frac{2a}{y'}. \quad (4123)$$

4207 (4204) is also the polar of any point  $x'y'$ , by (4124). Its intercepts are  $-x'$  and  $\frac{1}{2}y'$ .

Equations of the normal at  $x'y'$  :

$$4208 \quad y - y' = -\frac{y'}{2a}(x - x'). \quad (4122)$$

$$4209 \quad y'x + 2ay - (y'x' + 2ay') = 0.$$

$$4210 \quad y = mx - 2am - am^3. \quad (4123)$$

where

**PROOF.**—Geometrically, at (1239). Otherwise, let  $VQ = -VQ'$  be equal roots of opposite signs of the quadratic (4221),  $V$  being the point  $x'y'$ , therefore  $y^2$  or  $r^2 = (y^2 - 4ax') \operatorname{cosec}^2 \theta = 4a \operatorname{cosec}^2 \theta . x$ , since  $y^2 = 4a \times$  abscissa of  $P$ .

For the polar equation of the parabola, see (4336).

$$n_1^2 (y_2^2 - 4ax_2) + 2n_1n_2 \{y_1y_2 - 2a(x_1 + x_2)\} + n_2^2 (y_1^2 - 4ax_1) = 0. \quad (4131)$$

**4215**  $(y'^2 - 4ax')(y^2 - 4ax) = \{yy' - 2a(x + x')\}^2 = 0.$

Quadratics for the coordinates of the points of contact of tangents from  $x'y'$  :

4217  $y^2 - 2yy' + 4ax' = 0.$

Coordinates of the point of intersection of tangents at  $x_1, y_1$  and  $x_2, y_2$  :

4 F

---

Quadratic for  $m$  of the tangent from  $x'y'$ :

**4220**  $m^2x' - my' + a = 0.$  (4206)

---

**4221** General polar equation of the parabola, or quadratic for  $r$ , the segment intercepted between a point,  $x'y'$ , and the curve on a line drawn from that point at an inclination  $\theta$  to the  $x$  axis (4134),

$$r^2 \sin^2 \theta + 2r (y' \sin \theta - 2a \cos \theta) + y'^2 - 4ax' = 0.$$


---

Quadratics for the coordinates of the points of intersection of the line  $Ax + By + C$  and the parabola  $y^2 = 4ax$ : (4116)

**4222**  $A^2x^2 - 2(2B^2a - AC)x + C^2 = 0.$

**4223**  $Ay^2 + 4Bay + 4Ca = 0.$

---

Length of intercepted chord,

**4224**  $4\sqrt{\{(B^2a^2 - ACa)(A^2 + B^2)\}} \div A^2.$  (4034)

---

Equation of the secant through  $x_1y_1, x_2y_2$ , two points on the parabola:

**4225**  $y(y_1 + y_2) = y_1y_2 + 4ax,$  (4083)

**4226** or  $y(m_1 + m_2) = 2a + 2m_1m_2x.$

---

**4227** The subtangent  $NT = 2x.$  Fig. of (4201)

**4228** The subnormal  $NG = 2a.$

PROOF.—Put  $y = 0$  in (4205) and (4208).

---

**4229** The tangent  $PT^2 = 4x(a + x).$

**4230** The normal  $PG^2 = 4a(a + x).$

---

The perpendicular  $p$  from the focus upon the tangent at  $xy$ :

**4231**  $p = \sqrt{a(x + a)} = \sqrt{aa'}.$  (4212), (4095)

---

The part of the normal intercepted by the curve is equal to

$$4233 \quad \frac{4a(1+m^2)^{\frac{1}{2}}}{m^2} = \frac{4a}{\sin^2 \theta \cos \theta} \quad (4221), (4135)$$

$$4234 \quad \text{The minimum normal} = 6a\sqrt{3} \text{ and } m = \sqrt{2}.$$

Length of a chord through the focus

$$4235 \quad = \frac{4a}{\sin^2 \theta} = 4a'. \quad (4212)$$

Coordinates of its extremities, with the focus for origin :

$$4237 \quad x = -\frac{2a \cos \theta}{\cos \theta \mp 1}, \quad y = -\frac{2a \sin \theta}{\cos \theta \mp 1}.$$

Coordinates of its centre :

$$4239 \quad x = \frac{2a \cos^2 \theta}{\sin^2 \theta}, \quad y = 2a \cot \theta.$$

## THE ELLIPSE AND HYPERBOLA.

(See also p. 233, *et seq.*)

4250 Referring to the definition (4200); when  $e$  is less than unity, the conic is an *ellipse*; when greater than unity, an *hyperbola*.

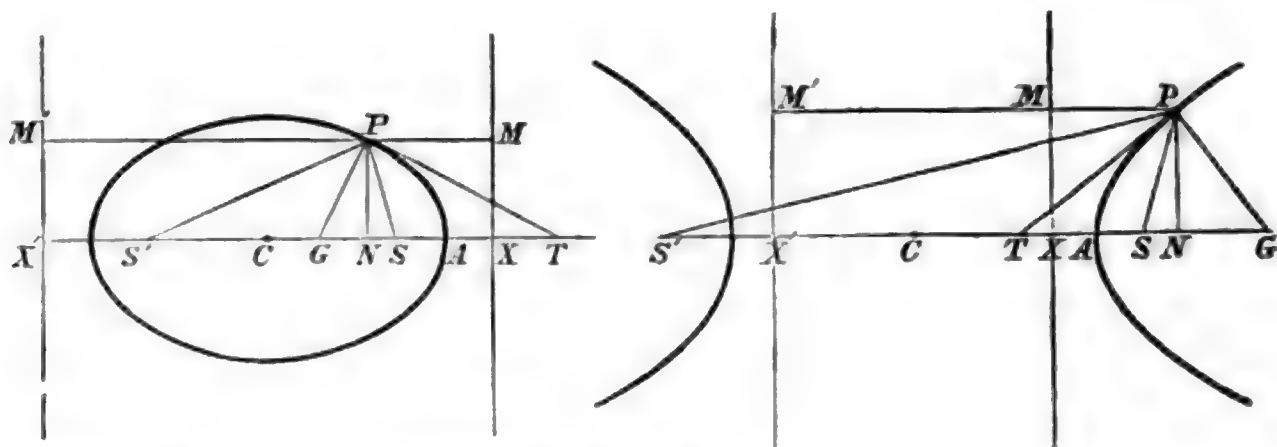
Equation of the ellipse with the origin of coordinates at  $X$  and  $SX = p$ .

$$4251 \quad y^2 + (x-p)^2 = e^2 x^2.$$

PROOF.—By the definition in (4200).

Abscissæ of vertices : (Supply  $A'$  in the following figure.)

$$4252 \quad XA = \frac{p}{1+e}, \quad XA' = \frac{p}{1-e}. \quad (4115)$$



**4254** Focal distances of vertices :

$$SA = \frac{ep}{1+e}, \quad SA' = \frac{ep}{1-e}. \quad (4251)$$

$$\mathbf{4256} \quad SL \equiv l = ep = a(1-e^2) = \frac{b^2}{a}. \quad (4251)$$

$$\mathbf{4260} \quad b^2 = a^2(1-e^2); \quad e^2 = \frac{a^2-b^2}{a^2}.$$

$$\mathbf{4262} \quad CX = \frac{p}{1-e^2} = \frac{a}{e}.$$

$$\mathbf{4264} \quad CA = \frac{ep}{1-e^2} = a.$$

$$\mathbf{4266} \quad CS = \frac{e^2 p}{1-e^2} = ae. \quad (4252)$$

**4268** If  $b = a \tan \alpha$ , then  $e = \sec \alpha$  in the hyperbola.

Equation with the origin at  $A$  :

$$\mathbf{4269} \quad y^2 = \frac{b^2}{a^2}(2ax - x^2) \quad \text{Ell.}$$

$$\mathbf{4270} \quad y^2 = \frac{b^2}{a^2}(2ax + x^2) \quad \text{Hyp.}$$

PROOF.—By (4200),  $y^2 + (x-SA)^2 = e^2(x+AX)^2$ , &c.

Equations with the origin at the centre  $C$  :

$$\mathbf{4271} \quad y^2 = \frac{b^2}{a^2}(a^2 - x^2) = (1-e^2)(a^2 - x^2). \quad (4269)$$

4273

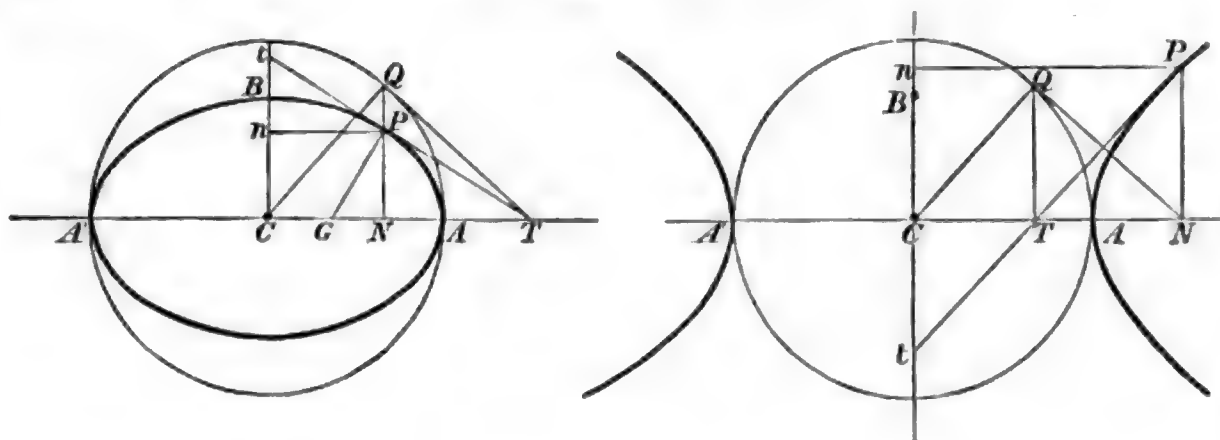
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

PROOF.—By (4200),  $y^2 + (x + CS)^2 = e^2(x + CX)^2$ , &c.

4274

$$PN : QN :: b : a.$$

(4271)



4275 DEF.— $QCN$  is the eccentric angle,  $\phi$ , of the point  $P$ .

$x$  and  $y$  in terms of the eccentric angle :

4276

$$x = a \cos \phi, \quad y = b \sin \phi. \quad (\text{Ell.})$$

4278

$$x = a \sec \phi, \quad y = b \tan \phi. \quad (\text{Hyp.})$$

Five forms of the equation of the tangent or polar of the point  $x'y'$  :

4280

$$y - y' = -\frac{b^2 x'}{a^2 y'}(x - x'). \quad (4120)$$

4281

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1.$$

4282

$$y = mx + \sqrt{a^2 m^2 + b^2}. \quad (4123)$$

4283

$$\frac{x \cos \phi}{a} + \frac{y \sin \phi}{b} = 1. \quad (\text{Ell.}) \quad (4276)$$

4284

$$\frac{x \sec \phi}{a} - \frac{y \tan \phi}{b} = 1. \quad (\text{Hyp.}) \quad (4278)$$

4285

$$x \cos \gamma + y \sin \gamma = \sqrt{a^2 \cos^2 \gamma + b^2 \sin^2 \gamma},$$

$\gamma$  being the inclination of  $p$ .

(4054) & (4372)



Five forms of the equation of the normal at  $x'y'$ :

$$4286 \quad y - y' = \frac{a^2 y'}{b^2 x'} (x - x'). \quad (4122)$$

$$4287 \quad \frac{a^2 x}{x'} - \frac{b^2 y}{y'} = a^2 - b^2, \quad \text{or} \quad hx - ky = a^2 - b^2,$$

where  $h$  and  $k$  are the intercepts of the tangent.

$$4289 \quad y = mx - \frac{m(a^2 - b^2)}{\sqrt{a^2 + b^2 m^2}}. \quad (4123)$$

$$4290 \quad ax \sec \phi - by \operatorname{cosec} \phi = a^2 - b^2. \quad (4276)$$

$$4291 \quad x_1 x - y_1 y = (x_1 x' - y_1 y'), \quad (4352)$$

where  $x_1 y_1$  is the extremity of the conjugate diameter.

Intercepts of the tangent or polar on the axes:

$$4292 \quad \frac{a^2}{x} \quad \text{and} \quad \frac{b^2}{y}. \quad (4115), (4281)$$

Intercepts of the normal: (4287)

$$4294 \quad \text{On the } x \text{ axis,} \quad \frac{a^2 - b^2}{a^2} x \quad \text{or} \quad e^2 x.$$

$$4296 \quad \text{On the } y \text{ axis,} \quad -\frac{a^2 - b^2}{b^2} y \quad \text{or} \quad -\frac{e^2}{1 - e^2} y.$$

Focal distances  $r, r'$  of a point  $xy$  on the curve:

$$4298 \quad (a \pm ex) \text{ in Ell.}$$

$$4299 \quad (ex \pm a) \text{ in Hyp.}$$

PROOF.—From  $r^2 = (ae \pm x)^2 + y^2$ , and (4272).

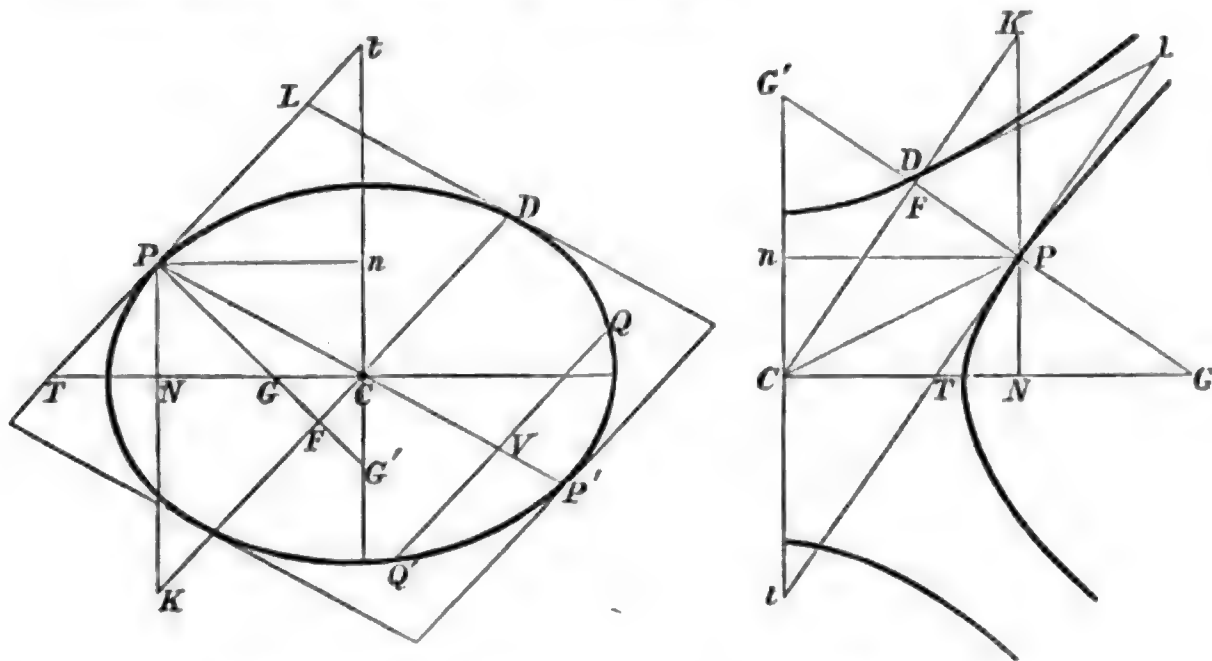
Perpendiculars from the foci upon the tangent:

$$4300 \quad p = b \sqrt{\frac{r}{r'}}, \quad p' = b \sqrt{\frac{r'}{r}}. \quad (4095, 4282)$$

$$\text{4302} \quad \therefore (\text{p. 588}) \sin SPT = \frac{p}{r} = \frac{p'}{r'} = \frac{b}{\sqrt{rr'}} = \frac{b}{b'}. \quad (4365)$$

$$b^2 = pp'. \quad (4300)$$

### Segments of tangent and normal:



$$4307 \quad PT = \frac{ay}{bx} \sqrt{rr'}, \quad Pt = \frac{bx}{ay} \sqrt{rr'}. \quad (4292)$$

**4309**  $PG = \frac{b}{a} \sqrt{rr'} = \frac{bb_1}{a}, \quad PG' = \frac{a}{b} \sqrt{rr'} = \frac{ab_1}{b}. \quad (4294)$

### Right Line and Ellipse.

Quadratic for the ratio  $n_1 : n_2$ , in which the line joining two given points  $x_1y_1, x_2y_2$  is cut by the ellipse (4131).

4310

$$\frac{n_1^2}{n_2^2} \left( \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 \right) + 2 \frac{n_1}{n_2} \left( \frac{x_1 x_2}{a^2} + \frac{y_2 y_1}{b^2} - 1 \right) + \left( \frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} - 1 \right) = 0.$$

**Equation of the two tangents drawn from  $x'y'$  :**

**4311**  $\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) - \left(\frac{xx'}{a^2} + \frac{yy'}{b^2} - 1\right)^2 = 0.$

**PROOF.**—By the condition for equal roots of (4310).

Quadratic for abscissæ of points of contact of the tangent from  $x'y'$  :

$$4312 \quad x^2(b^2x'^2 + a^2y'^2) - 2a^2b^2xx' + a^4(b^2 - y'^2) = 0. \quad (4282, 4125)$$

Quadratic for  $m$  of the tangent from  $xy$  :

$$4313 \quad m^2(x^2 - a^2) - 2mxy + y^2 - b^2 = 0. \quad (4282)$$

General polar equation of the ellipse, or quadratic for  $r$ , the segment intercepted between the point  $x'y'$  and the curve on the right line drawn from that point at an inclination  $\theta$  to the major axis and  $x$  axis of coordinates.

$$4314 \quad (a^2 \sin^2 \theta + b^2 \cos^2 \theta) r^2 + 2r(a^2 y' \sin \theta + b^2 x' \cos \theta) + (a^2 y'^2 + b^2 x'^2 - a^2 b^2) = 0.$$

4315 *Length of intercepted chord = difference of roots.*

4316 *Distance to middle point of chord = half sum of roots.*

4317 *Rectangle under segments = products of roots.*

COR.—If two chords be drawn to a conic at two constant inclinations to the major axis, the ratio of the rectangles under their segments is invariable.

For, if  $x'y'$  be their point of intersection, the ratio in question becomes  $a^2 \sin^2 \theta + b^2 \cos^2 \theta : a^2 \sin^2 \theta' + b^2 \cos^2 \theta'$ , which is constant if  $\theta$  and  $\theta'$  are constant.

Locus of centres of parallel chords :

$$4318 \quad a^2 y \sin \theta + b^2 x \cos \theta = 0. \quad (4314)$$

Quadratic for abscissæ of points of intersection of the line  $Ax + By + C = 0$  and the ellipse  $b^2 x^2 + a^2 y^2 - a^2 b^2 = 0$ . (4116)

$$4319 \quad (A^2 a^2 + B^2 b^2) x^2 + 2ACa^2 x + C^2 a^2 - B^2 a^2 b^2 = 0.$$

$$4320 \quad x = \frac{-ACa^2 \pm Bab\sqrt{A^2 a^2 + B^2 b^2 - C^2}}{A^2 a^2 + B^2 b^2}.$$

4321 For the ordinates transpose  $A, B$  and  $a, b$ .

Length of intercepted chord :

$$4322 \quad \frac{2ab\sqrt{(A^2+B^2)(A^2a^2+B^2b^2-C^2)}}{A^2a^2+B^2b^2}. \quad (4034)$$

Hence the condition that the line may touch the ellipse is

$$4323 \quad A^2a^2+B^2b^2=C^2.$$

The chord through two points  $x_1y_1, x_2y_2$  is

$$4324 \quad \frac{x(x_1+x_2)}{a^2} + \frac{y(y_1+y_2)}{b^2} = \frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} + 1;$$

or, denoting the points by their eccentric angles  $\alpha, \beta$ , the chord joining  $\alpha\beta$  is

$$4325 \quad \frac{x}{a} \cos \frac{\alpha+\beta}{2} + \frac{y}{b} \sin \frac{\alpha+\beta}{2} = \cos \frac{\alpha-\beta}{2}.$$

The coordinates of the pole of the chord or intersection of tangents at  $x_1y_1, x_2y_2$  (or  $\alpha\beta$  as above).

$$4326 \quad x = \frac{x_1y_2+x_2y_1}{y_1+y_2} = \frac{a^2(y_1-y_2)}{x_2y_1-x_1y_2} = a \frac{\cos \frac{1}{2}(\alpha+\beta)}{\cos \frac{1}{2}(\alpha-\beta)}.$$

$$4329 \quad y = \frac{x_1y_2+x_2y_1}{x_1+x_2} = \frac{b^2(x_1-x_2)}{x_1y_2-x_2y_1} = b \frac{\sin \frac{1}{2}(\alpha+\beta)}{\cos \frac{1}{2}(\alpha-\beta)}.$$

The following relations also subsist

$$4332 \quad \frac{a^2b^2}{b^2x^2+a^2y^2} = \frac{a^2 \sin \alpha \sin \beta}{a^2-x^2} = \frac{b^2 \cos \alpha \cos \beta}{b^2-y^2} \\ = \frac{b(\sin \alpha + \sin \beta)}{2y} = \frac{a(\cos \alpha + \cos \beta)}{2x},$$

“which are of use in finding the locus of  $(x, y)$  when  $\alpha, \beta$  are connected by some fixed equation.”

(Wolstenholme's Problems, p. 116.)

**4334** If  $\alpha, \beta, \gamma, \delta$  are the eccentric angles of the feet of the four normals drawn to an ellipse from a point  $xy$ , then

$$\alpha + \beta + \gamma + \delta = 3\pi \text{ or } 5\pi.$$

PROOF.—Equation (4290) gives the following biquadratic in  $z = \tan \frac{1}{2}\phi$ ,

$$byz^4 + 2(ax + a^3 - b^3)z^3 + 2(ax - a^3 + b^3)z - by = 0.$$

Let  $a, b, c, d$  be the roots. Eliminate  $d$  from  $ab + ac + \&c. = 0$  and  $abcd = -1$  (406). Thus  $ab + bc + ca = \frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab}$ ; from which, since  $a = \tan \frac{1}{2}\alpha$ ,

$$\&c., \text{ we get } \sin(\beta + \gamma) + \sin(\gamma + \alpha) + \sin(\alpha + \beta) = 0;$$

$$\text{and, since } 1 - (ab + ac + \&c.) + abcd = 0,$$

$$\tan \frac{1}{2}(\alpha + \beta + \gamma + \delta) = \infty, \quad \therefore \alpha + \beta + \gamma + \delta = 3\pi \text{ or } 5\pi.$$

**4335** The points on the curve where it is met by the normals drawn from a fixed point  $x'y'$  are determined by the intersections of the curve and the hyperbola

$$a^2x'y - b^2y'x = c^2xy. \quad (4287)$$

#### POLAR EQUATIONS OF THE CONIC.

The focus  $S$  being the pole (Fig. of 4201), the equation of any conic is

$$\mathbf{4336} \quad r(1 + e \cos \theta) = l,$$

$\theta$  being measured from  $A$ , the nearest vertex.

For the parabola, put  $e = 1$ .

PROOF.—

$$r = SP; \quad \theta = ASP; \quad l = SL; \quad r = e(SX + SN) \text{ (4200)} = l + er \cos \theta.$$

The secant through two points,  $P, P'$ , on the curve, whose angular coordinates are  $\alpha + \beta$  and  $\alpha - \beta$  (Fig. 28), is

$$\mathbf{4337} \quad r \{ e \cos \theta + \sec \beta \cos (\alpha - \theta) \} = l.$$

PROOF.—Let  $ASQ = \alpha$ ,  $PSQ = P'SQ = \beta$ .

*Analytically.* Take (4109) for the equation of  $PP'$ . Eliminate  $r_1$  and  $r_2$  by (4336), and substitute  $2\alpha$  for  $\theta_1 + \theta_2$  and  $2\beta$  for  $\theta_1 - \theta_2$ .

*Geometrically.* Let  $PP'$  cut the directrix in  $Z$ ; then  $QSZ$  is a right angle, by (1166). Take  $C$  any point in  $PP'$ ;  $SC = r$ ;  $ASC = \theta$ . Draw  $CD, CE, CF, CG$  parallel to  $SL, SP, SQ, SX$ , and  $DH$  parallel to  $XL$ . Then

$$l = SL = SH + HL.$$

$$SH = \frac{SL}{SX} SD = er \cos \theta. \quad \frac{CE}{CG} = \frac{SP}{PM} = \frac{SL}{SX} = \frac{HL}{DX} = \frac{HL}{CG},$$

$$\therefore HL = CE = r \sin CSF \sec \beta = r \cos (\alpha - \theta) \sec \beta,$$

$$\therefore l = er \cos \beta + r \sec \beta \cos (\alpha - \theta).$$

The equation of the tangent at the point  $a$  is, consequently,

$$4338 \quad r \{e \cos \theta + \cos (a - \theta)\} = l.$$

*A Focal Chord.*

$$4339 \quad \text{Length} = \frac{2l}{1 - e^2 \cos^2 \theta}. \quad (4336)$$

Coordinates of the extremities, the centre  $C$  being the origin :

$$4340 \quad x = \frac{a(e \pm \cos \theta)}{1 \pm e \cos \theta}, \quad y = \frac{l \sin \theta}{1 \pm e \cos \theta}.$$

4342 The lines joining the extremities of two focal chords meet in the directrix. [By (4337)]

Polar equation with vertex for pole :

$$4343 \quad r^2 (1 - e^2 \cos^2 \theta) = 2l \cos \theta. \quad (4200)$$

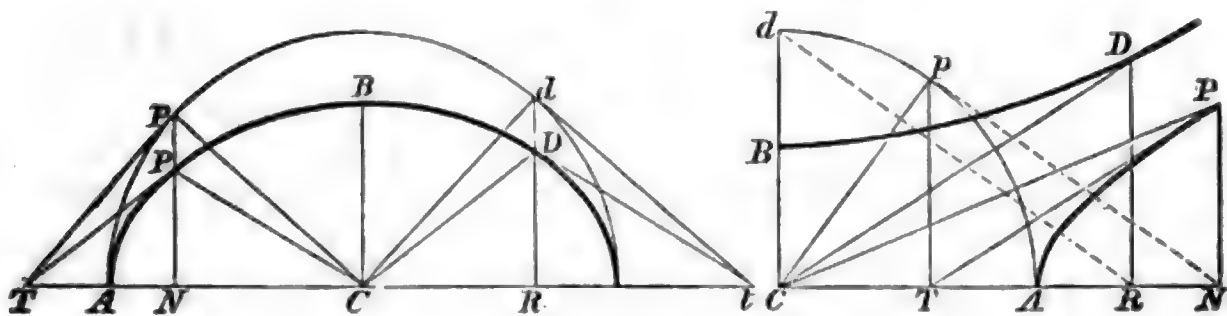
Polar equation with the centre for pole :

$$4344 \quad r^2 (a^2 \sin^2 \theta + b^2 \cos^2 \theta) = a^2 b^2,$$

$$4345 \quad \text{or} \quad r \sqrt{(1 - e^2 \cos^2 \theta)} = b.$$

PROOF.—By (4273). Otherwise, by (4314), with  $x' = y' = 0$ .

### CONJUGATE DIAMETERS.



Equation of the ellipse referred to conjugate diameters for coordinate axes :

$$4346 \quad \frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1,$$



where

$$4347 \quad a'^2 = \frac{a^2 b^2}{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}, \quad b'^2 = \frac{a^2 b^2}{a^2 \sin^2 \beta + b^2 \cos^2 \beta}.$$

Here  $a' = CD$ ,  $b' = CP$ ,  $\alpha$  is the angle  $DCR$ , and  $\beta$  the angle  $PCR$ .

PROOF.—Apply (4050) to the equation (4273), putting  $\omega = \frac{\pi}{2}$  and  $\tan \alpha \tan \beta = -\frac{b^2}{a^2}$ , by (4351).

When  $a' = b'$ ,  $\alpha + \beta = \pi$ , and equation (4346) becomes

$$4349 \quad x^2 + y^2 = a'^2 = \frac{1}{2} (a^2 + b^2).$$

Let the coordinates of  $D$  be  $x', y'$ , and those of  $P$   $x, y$ ; the equation of the diameter  $CP$  conjugate to  $CD$  is

$$4350 \quad \frac{xx'}{a^2} + \frac{yy'}{b^2} = 0.$$

$$4351 \quad \tan \alpha \tan \beta \quad \text{or} \quad mm' = -\frac{b^2}{a^2}. \quad (4318)$$

$xy$  in terms of  $x'y'$ , &c.

$$4352 \quad x = -\frac{a}{b} y', \quad y = \frac{b}{a} x'. \quad \text{Ell.}$$

$$4354 \quad x = \frac{a}{b} y', \quad y = \frac{b}{a} x'. \quad \text{Hyp.}$$

PROOF.—Solve (4350) with (4273).

$$4356 \quad x = dR, \quad x' = pN. \quad (4274, 4352)$$

$$4358 \quad x^2 + x'^2 = a^2, \quad y'^2 + y^2 = b^2. \quad \text{Ell.} \quad (4352)$$

$$4360 \quad x^2 - x'^2 = a^2, \quad y'^2 - y^2 = b^2. \quad \text{Hyp.} \quad (4354)$$

$$4362 \quad a^2 + b^2 = a'^2 + b'^2. \quad \text{Ell.} \quad (4358)$$

$$4363 \quad a^2 - b^2 = a'^2 - b'^2. \quad \text{Hyp.} \quad (4360)$$

$$4364 \quad a'^2 = b^2 + e^2 x^2. \quad (4271, '61)$$

$$4365 \quad b'^2 = a^2 - e^2 x^2 = rr'. \quad (4298)$$

The perpendicular from the centre upon the tangent at  $xy$  is given by

$$4366 \quad \frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4}. \quad (4281, 4064)$$

The area of the parallelogram  $PCDL$  (Fig. of 4307) is

$$4367 \quad pa' = ab = a'b' \sin \omega,$$

where  $p = PF$ ,  $a' = CD$ ,  $b' = CP$ ,  $\omega = \angle PCD$ .

PROOF.—From (4366), and (4352), and  $a'^2 = x'^2 + y'^2$ .

Other values of  $p^2$ :

$$4369 \quad p^2 = \frac{a^2 b^2}{a'^2} = \frac{a^2 b^2}{a^2 + b^2 - b'^2}. \quad (4362)$$

$$4371 \quad p^2 = a^2 \sin^2 \theta + b^2 \cos^2 \theta.$$

PROOF.—From (4344, '67), putting  $r = a'$ .

$$4372 \quad p^2 = a^2 \cos^2 \gamma + b^2 \sin^2 \gamma, \quad (4371)$$

$\gamma$  being the inclination of  $p$ .

$$4373 \quad p^2 = a^2 (1 - e^2 \sin^2 \gamma). \quad (4372, 4260)$$

Equations to the tangents at  $P$  and  $P'$ , the coordinates of  $D$  being  $x'$ ,  $y'$ :

$$4374 \quad xy' - yx' = \pm ab \quad (4073) \quad m = \frac{y'}{x'}.$$

#### DETERMINATION OF VARIOUS ANGLES.

$$4375 \quad pCd = \frac{\pi}{2}. \quad \text{Fig. p. 595.} \quad (4356)$$

$$4377 \quad \tan PCD = -\frac{a^2 b^2}{c^2 xy}, \quad (4070, 4352-3)$$

where  $c = \sqrt{a^2 - b^2} = CS$ .

$$4378 \quad \tan (SPT) = \frac{b^2}{cy} = \frac{1+e \cos \theta}{e \sin \theta}, \quad (4070, 4256, 4336)$$

where  $\theta = PST$ .

[See figure on page 588.]

If  $\psi$  be the inclination of the tangent to the  $x$  axis,

$$4380 \quad \tan \psi = -\frac{b^2 x}{a^2 y} = \frac{e + \cos \theta}{\sin \theta}. \quad (4280)$$

PROOF:  $\psi = \theta + SPT$ . Then by (631) and (4379).

$$4382 \quad \tan SPS' = \frac{2b^2 cy}{b^4 - c^2 y^2}. \quad (652, 4378)$$

$$4383 \quad \tan APA' = -\frac{2b^2}{ae^2 y}, \quad \tan CPG = \frac{e^2 xy}{b^2}.$$

If  $OP, OP'$  are tangents to an ellipse,

$$4385 \quad \cos POP' = \frac{CO^2 - a^2 - b^2}{OS \cdot OS'}.$$

PROOF.—By figure and construction of (1180),  $POP' = MOS'$ . Therefore

$$\cos POP' = \frac{OS^2 + OS'^2 - SM^2}{2OS \cdot OS'} = \frac{2CO^2 + 2CS^2 - 4a^2}{2OS \cdot OS'} = \&c.$$

If  $x', y'$  are the coordinates of  $O$ ,

$$4386 \quad \tan POP' = \frac{2\sqrt{(b^2 x'^2 + a^2 y'^2 - a^2 b^2)}}{x'^2 + y'^2 - a^2 - b^2}.$$

PROOF.—By (4311), taking terms of the second degree for the two parallel lines through the origin and  $\tan \phi$  from (4112).

It is worthy of remark that the substitutions (4276–8) may also be usefully employed when the axes of reference are conjugate diameters: though, in that case, the geometrical signification of  $\phi$  no longer exists.

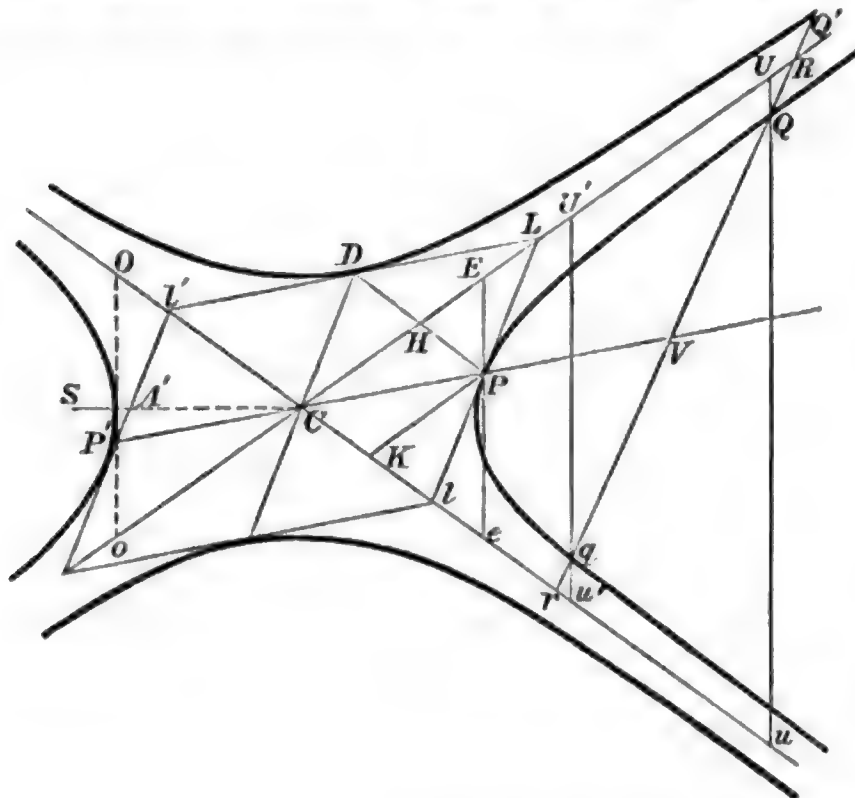
# THE HYPERBOLA

## REFERRED TO THE ASYMPTOTES.

4387

$$xy = \frac{1}{4}(a^2 + b^2).$$

**PROOF.**—By (4273) and (4050). Here  $x = CK$ ,  $y = PK$ .



Equations of the tangent at  $P, (x', y')$ .

**4388**  $xy' + x'y = \frac{1}{2}(a^2 + b^2).$  (4120)

**4389**  $y = mx + \sqrt{m(a^2 + b^2)}. \quad (4123)$

**4390**  $m = -\frac{y'}{x'}.$

**4391** Intercepts on the axes  $Cl = 2x'$ ,  $CL = 2y'$ .

## THE RECTANGULAR HYPERBOLA.

**4392** Here  $a = b$ ,  $e = \sqrt{2}$ ; and the equation with the ordinary axes is

**4393**  $x^2 - y^2 = a^2. \quad (4273)$

**4394** Tangent  $xx' - yy' = a^2$ . (4281)

Equation with the asymptotes for axes :

$$4395 \quad 2xy = a^2. \quad (4387)$$

$$4396 \quad \text{Tangent} \quad xy' + x'y = a^2. \quad (4388)$$

### THE GENERAL EQUATION.

The general equation of the second degree is

$$4400 \quad ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

$$4401 \quad \text{or } ax^2 + by^2 + cz^2 + 2fyz + 2gza + 2hxy = 0, \text{ with } z = 1.$$

The equation will be denoted by  $u$  or  $\phi(x, y) = 0$ .

### THE ELLIPSE AND HYPERBOLA.

When the general equation (4400), taken to rectangular axes of coordinates, represents a central conic, the coordinates of the centre,  $O'$  (Fig. 30), are

$$4402 \quad x' = \frac{hf - bg}{ab - h^2} = \frac{G}{C}, \quad y' = \frac{gh - af}{ab - h^2} = \frac{F}{C}. \quad (4665)$$

PROOF.—By changing the origin to the point  $x'y'$  and equating the new  $g$  and  $f$  each to zero (4048).

For the case in which  $ab = h^2$ , see (4430).

$$4404 \quad \text{The transformed equation is } ax'^2 + 2hxy' + by'^2 + c' = 0,$$

$$4405 \quad \text{where } c' = ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c.$$

$$4406 \quad = gx' + fy' + c.$$

$$4407 \quad = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{ab - h^2} = \frac{\Delta}{C}. \quad (4466)$$

The inclination  $\theta$  of the principal axis of the conic to the  $x$  axis is given by

$$4408 \quad \tan 2\theta = \frac{2h}{a - b}.$$

PROOF.—(Fig. 30.) By turning the axes in (4404) through the angle  $\theta$  (4049) and equating the new  $h$  to zero.

The transformed equation now becomes

$$4409 \quad a'x^2 + b'y^2 + c' = 0,$$

$$4410 \quad \text{in which } a' = \frac{1}{2} \{a + b + \sqrt{4h^2 + (a - b)^2}\},$$

$$4411 \quad b' = \frac{1}{2} \{a + b - \sqrt{4h^2 + (a - b)^2}\},$$

$a'$  and  $b'$  are found from the two equations

$$4412 \quad a' + b' = a + b, \quad a'b' = ab - h^2. \quad [\text{See (4418).}]$$

The semi-axes and excentricity are

$$4414 \quad \sqrt{-\frac{c'}{a'}}, \quad \sqrt{-\frac{c'}{b'}}, \quad \text{and} \quad e = \sqrt{\left(1 - \frac{a'}{b'}\right)}. \quad (4273) \quad (4261)$$

For the coordinates of the foci, see (5008).

**4416** NOTE.—If  $\theta$  be the acute angle determined by equation (4408), we have to choose between  $\theta$  and  $\theta + \frac{\pi}{2}$  for the inclination in question, since  $\tan 2\theta$  is also equal to  $\tan (2\theta + \pi)$ .

**RULE.\***—For the ellipse, the inclination of the major axis to the  $x$  axis of coordinates will be the acute angle  $\theta$  or  $\theta + \frac{1}{2}\pi$ , according as  $h$  and  $c'$  have the same or different signs. For the hyperbola, read “different or the same.”

**PROOF.**—Let the transformed equation (4409) be written in terms of the semi-axes  $p, q$ ; thus  $q^2x^2 + p^2y^2 = p^2q^2$ , representing an ellipse. Now turn the axes back again through the angle  $-\theta$ , and we get

$$(q^2 \cos^2 \theta + p^2 \sin^2 \theta) x^2 - (p^2 - q^2) \sin 2\theta xy + (q^2 \sin^2 \theta + p^2 \cos^2 \theta) y^2 = p^2 q^2.$$

Comparing this with the identical equation (4404),  $ax^2 + 2hxy + by^2 = -c'$ , we have

$$(p^2 - q^2) \sin 2\theta = -2h, \quad p^2 q^2 = -c';$$

$$\therefore \sin 2\theta = \frac{2h}{c'} \cdot \frac{p^2 q^2}{p^2 - q^2}. \quad \text{Hence } \theta \text{ is } < \frac{\pi}{2}$$

when  $h$  and  $c'$  have the same sign,  $p$  being  $> q$ . A similar investigation applies to the hyperbola by changing the sign of  $q^2$ .

\* This rule and the demonstration of it are due to Mr. George Heppel, M.A., of Hammersmith.



## INVARIANTS OF THE CONIC.

**4417** Transformation of the origin of coordinates alone does not alter the values of  $a$ ,  $h$ , or  $b$ , whether the axes be rectangular or oblique. This is seen in (4404).

When the axes are rectangular, turning each through an angle  $\theta$  does not affect the values of

$$\mathbf{4418} \quad ab-h^2, \quad a+b, \quad g^2+f^2, \quad \text{or } c.$$

When the axes are oblique (inclination  $\omega$ ), transformation in any manner does not affect the values of the expressions

$$\mathbf{4422} \quad \frac{ab-h^2}{\sin^2 \omega} \quad \text{and} \quad \frac{a+b-2h \cos \omega}{\sin^2 \omega}.$$

These theorems may be proved by actual transformation by the formulæ in (4048-50). For other methods and additional invariants of the conic, see (4951).

**4424** If the axes of coordinates are oblique, equation (4400) is transformed to the centre in the same way, and equations (4402-6) still hold good. If the final equation referred to axes coinciding with those of the conic be

$$\mathbf{4425} \quad a'x^2 + b'y^2 + c' = 0,$$

and  $\theta$  the inclination of the new axis of  $x$  to the old one, we shall have  $c'$  unaltered,

$$\mathbf{4426} \quad \tan 2\theta = \frac{2h \sin \omega - a \sin 2\omega}{2h \cos \omega - a \cos 2\omega - b};$$

$$\mathbf{4427} \quad a' = \frac{a+b-2h \cos \omega + \sqrt{Q}}{2 \sin^2 \omega}; \quad b' = \frac{a+b-2h \cos \omega - \sqrt{Q}}{2 \sin^2 \omega};$$

where  $Q = a^2 + b^2 + 2ab \cos 2\omega + 4h(a+b) \cos \omega + 4h^2$ .

PROOF.—(4404) is now transformed by the substitutions in (4050), putting  $\beta = \theta + 90^\circ$ , and equating the new  $h$  to zero to determine  $\tan 2\theta$ .  $a'$  and  $b'$  are most readily found from the invariants in (4422). Thus, putting the new  $h = 0$  and the new  $\omega = 90^\circ$ ,

$$a + b' = \frac{a+b-2h \cos \omega}{\sin^2 \omega} \quad \text{and} \quad a'b' = \frac{ab-h^2}{\sin^2 \omega},$$

equations which determine  $a'$  and  $b'$ .

The eccentricity of the general conic (4400) is given by the equation

$$4429 \quad \frac{e^4}{1-e^2} = \frac{(a+b-2h \cos \omega)^2}{(ab-h^2) \sin^2 \omega} - 4.$$

PROOF.—By (4415), and the invariants in (4422).

### THE PARABOLA.

**4430** When  $ab-h^2=0$ , the general equation (4400) represents a parabola.

For  $x', y'$  in (4402) then become infinite and the curve has no centre, or the centre may be considered to recede to infinity.

Turn the axes of coordinates at once through an angle  $\theta$  (4049), and in the transformed equation let the new coefficients be  $a', 2h', b', 2g', 2f', c'$ . Equate  $h'$  to zero; this gives (4408) again,  $\tan 2\theta = \frac{2h}{a-b}$ . If  $\theta$  be the acute angle deter-

mined by this equation, we can decide whether  $\theta$  or  $\theta + \frac{1}{2}\pi$  is the angle between the  $x$  axis and the axis of the parabola by the following rule.

**4431** RULE.—*The inclination of the axis of the parabola to the  $x$  axis of coordinates will be the acute angle  $\theta$  if  $h$  has the opposite sign to that of  $a$  or  $b$ , and  $\theta + \frac{1}{2}\pi$  if it has the same sign.*

PROOF.—Since  $ab-h^2=0$ ,  $a$  and  $b$  have the same sign. Let that sign be positive, changing signs throughout if it is not. Then, for a point at infinity on the curve,  $x$  and  $y$  will take the same sign when the inclination is the acute angle  $\theta$ , and opposite signs when it is  $\theta + \frac{1}{2}\pi$ . But, since  $ax^2+by^2=+\infty$ , we must have  $2hxy=-\infty$ , the terms of the first degree vanishing in comparison. Hence the sign of  $h$  determines the angle as stated in the rule.

$$4432 \quad \sin \theta = \sqrt{\frac{b}{a+b}}, \quad \cos \theta = \sqrt{\frac{a}{a+b}}.$$

PROOF.—From the value of  $\tan 2\theta$  above,  $\theta$  being the acute angle obtained, and from  $h^2=ab$ .

$$4434 \quad \text{Also} \quad a' = 0 \quad \text{and} \quad b' = a+b.$$

For  $a'b' = ab-h^2=0$ , and we ensure that  $a'$  and not  $b'$  vanishes by (4431). Also  $a'+b' = a+b$  (4412).

$$4436 \quad g' = g \cos \theta + f \sin \theta = \frac{g\sqrt{a} + f\sqrt{b}}{\sqrt{a+b}}.$$

$$4438 \quad f' = f \cos \theta - g \sin \theta = \frac{f\sqrt{a} - g\sqrt{b}}{\sqrt{a+b}}.$$

4440 But if  $h$  has the same sign as  $a$  and  $b$ , change  $\theta$  into  $\theta + \frac{1}{2}\pi$ . (4431)

PROOF.—By (4418, 4432-3).

The coordinates of the vertex are

$$4441 \quad x' = \frac{f'^2 - b'c}{2b'g'}, \quad y' = -\frac{f'}{b'}.$$

Obtained by changing the origin to the point  $x'y'$  and equating to zero the coefficient of  $y$  and the absolute term. The coefficient of  $x$  then gives the latus rectum of the parabola; viz.:

$$4443 \quad L = -\frac{2g'}{b'} = -2\frac{g\sqrt{a} + f\sqrt{b}}{\sqrt{a+b}^3}. \quad (4437)$$

#### METHOD WITHOUT TRANSFORMATION OF THE AXES.

4445 Let the general equation (4400) be solved as a quadratic in  $y$ . The result may be exhibited in either of the forms

$$4446 \quad y = ax + \beta \pm \sqrt{\mu(x^2 - 2px + q)},$$

$$4447 \quad y = ax + \beta \pm \sqrt{\mu\{(x-p)^2 + (q-p^2)\}},$$

$$4448 \quad y = ax + \beta \pm \sqrt{\mu(x-\gamma)(x-\delta)},$$

$$4449 \quad \text{where } a = -\frac{h}{b}, \quad \beta = -\frac{f}{b}, \quad \mu = \frac{h^2 - ab}{b^2}.$$

$$4452 \quad p = \frac{hf - bg}{ab - h^2} \quad \text{or} \quad \frac{G}{C}, \quad (1642) \quad q = \frac{bc - f^2}{ab - h^2} \quad \text{or} \quad \frac{A}{C}.$$

$$4454 \quad q - p^2 = \frac{b(abc + 2fgh - af^2 - bg^2 - ch^2)}{(ab - h^2)^2} = \frac{b\Delta}{C^2}.$$

$$4456 \quad \gamma \text{ and } \delta = p \pm \sqrt{(p^2 - q)}.$$

4458 Here  $y = ax + \beta$  is the equation to the diameter  $DD$

(Fig. 31),  $\gamma$  and  $\delta$  are the abscissæ of  $D$  and  $D'$ , its extremities, the tangents at those points being parallel to the  $y$  axis. The surd  $= PN = P'N$  when  $x = OM$ . The axes may be rectangular or oblique.

When  $ab - h^2 = 0$ , equation (4446) becomes

$$4459 \quad y = ax + \beta \pm \frac{1}{b} \sqrt{q' - 2p'x},$$

$$4460 \quad \text{where } p' = bg - hf, \quad q' = f^2 - bc.$$

4462 In this case,  $\frac{q'}{2p'}$  is the abscissa of the extremity of the diameter whose equation is  $y = ax + \beta$  and the curve has infinite branches.

#### RULES FOR THE ANALYSIS OF THE GENERAL EQUATION.

First examine the value of  $ab - h^2$ , and, if this is not zero, calculate the numerical value of  $c'$  (4407), and proceed as in (4400) et seq. If  $ab - h^2$  is zero, find the values of  $p'$  and  $q'$  (4459). The following are the cases that arise.

4464  $ab - h^2$  positive—Locus an ellipse.

##### Particular Cases.

4465  $\Delta = 0$ —Locus the point  $x'y'$ .

See (4402). For, by (4404), the conjugate axes vanish.

4466  $b\Delta$  positive—No locus.

By (4447-54), since  $q - p^2$  is then positive.

4467  $h = 0$  and  $a = b$ —Locus a circle.

By (4144). In other cases proceed as in (4400-14).

4468  $ab - h^2$  negative—Locus an hyperbola.

##### Particular Cases.

4469  $\Delta = 0$ —Locus two right lines intersecting in the point  $x'y'$ .

By (4447), since  $q - p^2$  then vanishes. In this case solve as in (4447).

**4470**  $b\Delta$  negative—Locus the conjugate hyperbola.

**4471**  $a+b=0$ —Locus the rectangular hyperbola.

By (4414), since  $a' = -b'$ .

**4472**  $a=b=0$ —Locus an hyperbola, with its asymptotes parallel to the coordinate axes. The coordinates of the centre are now  $-\frac{f}{h}$  and  $-\frac{g}{h}$ , by (4402). Transfer the origin to the centre, and the equation becomes

$$\mathbf{4473} \quad xy = \frac{2fg - ch}{2h^2}.$$

In other cases proceed as in (4400-14).

**4474**  $ab-h^2=0$ —Locus a parabola.

*Particular Cases.*

**4475**  $p'=0$ —Locus two parallel right lines. By (4459).

**4476**  $p'=q'=0$ —Locus two coinciding right lines.

By (4459).

**4477**  $p'=0$  and  $q'$  negative—No locus.

By (4459). In other cases proceed as in (4430-43).

Ex. 1.:  $2x^2 - 2xy + y^2 + 3x - y - 1 = 0.$

Here the values of  $a, h, b, g, f, c$  are respectively 2, -1, 1,  $\frac{3}{2}$ ,  $-\frac{1}{2}$ , -1,

$$ab-h^2=1; \quad c' = \frac{\Delta}{C} = \frac{abc+2fgh-af^2-bg^2-ch^2}{ab-h^2} = -\frac{9}{4} \quad (4406)$$

The locus is therefore an ellipse, none of the exceptions (4465-7) occurring here. The coordinates of the centre, by (4402), are

$$x' = \frac{hf-bg}{ab-h^2} = -1, \quad y' = \frac{gh-af}{ab-h^2} = -\frac{1}{2}.$$

Hence the equation transformed to the centre is

$$2x^2 - 2xy + y^2 - \frac{9}{4} = 0.$$

Turning the axes of coordinates through an angle  $\theta$  so that  $\tan 2\theta = -2$  (4408), we find the new  $a$  and  $b$  from

$$a'+b'=3, \quad a'b'=1; \quad (4412)$$

therefore  $a' = \frac{1}{2}(3 - \sqrt{5})$ ,  $b' = \frac{1}{2}(3 + \sqrt{5})$ ,  
and the final equation becomes  $2(3 - \sqrt{5})x^2 + 2(3 + \sqrt{5})y^2 = 9$ .

The inclination of the major axis to the original  $x$  axis of coordinates is the acute angle  $\frac{1}{2}\tan^{-1}(-2)$ , by the rule in (4416).

Ex. (2):  $12x^2 + 60xy + 75y^2 - 12x - 8y - 6 = 0$ .

The values of  $a, h, b, g, f, c$  are respectively 12, 30, 75, -6, -4, -6,

$$ab - h^2 = 0; \quad p' = bg - hf = -330; \quad q' = f^2 - bc. \quad (4460)$$

Since  $p'$  does not vanish (4475-7), the locus is a parabola. Proceeding, therefore, by (4430-43), we have

$$\tan 2\theta = \frac{2h}{a-b} = -\frac{20}{21}; \quad \sin \theta = \frac{5}{\sqrt{29}}, \quad \cos \theta = \frac{2}{\sqrt{29}}. \quad (4432)$$

By the rule (4431), we must take  $\theta + \frac{1}{2}\pi$  for the angle, instead of  $\theta$ . There-

fore 
$$g' = -g \sin \theta + f \cos \theta = \frac{-g\sqrt{b} + f\sqrt{a}}{\sqrt{(a+b)}} = \frac{22}{\sqrt{29}},$$

$$f' = -f \sin \theta - g \cos \theta = \frac{-f\sqrt{b} - g\sqrt{a}}{\sqrt{(a+b)}} = \frac{32}{\sqrt{29}}.$$

and  $b' = a + b = 87$  (4435).

Consequently the transformed equation is

$$87y'^2 + \frac{44}{\sqrt{29}}x' + \frac{64}{\sqrt{29}}y' - 6 = 0.$$

The coordinates of the vertex are computed by (4441), and the final equation with the vertex for origin is  $y'^2 = \frac{44}{87\sqrt{29}}x'$ .

Ex. (3):  $x^2 + 6xy + 9y^2 + 5x + 15y + 6 = 0$ .

The values of  $a, h, b, g, f, c$  are respectively 1, 3, 9,  $\frac{5}{2}$ ,  $\frac{15}{2}$ , 6,

$$ab - h^2 = 0 \quad \text{and} \quad p' = bg - hf = 0,$$

therefore, by (4475-7), if there is a locus at all, it consists of two parallel or coinciding lines. Solving the equation therefore as a quadratic in  $y$ , we obtain it in the form  $(x + 3y + 2)(x + 3y + 3) = 0$ ,  
the equation of two parallel right lines.

The equation of the tangent or polar of  $x'y'$  is

**4478**  $u_x x + u_y y + u_z z = 0$  or  $u_x x' + u_y y' + u_z z' = 0$ ;

(4401, 1405) obtained by (4120) in the form

**4479**  $(ax' + hy' + g)x + (hx' + by' + f)y + gx' + fy' + c = 0$ ,

**4480** or  $(ax + hy + g)x' + (hx + by + f)y' + gx + fy + c = 0$ ,



**4481** or

$$axx' + h(xy' + x'y) + byy' + g(x + x') + f(y + y') + c = 0.$$

When the curve passes through the origin, the tangent at the origin is

$$\mathbf{4482} \quad gx + fy = 0. \quad (4479)$$

And the normal at the same point is

$$\mathbf{4483} \quad fx - gy = 0.$$

$$\mathbf{4484} \quad \text{Intercepts of the curve on the axes, } -\frac{2g}{a}, -\frac{2f}{b}.$$

**4486** Length of normal intercepted between the origin and the chord

$$= \frac{\sqrt{g^2 + f^2}}{a + b}. \quad (4483-4)$$

*Right Line and Conic with the general Equation.*

**4487** Quadratic for  $n : n'$ , the ratio in which the line joining  $xy, x'y'$  is cut by the curve.

Let the equation of the curve (4400) be denoted by  $\phi(x, y) = 0$ , and the equation of the tangent (4479) by  $\psi(x, y, x', y') = 0$ ; then the quadratic required will be found, by the method of (4131), to be

$$n^2\phi(x', y') + 2nn'\psi(x, y, x', y') + n'^2\phi(x, y) = 0.$$

The equation of the tangents from  $x'y'$  is

$$\mathbf{4488} \quad \phi(x', y')\phi(x, y) = \{\psi(x, y, x', y')\}^2.$$

PROOF.—By the condition for equal roots in (4487).

COR.—The equation of two tangents through the origin is

$$\mathbf{4489} \quad Bx^2 - 2Hxy + Cy^2 = 0. \quad (4665)$$

The equation of the asymptotes of  $u$  (4400) is

$$\mathbf{4490} \quad au_v^2 + 2hu_xu_v + bu_x^2 = 0.$$

The equation of the equi-conjugates of the conic  $ax^2 + 2hxy + by^2 = 1$  is

$$4491 \quad (a+b)(ax^2 + 2hxy + by^2) = 2(ab - h^2)(x^2 + y^2).$$

PROOF.—When the conic is  $ax^2 + by^2 = 1$ , the similar equation is

$$(a+b)(ax^2 + by^2) = 2ab(x^2 + y^2) \quad \text{or} \quad (ax^2 - by^2) = 0,$$

given by the intersections of the conic and a circle. Transformation of the axes then produces the above by the invariants in (4418).

4492 When the coordinate axes are oblique, the equation becomes

$$(a-b)(ax^2 - by^2) + 2x(hx + by)(h - a \cos \omega) + 2y(ax + hy)(h - b \cos \omega) = 0.$$

General polar equation :

$$4493 \quad (a \cos^2 \theta + 2h \sin \theta \cos \theta + b \sin^2 \theta) r^2 + 2(g \cos \theta + f \sin \theta) r + c = 0.$$

Polar equation with  $(x, y)$  for the pole : (4134)

$$4494 \quad (a \cos^2 \theta + 2h \sin \theta \cos \theta + b \sin^2 \theta) r^2 + 2\{(ax + hy + g) \cos \theta + (by + hx + f) \sin \theta\} r + F(xy) = 0.$$

Equation of the line through  $x'y'$  parallel to the conjugate diameter :

$$4495 \quad (x - x')(ax' + hy' + g) + (y - y')(hx' + by' + f) = 0.$$

PROOF.—By the condition for equal roots of opposite signs (4494).

Equation of the conic with the origin at the extremity of the major axis,  $L$  being the latus rectum.

$$4496 \quad y^2 = Lx - (1 - e^2)x^2. \quad (4269, '59)$$

Equation when the point  $ab$  is the focus and

$Ax + By + C = 0$  the directrix :

$$4497 \quad \sqrt{\{(x-a)^2 + (y-b)^2\}} = e \frac{Ax + By + C}{\sqrt{A^2 + B^2}}. \quad (4200, 4095)$$

## INTERCEPT EQUATION OF A CONIC.

The equation of a conic passing through four points whose intercepts on oblique axes of coordinates are  $s, s'$  and  $t, t'$ , is

$$4498 \quad \frac{x^2}{ss'} + 2hxy + \frac{y^2}{tt'} - x\left(\frac{1}{s} + \frac{1}{s'}\right) - y\left(\frac{1}{t} + \frac{1}{t'}\right) + 1 = 0.$$

Equation of a conic touching oblique axes in the points whose intercepts are  $s$  and  $t$ :

$$4499 \quad \frac{x^2}{s^2} + 2hxy + \frac{y^2}{t^2} - \frac{2x}{s} - \frac{2y}{t} + 1 = 0,$$

$$4500 \quad \text{or} \quad \left(\frac{x}{s} + \frac{y}{t} - 1\right)^2 + vxy = 0.$$

Comparing with the general equation (4400), we have

$$4501 \quad s = -\frac{c}{g}, \quad t = -\frac{c}{f}, \quad v = 2h - \frac{2}{st} = 2 \frac{hc^2 - gf}{c^2}.$$

Perpendicular  $p$  from  $xy$ , any point on the curve, to the chord of contact:

$$4505 \quad p^2 = \frac{rs^2t^2xy \sin^2 \omega}{s^2 + t^2 - 2st \cos \omega}. \quad (4096, 4500)$$

Equation of the tangent at  $x'y'$ :

$$4507 \quad 2\left(\frac{x'}{s} + \frac{y'}{t} - 1\right)\left(\frac{x}{s} + \frac{y}{t} - 1\right) + v(xy' + x'y) = 0.$$

4508 The equation of the director-circle is

$$(1 + \frac{1}{4}stv)(x^2 + y^2 + 2xy \cos \omega) - h(x + y \cos \omega) - k(y + x \cos \omega) + hk \cos \omega = 0.$$

The parabola with the same coordinate axes as in (4499):

$$4509 \quad \left(\frac{x}{s} + \frac{y}{t} - 1\right)^2 = \frac{4xy}{st} \quad \text{or} \quad \sqrt{\frac{x}{s}} + \sqrt{\frac{y}{t}} = 1.$$

PROOF.—From (4500), putting  $h = -\frac{1}{st}$  (4474), and therefore  $v = -\frac{4}{st}$ .

Equation of the tangent at  $x', y'$  :

$$4510 \quad \frac{x}{\sqrt{sx'}} + \frac{y}{\sqrt{ty'}} = 1, \quad (4509)$$

$$4511 \quad \text{or} \quad y = mx + \frac{mst}{ms+t}, \quad m = -\sqrt{\frac{ty'}{sx'}}. \quad (4123)$$

Equation of the normal at  $x', y'$  :

$$4512 \quad y = mx + \frac{s^2t - m^3st^3}{(mt+s)^2}, \quad m = \sqrt{\frac{sx'}{ty'}}. \quad (4122)$$

$$4513 \quad \text{Normal through the origin} \quad x\sqrt{s} = y\sqrt{t}.$$

The equations of two diameters are, with any axes,

$$4514 \quad \frac{x}{s} - \frac{y}{t} = 1 \quad \text{and} \quad \frac{x}{s} - \frac{y}{t} = -1.$$

PROOF.—Diameter through  $0t$ ,  $\frac{y-t}{x} = \frac{t}{s}$  by the property  $OR = RQ$ , in the figure of (4211).

Coordinates of the focus :

$$4516 \quad x = \frac{st^3}{s^2+t^2+2st \cos \omega}, \quad y = \frac{s^2t}{s^2+t^2+2st \cos \omega}. \quad (5009)$$

Equation of the directrix :

$$4518 \quad x(s+t \cos \omega) + y(t+s \cos \omega) = st \cos \omega.$$

PROOF.—Expand (4509), and form the equation of the polar of the focus by (4479) and (4516).

When the axes are also rectangular, the latus rectum

$$4519 \quad L = \frac{4s^2t^2}{(s^2+t^2)^{\frac{3}{2}}}. \quad (4095, 4516-8)$$

4520 Locus of the centre of the conic which touches the axes at the points  $s0, 0t$  :

$$tx = sy. \quad (4500, 4402)$$

4521 To make the conic pass through a point  $x'y'$ ; substitute  $x'y'$  in (4500), and determine  $v$ .

## SIMILAR CONICS.

**4522** DEFINITION.—If two radii, drawn from two fixed points, maintain a constant ratio and a constant mutual inclination, they will describe *similar curves*.

**4523** If the proportional radii be always parallel, the curves are also *similarly situated*.

If there be two conics (1) and (2), with equations of the form (4400), then—

The condition of their being similar and similarly situated is

$$\mathbf{4524} \quad \frac{a}{a'} = \frac{h}{h'} = \frac{b}{b'}.$$

PROOF.—By (4404), changing to polar coordinates,  $r : r' = \text{constant}$ .

The condition of similarity only is

$$\mathbf{4525} \quad \frac{(a+b)^2}{h^2-ab} = \frac{(a'+b')^2}{h'^2-a'b'}; \quad (4418-9)$$

or, with oblique axes,

$$\mathbf{4526} \quad \frac{(a+b-2h \cos \omega)^2}{h^2-ab} = \frac{(a'+b'-2h' \cos \omega)^2}{h'^2-a'b'}. \quad (4422-3)$$

## CIRCLE OF CURVATURE.

## CONTACT OF CONICS.

**4527** DEF.—When two points of intersection of two curves coincide on a common tangent, the curves have a contact of the *first order*; when three such points coincide, a contact of the *second order*; and so on. To osculate, is to have a contact higher than the first.

**4528** The two conics (Fig. 32) whose equations are

$$ax^2 + 2hxy + by^2 + 2gx = 0 \dots\dots\dots(1),$$

$$a'x^2 + 2h'xy + b'y^2 + 2g'x = 0 \dots\dots\dots(2),$$

touch the  $y$  axis at the origin,  $O$ , by (4482). Eliminate the third terms from (1) and (2), and we obtain  $x = 0$ , the line through two coincident points, and

$$4529 \quad (ab' - a'b)x + 2(hb' - h'b)y - 2(bg' - b'g) = 0,$$

the equation to  $LM$ , the line passing through the two remaining points of intersection of (1) and (2). (4099)

Again, eliminate the last terms from (1) and (2), and we obtain

$$4530 \quad (ag' - a'g)x^2 + 2(hg' - h'g)xy + (bg' - b'g)y^2 = 0,$$

the equation of the two lines  $OL$ ,  $OM$ . [By (4111) and (4099)]

4531 If the points  $L$ ,  $M$  coincide, the conics have contact of the first order. The condition for this is that (4530) must have equal roots; therefore

$$4532 \quad (ag' - a'g)(bg' - b'g) = (hg' - h'g)^2.$$

4533 If the conics (1) and (2) are to osculate,  $M$  must coincide with  $O$ . Therefore, in (4529),  $bg' = b'g$ .

If in (4532)  $bg' = b'g$ , the conics have a contact of the third order.

### CIRCLE OF CURVATURE.

(See also 1254 *et seq.*)

The radius of curvature at the origin for the conic

$$ax^2 + 2hxy + by^2 + 2gx = 0,$$

the axes of coordinates including an angle  $\omega$ , is

$$4534 \quad \rho = -\frac{g}{b \sin \omega}.$$

PROOF.—The circle touching the curve at the origin is

$$x^2 + 2xy \cos \omega + y^2 - 2rx \sin \omega = 0,$$

by (4148), and the geometry of the figure,  $2r \sin \omega$  being the intercept on the  $x$  axis. The condition of osculating (4533) gives the value of  $\rho$ .

$\rho$  is positive when the convexity of the curve is towards the  $y$  axis.

Radius of curvature for a central conic at the extremity  $P$  of a semi-diameter  $a'$ , the conjugate being  $b'$ .



$$4535 \quad \rho = \frac{b'^2}{a' \sin \omega} = \frac{b'^2}{p} = \frac{a^2 b^2}{p^3} = \frac{b^3}{ab}. \quad (4367)$$

PROOF.—Take the equation and figure of (4346) ( $a' = CP$ ). Transform to parallel axes through  $P$ . Then by (4534).

The same in terms of  $x, y$ , the coordinates of the point  $P$ .

$$4539 \quad \rho = \frac{(b^4 x^2 + a^4 y^2)^{\frac{3}{2}}}{a^2 b^2}.$$

PROOF.—By (5138), or from (4538) and the value of  $b$  at (4365).

The coordinates of the centre of curvature  $O$  for  $P$ , the point  $xy$ , are

$$4540 \quad \xi = \frac{c^2 x^3}{a^4}, \quad \eta = -\frac{c^2 y^3}{b^4}, \quad \text{where } c^2 = a^2 - b^2.$$

PROOF.—(Fig. 33.) From  $\frac{OD}{OP} = \frac{x - \xi}{\rho} = \frac{x}{PG'}$  and  $\frac{PD}{OP} = \frac{y - \eta}{\rho} = \frac{y}{PG'}$ , with the values of  $\rho$ ,  $PG$ , and  $PG'$  at (4535) and (4309).

Radius of curvature for the parabola.

Taking the diameter and tangent through the point for axes,

$$4542 \quad \rho = \frac{2a'}{\sin \theta} = \frac{2a}{\sin^3 \theta} = \frac{2SP^2}{SY}. \quad (\text{Fig. of 4201})$$

By (4534), and equation (4211).

Coordinates of the centre of curvature at  $xy$  (rectangular axes):

$$4545 \quad \xi = 3x + 2a, \quad \eta = -\frac{y^3}{4a^2}.$$

PROOF.—From  $y - \eta : \rho = y : PG$  and  $\rho = 2a \operatorname{cosec}^3 \theta$ ,  $PG = 2a \operatorname{cosec} \theta$  and  $y = 2a \cot \theta$ .

The evolute of a central conic (Fig. 33):

$$4547 \quad (ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}},$$

$$4548 \quad \text{or} \quad (a^2 x^2 + b^2 y^2 - c^4)^3 + 27a^2 b^2 c^4 x^2 y^2 = 0,$$

where  $c^2 = a^2 - b^2$ .

PROOF.—Substitute for  $x, y$  in the equation of the conic (4273) their values in terms of  $\xi, \eta$  from (4540). Otherwise as in (4958), or by the method of (5157).

The curve has cusps at  $L, H, M$ , and  $K$ .

The evolute of the parabola :

$$4549 \quad a \left( \frac{y}{2} \right)^2 = \left( \frac{x-2a}{3} \right)^3.$$

PROOF.—As in (4548), from the equations (4201) and (4545).

## CONFOCAL CONICS.

$$4550 \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$$

are confocal conics, if

$$a^2 - a'^2 = b^2 - b'^2,$$

or the sign of  $b'^2$  may be changed.

For the confocal of the general conic, see (5007).

**4551** Confocal conics intersect, if at all, at right angles.

PROOF.—If  $u, u'$  are the two conics in (4550), changing the sign of  $b'^2$  to make the second conic an hyperbola,  $u - u' = 0$  will be satisfied at their point of intersection; and (by  $a^2 - a'^2 = b^2 + b'^2$ ) this proves the tangents at that point to be at right angles (4078, 4280).

Otherwise geometrically by (1168).

**4552** Tangents from a point  $P$  on one conic to a confocal conic make equal angles with the tangent at  $P$ . [Proof at (1291)]

**4553** The locus of the pole of the line  $Ax + By + C$  with respect to a series of confocal conics in which  $a^2 - b^2 = \lambda$ , is the right line perpendicular to the given one,

$$BCx - ACy + AB\lambda = 0.$$

PROOF.—The pole of the line for any of the conics being  $xy$ ;  $Aa^2 = -Cx$  and  $Bb^2 = -Cy$  (4292); also  $a^2 - b^2 = \lambda$ . Eliminate  $a^2$  and  $b^2$ .

**4554** COR.—If the given line touch one of the conics, the locus is the normal at the point of contact.

**4555 Graves' Theorem.**—The two tangents drawn to an ellipse from a point on a confocal ellipse together exceed the intercepted arc by a constant quantity.

PROOF.—(Fig. 132.) Let  $P, P'$  be consecutive points on the confocal from which the tangents are drawn. Let fall the perpendiculars  $PN, P'N'$ . From (1291), it follows that  $\angle PP'N = P'PN'$ , and therefore  $P'N = PN'$ . The increment in the sum of the tangents in passing from  $P$  to  $P'$  is

$$RR' - QQ' + P'N - PN = RR' - QQ'.$$

But this is also the increment in the arc  $QR$ , which proves the theorem.

**4556** If the tangents are drawn from a confocal hyperbola, as in (Fig. 133), the difference of the tangents  $PQ, PR$  is equal to the difference of the arcs  $QT, RT$ .

The proof is quite similar to the foregoing.

**4557** At the intersection of two confocal conics, the centre of curvature of either is the pole of its tangent with respect to the other.

PROOF.—Take  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  (i.) and  $\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1$  (ii.) for confocal conics. At the point of intersection,  $x^2 = \frac{a^2 a'^2}{c^2}$  and  $y^2 = -\frac{b^2 b'^2}{c^2}$  (where  $c^2 = a^2 - b^2$ ); by  $a^2 - a'^2 = b^2 + b'^2$ . The coordinates of the centre of curvature of  $x'y'$  in (i.) are  $x'' = \frac{c^2 x^3}{a^4}$ ,  $y'' = -\frac{c^2 y^3}{b^4}$  (4540-1). The polar of this point with respect to (ii.) will be  $\frac{xx''}{a'^2} + \frac{yy''}{b'^2} = 1$ . Substitute the values of  $x'', y''$ ; and we see, by the values of  $x', y'$ , that this is also the tangent of (i.) at  $P$ .

**4558** A system of coaxal circles (4161), reciprocated with respect to one of the limiting points  $D$  or  $D'$ , becomes a system of confocal conics.

PROOF.—The origin  $D$  is one common focus of the reciprocal conics, by (4844). The polar of  $D$  with respect to any of the circles is the *same* line, by (4166).  $D$  and its polar (both fixed) reciprocate (4858) into the line at infinity and its polar, which is the centre of the conic. The centre and one focus being the same for all, the conics are confocal.

# ANALYTICAL CONICS

## IN

## TRILINEAR COORDINATES.

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### THE RIGHT LINE.

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For a description of this system of coordinates, see (4006). The square of the distance between two points  $a\beta\gamma$ ,  $a'\beta'\gamma'$  is, with the notation of (4008),

**4601**

$$\frac{abr}{\Sigma^2} \{a(\beta - \beta')(\gamma - \gamma') + b(\gamma - \gamma')(a - a') + r(a - a')(\beta - \beta')\},$$

**4602**

$$= \frac{abr}{\Sigma^2} \{a \cos A (a - a')^2 + b \cos B (\beta - \beta')^2 + r \cos C (\gamma - \gamma')^2\}.$$

PROOF.—Let  $P$ ,  $Q$  be the points. By drawing the coordinates  $\beta\gamma$ ,  $\beta'\gamma'$ , it is easily seen, by (702), that

$$PQ^2 = [(\beta - \beta')^2 + (\gamma - \gamma')^2 + 2(\beta - \beta')(\gamma - \gamma') \cos A] \operatorname{cosec}^2 A \dots (1).$$

Now, by (4007),  $a(a - a') + b(\beta - \beta') + r(\gamma - \gamma') = 0$ ,

from which  $b(\beta - \beta')^2 = -a(a - a')(\beta - \beta') - r(\beta - \beta')(\gamma - \gamma')$ ,

and a similar expression for  $r(\gamma - \gamma')^2$ . Substitute these values of the square terms in (1), reducing by (702).

Coordinates of the point which divides the straight line joining the points  $a\beta\gamma$ ,  $a'\beta'\gamma'$  in the ratio  $l : m$ :

**4603**  $\frac{la + ma'}{l + m}, \quad \frac{l\beta + m\beta'}{l + m}, \quad \frac{l\gamma + m\gamma'}{l + m}.$  By (4032).

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$ABC$  being the triangle of reference, and  $a = 0$ ,  $\beta = 0$ ,  $\gamma = 0$  the equations of its sides, the equation of a line passing through the intersection of the lines  $a = 0$ ,  $\beta = 0$  is

**4604**

$$la - m\beta = 0 \quad \text{or} \quad a - k\beta = 0.$$

4 K

PROOF.—For this is the locus of a point whose coordinates  $\alpha, \beta$  are in the constant ratio  $m : l$  or  $k$  (4099).

When  $l$  and  $m$  have the same sign, the line divides the external angle  $C$  of the triangle  $ABC$ ; when of opposite sign, the internal angle  $C$ .

The general equation of a straight line is

$$4605 \quad l\alpha + m\beta + n\gamma = 0,$$

and it may be referred to as the line  $(l, m, n)$ .

PROOF:  $l\alpha + m\beta = 0$  is any line through the point  $C$ , and  $(l\alpha + m\beta) + n\gamma = 0$  is any line through the intersection of the former line and the line  $\gamma = 0$  (4604), and therefore any line whatever according to the values of the arbitrary constants  $l, m, n$ .

The same straight line in Cartesian coordinates is

$$4606 \quad (l \cos \alpha + m \cos \beta + n \cos \gamma) x \\ + (l \sin \alpha + m \sin \beta + n \sin \gamma) y - (lp_1 + mp_2 + np_3) = 0.$$

PROOF.—By substituting the values of  $\alpha, \beta, \gamma$  at (4009).

Or, if the equations of the sides of  $ABC$  are given in the form  $A_1x + B_1y + C_1 = 0$ , &c., the line becomes

$$4607 \quad (lA_1 + mA_2 + nA_3)x + (lB_1 + mB_2 + nB_3)y \\ + lC_1 + mC_2 + nC_3 = 0.$$

PROOF.—By (4095), the denominators like  $\sqrt{(A_1^2 + B_1^2)}$  being included in the constants  $l, m, n$ .

4608 If  $u = 0, v = 0, w = 0$  are the general equations of the lines  $\alpha, \beta, \gamma$ , then it is obvious that  $lu + mv = 0$  is, like (4604), a line passing through the intersection of  $u$  and  $v$ , and  $lu + mv + nw = 0$  represents any straight line whatever.

To make an equation such as  $\alpha = p$  (a constant) homogeneous in  $\alpha, \beta, \gamma$ ; multiply by the equation  $\Sigma = a\alpha + b\beta + c\gamma$  (4007), thus

$$(ap - \Sigma)\alpha + bp\beta + cp\gamma = 0,$$

which is of the same form as (4605).

**4610** The point of intersection of the lines

$$la + m\beta + n\gamma = 0 \quad \text{and} \quad l'a + m'\beta + n'\gamma = 0$$

is determined by the ratios

$$\frac{a}{mn' - m'n} = \frac{\beta}{nl' - n'l} = \frac{\gamma}{lm' - l'm} \quad \text{and (4017).}$$

The values of  $a$ ,  $\beta$ ,  $\gamma$  are therefore

$$\mathbf{4611} \quad \frac{\Sigma(mn' - m'n)}{D}, \quad \frac{\Sigma(nl' - n'l)}{D}, \quad \frac{\Sigma(lm' - l'm)}{D},$$

where  $D = a(mn' - m'n) + b(nl' - n'l) + c(lm' - l'm)$ .

PROOF.—By (4017), or by solving the three equations

$$aa + b\beta + c\gamma = \Sigma, \quad la + m\beta + n\gamma = 0, \quad l'a + m'\beta + n'\gamma = 0.$$

The equation

$$\mathbf{4612} \quad aa + b\beta + c\gamma = 0 \quad \text{or} \quad a \sin A + \beta \sin B + \gamma \sin C = 0$$

represents a straight line at infinity.

PROOF.—The coordinates of its intersection with any other line  $la + m\beta + n\gamma = 0$  are infinite by (4611).

**4613** NOTE:  $aa + b\beta + c\gamma = \Sigma$ , a quantity not zero. The equation  $aa + b\beta + c\gamma = 0$  is therefore in itself impossible, and so is a line infinitely distant. The two conceptions are, however, *together consistent*; the one involves the other. And if, in the equation  $la + m\beta + n\gamma = 0$ , the ratios  $l : m : n$  approach the values  $a : b : c$ , the line it represents recedes to an unlimited distance from the trigon.

**4614** The equation corresponding to (4612) in Cartesian coordinates is  $0x + 0y + C = 0$ , the intercepts on the axes being both infinite. Cartesian coordinates may therefore be regarded as trilinear with the  $x$  and  $y$  axes for two sides of the trigon and the other side at an infinite distance.

**4615** The condition that three points  $a_1\beta_1\gamma_1$ ,  $a_2\beta_2\gamma_2$ ,  $a_3\beta_3\gamma_3$  may lie on the same straight line is the determinant equation,

$$\begin{vmatrix} a_1 & \beta_1 & \gamma_1 \\ a_2 & \beta_2 & \gamma_2 \\ a_3 & \beta_3 & \gamma_3 \end{vmatrix} = 0.$$

PROOF.—For it is the eliminant of the three simultaneous equations,

$$la_1 + m\beta_1 + n\gamma_1 = 0, \quad la_2 + m\beta_2 + n\gamma_2 = 0, \quad la_3 + m\beta_3 + n\gamma_3 = 0. \quad (583)$$



**4616** COR.—The above is also the equation of a straight line passing through two of the fixed points if the third point be considered variable.

**4617** Similarly, the condition that the three following straight lines may pass through the same point, is the determinant equation on the right,

$$\begin{array}{l} l_1\alpha + m_1\beta + n_1\gamma = 0 \\ l_2\alpha + m_2\beta + n_2\gamma = 0 \\ l_3\alpha + m_3\beta + n_3\gamma = 0, \end{array} \quad \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.$$

**4618** The condition of parallelism of the two straight lines

$$\begin{array}{l} l\alpha + m\beta + n\gamma = 0, \\ l'\alpha + m'\beta + n'\gamma = 0, \end{array} \quad \begin{vmatrix} l & m & n \\ l' & m' & n' \\ a & b & c \end{vmatrix} = 0,$$

is the determinant equation

PROOF.—By taking the line at infinity (4612) for the third line in (4617).

**4619** Otherwise the equations of two parallel lines differ by a constant (4076). Thus

$$l\alpha + m\beta + n\gamma + k(a\alpha + b\beta + c\gamma) = 0 \quad (4007)$$

or  $(l+ka)\alpha + (m+kb)\beta + (n+kc)\gamma = 0$

represents any line parallel to  $l\alpha + m\beta + n\gamma = 0$  by varying the value of  $k$ .

The condition of perpendicularity of the two lines in (4618) is

$$\begin{aligned} \mathbf{4620} \quad ll' + mm' + nn' - (mn' + m'n) \cos A - (nl' + n'l) \cos B \\ - (lm' + l'm) \cos C = 0, \end{aligned}$$

$$\begin{aligned} \mathbf{4621} \quad \text{or } l'(l - m \cos C - n \cos B) + m'(m - n \cos A - l \cos C) \\ + n'(n - l \cos B - m \cos A) = 0. \end{aligned}$$

PROOF.—Transform the two equations into Cartesians, by (4606), and apply the test  $AA' + BB' = 0$  (4078), remembering that

$$\cos(\beta - \gamma) = -\cos A, \text{ \&c. (4011).}$$

When the second line is  $AB$  or  $\gamma = 0$ , the condition is

$$4622 \quad n = m \cos A + l \cos B.$$

It also appears, by (4676), that (4620) is the condition that the two lines may be conjugate with respect to the conic whose tangential equation is

$$4623 \quad l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C = 0.$$

The length of the perpendicular from a point  $a'\beta'\gamma'$  to the line  $la + m\beta + n\gamma = 0$ :

$$4624 \quad \frac{la' + m\beta' + n\gamma'}{\sqrt{l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C}}.$$

PROOF.—By (4095) the perpendicular is equal to the form in (4606), with  $x', y'$  in the place of  $x, y$ , divided by the square root of sum of squares of coefficients of  $x$  and  $y$ . The numerator  $= la' + m\beta' + n\gamma'$ . The denominator reduces by  $\cos(\beta - \gamma) = -\cos A$ , &c.

4625 Equation of the same perpendicular:

$$\begin{vmatrix} a & a' & l - m \cos C - n \cos B \\ \beta & \beta' & m - n \cos A - l \cos C \\ \gamma & \gamma' & n - l \cos B - m \cos A \end{vmatrix} = 0.$$

PROOF.—This is the eliminant of the three conditional equations  $La + M\beta + N\gamma = 0$ ,  $La' + M\beta' + N\gamma' = 0$ , and equation (4621).

4626 Equation of a line drawn through  $a'\beta'\gamma'$  parallel to the line  $(l, m, n)$ :

$$\begin{vmatrix} a & a' & am - bn \\ \beta & \beta' & an - bl \\ \gamma & \gamma' & bl - am \end{vmatrix} = 0.$$

PROOF.—It is the eliminant of the three conditional equations  $la + m\beta + n\gamma = 0$ ,  $la' + m\beta' + n\gamma' = 0$ , and the equation at (4618).

4627 The tangent of the angle between the lines  $(l, m, n)$  and  $(l', m', n')$  is

$$\frac{(mn' - m'n) \sin A + (nl' - n'l) \sin B + (lm' - l'm) \sin C}{ll' + mm' + nn' - (mn' + m'n) \cos A - (nl' + n'l) \cos B - (lm' + l'm) \cos C}$$

PROOF.—By (4071) applied to the transformed equations of the lines, (4606), observing (4007).

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EQUATIONS OF PARTICULAR LINES AND COORDINATE RATIOS  
OF PARTICULAR POINTS IN THE TRIGON.

**4628** Bisectors of the angles  $A, B, C$ :

$$\beta - \gamma = 0, \quad \gamma - \alpha = 0, \quad \alpha - \beta = 0.$$

**4629** Centre of inscribed circle (or *in-centre*)\*  $1 : 1 : 1$ .

The coordinates are obtained from their mutual ratios by the formula (4017).

**4630** Bisectors of the angles  $A, \pi - B, \pi - C$ :

$$\beta - \gamma = 0, \quad \gamma + \alpha = 0, \quad \alpha + \beta = 0.$$

Centre of the escribed circle which touches the side  $a$  (or  
*a ex-circle*)  $-1 : 1 : 1$ .

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**4631** Bisectors of sides drawn through opposite vertices:

$$\beta \sin B = \gamma \sin C, \quad \gamma \sin C = \alpha \sin A, \quad \alpha \sin A = \beta \sin B.$$

**4632** Point of intersection (or *mass-centre*):

$$\operatorname{cosec} A : \operatorname{cosec} B : \operatorname{cosec} C.$$

PROOF.—Assume  $m\beta - n\gamma = 0$ , by (4604), as the form of the equation of a line through  $A$ , and determine the ratio  $m : n$  from the value of  $\gamma : \beta$  when  $\alpha = 0$ .

The coordinates of the point of intersection may be found by (4610), or thus:

$$\begin{aligned} \alpha : \beta &= \sin B : \sin A = \operatorname{cosec} A : \operatorname{cosec} B, \\ \beta : \gamma &= \sin C : \sin B = \operatorname{cosec} B : \operatorname{cosec} C, \end{aligned}$$

therefore

$$\alpha : \beta : \gamma = \operatorname{cosec} A : \operatorname{cosec} B : \operatorname{cosec} C.$$


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**4633** Perpendiculars to sides drawn through opposite vertices:

$$\beta \cos B = \gamma \cos C, \quad \gamma \cos C = \alpha \cos A, \quad \alpha \cos A = \beta \cos B.$$

**4634** Orthocentre:  $\sec A : \sec B : \sec C$ .

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\* This nomenclature is suggested by Professor Hudson, who proposes the following:—  
“*In-circle*, *circum-circle*, *a ex-circle* ... *mid-circle* for inscribed circle, circumscribed circle, circle escribed to the side  $a$ , and nine-point circle; also *in-centre*, *circum-centre*, *a ex-centre*, ... *mid-centre*, for the centres of these circles; and *in-radius*, *circum-radius*, *a ex-radius*, ... *mid-radius*, for their radii; *central line*, for the line on which the circum-centre, mid-centre, ortho-centre, and mass-centre lie; and *central length* for the distance between the circum-centre and the ortho-centre.”

If the Cartesian coordinates of  $A, B, C$  be  $x_1y_1, x_2y_2, x_3y_3$ , the coordinates of the centre of the inscribed circle are

$$4635 \quad x = \frac{ax_1 + bx_2 + cx_3}{a + b + c}, \quad y = \frac{ay_1 + by_2 + cy_3}{a + b + c}.$$

PROOF.—By (4032). Find the coordinates of  $D$  where the bisector of the angle  $A$  cuts  $BC$  in the ratio  $b : c$  (VI. 3), and then the coordinates of  $E$  where the bisector of  $B$  cuts  $AD$  in the ratio  $b + c : a$ .

4636 For the coordinates of the centre of the  $a$  ex-circle, change the sign of  $a$  in the above values of  $x$  and  $y$ .

4637 The coordinates of the mass-centre are

$$x = \frac{1}{3}(x_1 + x_2 + x_3), \quad y = \frac{1}{3}(y_1 + y_2 + y_3).$$

4638 The coordinates of the orthocentre are obtained from the equations of the perpendiculars from  $x_2y_2, x_3y_3$ , viz.,

$$(x_1 - x_3)x + (y_1 - y_3)y = x_2(x_1 - x_3) + y_2(y_1 - y_3),$$

$$(x_1 - x_2)x + (y_1 - y_2)y = x_3(x_1 - x_2) + y_3(y_1 - y_2).$$

Perpendicular bisector of the side  $AB$ :

$$4639 \quad a \sin A - \beta \sin B + \gamma \sin(A - B) = 0,$$

$$4640 \quad \text{or} \quad a \cos A - \beta \cos B - \frac{c}{2} \sin(A - B) = 0,$$

4641 or

$$\left(a + \frac{a \sin B \sin C}{2 \sin A}\right) \cos A - \left(\beta + \frac{b \sin C \sin A}{2 \sin B}\right) \cos B = 0.$$

4642 Centre of circumscribed circle (or *circum-centre*):

$$\cos A : \cos B : \cos C.$$

PROOF.—A line through the intersection of  $\gamma$  and  $a \sin A - \beta \sin B$  (4631) is of the form  $a \sin A - \beta \sin B + n\gamma = 0$ , and, by (4622),

$$n = -\sin B \cos A + \sin A \cos B = \sin(A - B).$$

Otherwise, by (4633) and (4619),

$$a \cos A - \beta \cos B + k = 0$$

is any line perpendicular to  $AB$ ; and the constant  $k$  is found by giving  $a : \beta$  the value which it has at the centre of  $AB$ .

**4643** Centre of the nine-point circle (or *mid-centre*):

$$\cos (B-C) : \cos (C-A) : \cos (A-B).$$

PROOF.—By (955) the coordinates are the arithmetic means of the corresponding coordinates of the orthocentre and circum-centre. Therefore, by (4634, '42) and (4017),

$$a = k \left\{ \frac{\sec A}{\sin A \sec A + \sin B \sec B + \sin C \sec C} + \frac{\cos A}{\sin A \cos A + \sin B \cos B + \sin C \cos C} \right\},$$

which reduces to  $\cos (B-C) \times \text{constant}$ .

**4644** Ex. 1.—In any triangle  $ABC$  (Fig. of 955), the mass-centre  $R$ , the orthocentre  $O$ , and the circum-centre  $Q$  lie on the same straight line;\* for the coordinates of these points given at (4632, '34, '42), substituted in (4615), give for the value of the determinant

$$\operatorname{cosec} A (\sec B \cos C - \cos B \sec C) + \&c.,$$

which vanishes.

Similarly, by the coordinates in (4643), it may be shown that the mid-centre  $N$  lies on the same line.

Equation of the central line:

Ex. 2.—To find the line drawn through the orthocentre and mass-centre of  $ABC$ . The coordinates of these points are given at (4632, '34). Substituting in the determinant (4616) and reducing, the equation becomes

$$a \sin 2A \sin (B-C) + \beta \sin 2B \sin (C-A) + \gamma \sin 2C \sin (A-B) = 0.$$

Ex. 3.—Similarly, from (4629, '42), the line drawn through the centres of the inscribed and circumscribed circles is

$$a (\cos B - \cos C) + \beta (\cos C - \cos A) + \gamma (\cos A - \cos B) = 0.$$

Ex. 4.—A parallel to  $AB$  drawn through  $C$ :

$$a \sin A + \beta \sin B = 0.$$

For this is a line through  $a\beta$ , by (4604), and the equation differs only by a constant from  $\gamma = 0$ , for it may be written

$$(a \sin A + \beta \sin B + \gamma \sin C) - \gamma \sin C = 0.$$

Ex. 5.—A perpendicular to  $BC$  drawn through  $C$  is

$$a \cos C + \beta = 0.$$

For a perpendicular is  $\beta \cos B - \gamma \cos C = 0$  (4633) ..... (1), and a line through  $C$  is of the form  $la + m\beta = 0$ . Hence, by (4619), the constant  $k(a \sin A + \beta \sin B + \gamma \sin C)$  must be added to (1) so as to eliminate  $\gamma$ . Thus

$$\begin{aligned} \beta \sin C \cos B + a \sin A \cos C + \beta \sin B \cos C &= 0, \\ \beta \sin (B+C) + a \sin A \cos C &= 0 \quad \text{or} \quad \beta + a \cos C = 0. \end{aligned}$$

\* The central line. See note to (4629).

ANHARMONIC RATIO.

For the definition, see (1052).

**4648** The three ratios of that article are the values of the ratio  $k : k'$  in the three following pencils of four lines respectively—

$$\alpha = 0, \quad \alpha - k\beta = 0, \quad \beta = 0, \quad \alpha + k'\beta = 0 \dots \text{(i.) (Fig. 34),}$$

$$\alpha = 0, \quad \alpha - k\beta = 0, \quad \alpha - k'\beta = 0, \quad \beta = 0 \dots \text{(ii.) (Fig. 35),}$$

$$\alpha = 0, \quad \beta = 0, \quad \alpha + k\beta = 0, \quad \alpha + k'\beta = 0 \dots \text{(iii.) (Fig. 36).}$$

**4649** The anharmonic ratio (i.) becomes harmonic when  $k = k'$ . Hence the lines  $\alpha + k\beta$ ,  $\alpha - k\beta$  form a harmonic pencil with the lines  $\alpha$ ,  $\beta$ , the first dividing the external and the second the internal angle between  $\alpha$  and  $\beta$  (Fig. 37).

**4650** Similarly, the anharmonic ratio of four lines whose equations are

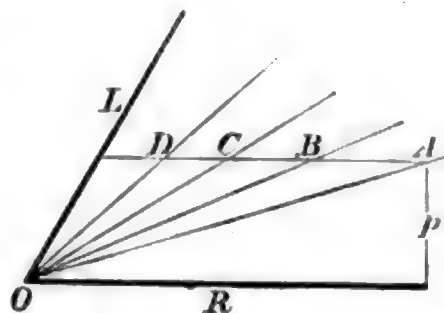
$$\alpha - \mu_1\beta = 0, \quad \alpha - \mu_2\beta = 0, \quad \alpha - \mu_3\beta = 0, \quad \alpha - \mu_4\beta = 0,$$

is the fraction 
$$\frac{(\mu_1 - \mu_2)(\mu_3 - \mu_4)}{(\mu_1 - \mu_4)(\mu_2 - \mu_3)}.$$

PROOF.—Let  $OL$  be the line  $\alpha = 0$ , and  $OR$ ,  $\beta = 0$ .

$\mu_1 - \mu_2$  = difference of perpendiculars from  $A$  and  $B$  upon  $OL$ , divided by  $p$ .

Similarly,  $\mu_3 - \mu_4$ , &c. These differences are proportional to the segments  $AB$ ,  $CD$ ,  $AD$ ,  $BC$ , and  $p$  is a common divisor.



**4651** *Homographic* pencils of lines are those which have the same anharmonic ratio. Thus the two pencils

$$\alpha - \mu_1\beta, \quad \alpha - \mu_2\beta, \quad \alpha - \mu_3\beta, \quad \alpha - \mu_4\beta,$$

and 
$$\alpha' - \mu_1\beta', \quad \alpha' - \mu_2\beta', \quad \alpha' - \mu_3\beta', \quad \alpha' - \mu_4\beta',$$

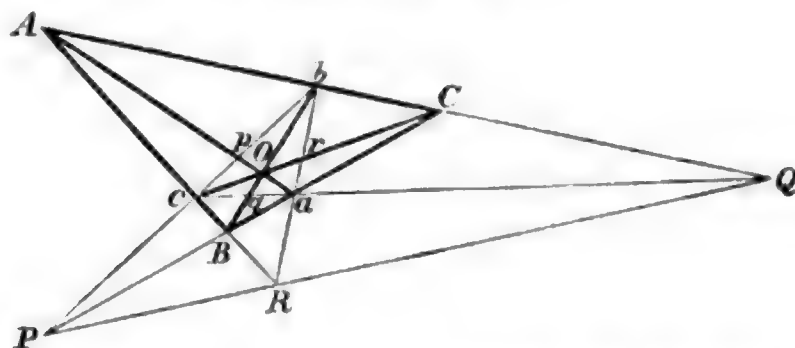
are homographic pencils.



## THE COMPLETE QUADRILATERAL.

**4652** DEF.—Any four right lines together with the three, called diagonals, which join the points of intersection, make a figure called a complete quadrilateral.

**4653** Let  $O$  be any point in the plane of the trigon  $ABC$ . Draw  $AOa$ ,  $BOb$ ,  $COc$ , and complete the figure. The equations of the different lines may be written as under, with the aid of proposition (4604), the ratios  $l : m : n$  being arbitrary and dependent upon the position of  $O$ .



$$\begin{array}{ll} Aa, & m\beta - n\gamma = 0, & AP, & m\beta + n\gamma = 0, \\ Bb, & n\gamma - la = 0, & BQ, & n\gamma + la = 0, \\ Cc, & la - m\beta = 0, & CR, & la + m\beta = 0; \end{array}$$

$$\begin{array}{ll} bc, & m\beta + n\gamma - la = 0, & OP, & m\beta + n\gamma - 2la = 0, \\ ca, & n\gamma + la - m\beta = 0, & OQ, & n\gamma + la - 2m\beta = 0, \\ ab, & la + m\beta - n\gamma = 0, & OR, & la + m\beta - 2n\gamma = 0, \end{array}$$

$$PQR, \quad la + m\beta + n\gamma = 0.$$

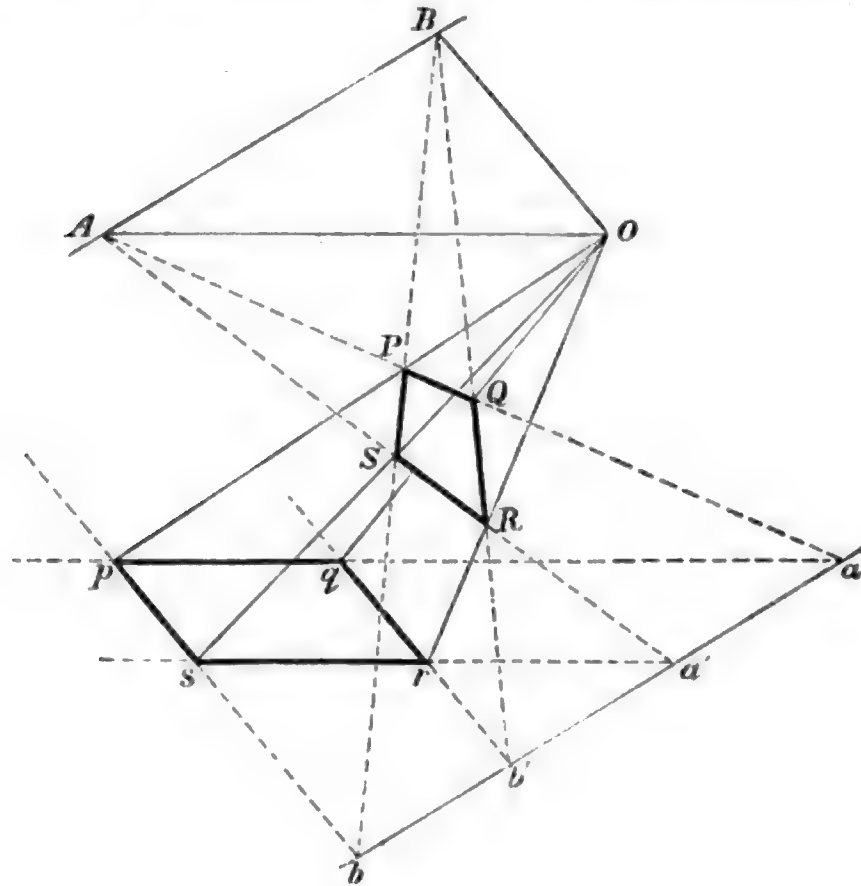
PROOF.— $Aa$ ,  $Bb$ ,  $Cc$  are concurrent by addition.  $bc$  is concurrent with  $Bb$  and  $\beta$ , and with  $Cc$  and  $\gamma$ , by (4604).  $AP$  and  $OP$  are each concurrent with  $bc$  and  $a$ .  $PQR$  is concurrent with each pair of lines  $bc$  and  $a$ ,  $ca$  and  $\beta$ ,  $ab$  and  $\gamma$ . Similarly for the rest.

**4654** Every pencil of four lines in the above figure (supplying  $AP$ ,  $BQ$ ,  $CR$ ) is a harmonic pencil.

PROOF.—By the test in (4649), the alternate pairs of equations being the sum and difference of the other two in every case.

Otherwise by projection. Let  $PQRS$  be the quadrilateral, with diagonals  $RP$ ,  $QS$  meeting in  $O$ . (Supply the lines  $AC$ ,  $BC$  in the figure.) Taking the plane of projection parallel to  $OAB$ , the figure projects into the parallelogram  $pqrs$ ; the points  $A$ ,  $B$  pass to infinity, and therefore the lines  $AO$ ,  $BC$  become

lines harmonically divided by the sides of the parallelogram, the centre, and the points at infinity.



**4655** Theorem (974) may be proved by taking  $a, \beta, \gamma$  for the lines  $BC, CA, AB$ , and  $l'a + m\beta + n\gamma, la + m'\beta + n'\gamma, la + m\beta + n'\gamma$  for  $bc, ca, ab$ , the last form being deduced from the preceding by the concurrence of  $Aa, Bb$ , and  $Cc$ .

## THE GENERAL EQUATION OF A CONIC.

The general equation of the second degree is

**4656**  $aa^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma a + 2ha\beta = 0.$

This equation will be denoted by  $\phi(a, \beta, \gamma) = 0$  or  $u = 0$ .

Equation of the tangent or polar :

**4657**  $u_a a + u_\beta \beta + u_\gamma \gamma = 0$  or  $u_a a' + u_\beta \beta' + u_\gamma \gamma' = 0,$

the two forms being equivalent and the notation being that of (1405). The first equation written in full is

**4659**

$$(aa' + h\beta' + g\gamma') a + (ha' + b\beta' + f\gamma') \beta + (ga' + f\beta' + c\gamma') \gamma = 0.$$

PROOF.—By the methods in (4120). Otherwise by (4678); let  $a\beta\gamma$  be on the curve; then  $\phi(a, \beta, \gamma) = 0$ . Next let the point where the line cuts the curve move up to  $a\beta\gamma$ . Then the line becomes a tangent and the ratio  $n : n'$  vanishes; the condition for this gives equation (4658).

COR.—The polars of the vertices of the triangle of reference are

$$4660 \quad aa + h\beta + g\gamma = 0, \quad ha + b\beta + f\gamma = 0, \quad ga + f\beta + c\gamma = 0.$$

4661 The condition that  $u$  may break up into two linear factors representing two right lines is, by (4469),  $\Delta = 0$ , where

$$4662 \quad \Delta = abc + 2fgh - af^2 - bg^2 - ch^2. \quad (4454)$$

4663 The general tangential equation of the conic (4656) expresses the condition that the line  $\lambda a + \mu \beta + \nu \gamma$  may touch the curve and is the determinant equation annexed. The same written in full is

$$\begin{vmatrix} a & h & g & \lambda \\ h & b & f & \mu \\ g & f & c & \nu \\ \lambda & \mu & \nu & \end{vmatrix} = 0.$$

$$4664 \quad (bc - f^2) \lambda^2 + (ca - g^2) \mu^2 + (ab - h^2) \nu^2 \\ + 2(gh - af) \mu \nu + 2(hf - bg) \nu \lambda + 2(fg - ch) \lambda \mu = 0,$$

$$4665 \quad \text{or } A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu = 0;$$

writing, as in (1642),

$$\begin{aligned} A &= bc - f^2, & B &= ca - g^2, & C &= ab - h^2, \\ F &= gh - af, & G &= hf - bg, & H &= fg - ch. \end{aligned}$$

The tangential equation will be denoted by  $\Phi(\lambda, \mu, \nu) = 0$  or  $U = 0$ , to correspond with (4656).

PROOF.—The determinant is the eliminant of the equation of the line  $\lambda a + \mu \beta + \nu \gamma = 0$ , and the three equations obtained by equating  $\lambda, \mu, \nu$  to the coefficients of  $a, \beta, \gamma$  in (4659).

Otherwise.—Assume  $\lambda a + \mu \beta + \nu \gamma = 0$  for the tangent. Substitute the value of the ratio  $\beta : \gamma$  obtained from it in the equation of the curve, and express the condition for equal roots (4119).

4666 Conversely, if the line  $\lambda a + \mu \beta + \nu \gamma$  has the coefficients  $\lambda, \mu, \nu$  connected by the equation of the second degree  $U = 0$  (4664), then the envelope of the line is the conic in the

$$\begin{vmatrix} A & H & G & a \\ H & B & F & \beta \\ G & F & C & \gamma \\ a & \beta & \gamma & \end{vmatrix} = 0.$$

determinant form annexed corresponding to (4663), or in full

$$\begin{aligned} 4667 \quad & (BC-F^2) \alpha^2 + (CA-G^2) \beta^2 + (AB-H^2) \gamma^2 \\ & + 2(GH-AF) \beta\gamma + 2(HF-BG) \gamma\alpha + 2(FG-CH) \alpha\beta = 0. \end{aligned}$$

$$4668 \quad \text{or } \Delta (a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2ha\beta) = 0.$$

PROOF.—Eliminate  $\nu$  from  $U = 0$  and the given line. The result is of the form  $L\lambda^3 + 2R\lambda\mu + M\mu^3 = 0$ , and therefore the envelope is  $LM = R^2$ , by (4792). This produces equation (4667). The coefficients are the first minors of the reciprocal determinant of  $\Delta$  (1643), and therefore, by (585), are equal to  $a\Delta$ ,  $b\Delta$ , &c.

4669 The condition that  $U$  may consist of two linear factors is, as in (4661),  $D = 0$ , where

$$4670 \quad D = ABC + 2FGH - AF^2 - BG^2 - CH^2 = \Delta^2. \quad (1643)$$

In this case  $U$  becomes the equation of two points, since the line  $\lambda\alpha + \mu\beta + \nu\gamma$  must pass through one or other of two fixed points. See (4913).

4671 The coordinates of the pole of  $\lambda\alpha + \mu\beta + \nu\gamma$  are as

$$A\lambda + H\mu + G\nu : H\lambda + B\mu + F\nu : G\lambda + F\mu + C\nu,$$

4672 or  $U_\lambda : U_\mu : U_\nu$ .

PROOF.—By (4659) we have the equations in the margin, the solution of which gives the ratios of  $\alpha : \beta : \gamma$ .

$$\begin{aligned} a\alpha + h\beta + g\gamma &= k\lambda \\ h\alpha + b\beta + f\gamma &= k\mu \\ g\alpha + f\beta + c\gamma &= k\nu. \end{aligned}$$

$$4673 \quad \frac{\alpha}{A\lambda + H\mu + G\nu} = \frac{\beta}{H\lambda + B\mu + F\nu} = \frac{\gamma}{G\lambda + F\mu + C\nu} = \frac{k}{\Delta}.$$

Hence the tangential equation of the pole of  $\lambda'\alpha + \mu'\beta + \nu'\gamma$ , i.e., the condition that  $\lambda\alpha + \mu\beta + \nu\gamma$  may pass through the pole; or, in other words, that the two lines may be mutually conjugate, is

$$4674 \quad \lambda U_{\lambda'} + \mu U_{\mu'} + \nu U_{\nu'} = 0 \quad \text{or} \quad \lambda' U_\lambda + \mu' U_\mu + \nu' U_\nu = 0,$$

the two forms being equivalent, and each

$$\begin{aligned} 4676 \quad & = A\lambda\lambda' + B\mu\mu' + C\nu\nu' \\ & + F(\mu\nu' + \mu'\nu) + G(\nu\lambda' + \nu'\lambda) + H(\lambda\mu' + \lambda'\mu). \end{aligned}$$

The coordinates of the centre  $\alpha_0, \beta_0, \gamma_0$  are in the ratios

$$4677 \quad Aa + Hb + Gc : Ha + Bb + Fc : Ga + Fb + Cc,$$

where  $a, b, c$  are the sides of the trigon.

PROOF.—By (4671), since the centre is the pole of the line at infinity  $aa + b\beta + c\gamma = 0$  (4612).

The quadratic for the ratio  $n : n'$  of the segments into which the line joining two given points  $a\beta\gamma, a'\beta'\gamma'$  is divided by the conic is, with the notation of (4656–7),

**4678**

$$\phi(a', \beta', \gamma') n^2 + 2(\phi_a a' + \phi_\beta \beta' + \phi_\gamma \gamma') nn' + \phi(a, \beta, \gamma) n'^2 = 0.$$

PROOF.—By the method of (4131).

The equation of the pair of tangents at the points where  $\gamma$  meets the general conic  $u$  (4656), is

**4679** 
$$au_\beta^2 + 2hu_a u_\beta + bu_a^2 = 0.$$

PROOF.—The point  $a'\beta'$ , where  $\gamma$  meets the curve, is found from  $aa'^2 + 2ha'a'\beta' + b\beta'^2 = 0$  [ $\gamma = 0$  in (4656)]. The tangent at such a point is  $u_a a' + u_\beta \beta' = 0$  (4658). Eliminate  $a', \beta'$ .

The equation of a pair of tangents from  $a'\beta'\gamma'$  is

**4680** 
$$\phi(a'\beta'\gamma') \phi(a\beta\gamma) = (\phi_a a' + \phi_\beta \beta' + \phi_\gamma \gamma')^2.$$

PROOF.—By the condition for equal roots of (4678).

By actual expansion the equation becomes

**4681** 
$$\begin{aligned} (B\gamma^2 + C\beta^2 - 2F\beta\gamma) a'^2 + (Ca^2 + A\gamma^2 - 2G\gamma a) \beta'^2 + (A\beta^2 + Ba^2 - 2Ha\beta) \gamma'^2 \\ + 2(-A\beta\gamma + H\gamma a + Ga\beta - Fa^2) \beta' \gamma' \\ + 2(H\beta\gamma - B\gamma a + Fa\beta - G\beta^2) \gamma' a' \\ + 2(G\beta\gamma + F\gamma a - Ca\beta - H\gamma^2) a' \beta' = 0. \end{aligned}$$

In which either  $a', \beta', \gamma'$  or  $a, \beta, \gamma$  may be the variables, for the forms are convertible.

Otherwise the equation of the two tangents is

**4682** 
$$\Phi(\beta\gamma' - \beta'\gamma, \gamma a' - \gamma' a, a\beta' - a'\beta) = 0. \quad (4665)$$

PROOF.—By substituting  $\beta\gamma' - \beta'\gamma$ , &c. for  $\lambda, \mu, \nu$  in (4664), the condition that the line joining  $a'\beta'\gamma'$  to any point  $a\beta\gamma$  on either tangent (see 4616) should touch the conic is fulfilled. The expansion produces the preceding equation (4681).

The equation of the asymptotes is

**4683** 
$$\phi(a, \beta, \gamma) = \phi(a_0, \beta_0, \gamma_0) = k\Sigma \dots\dots\dots (1),$$

where  $a_0, \beta_0, \gamma_0$  are the coordinates of the centre.

Otherwise the equation, in a form homogeneous in  $a, \beta, \gamma$ , is

$$4684 \quad (a\alpha_0 + b\beta_0 + c\gamma_0) \phi(a, \beta, \gamma) = k(a\alpha + b\beta + c\gamma)^2 \dots\dots (2),$$

where  $a, b, c$  are the sides of the trigon.

And, finally, if the tangential equation (4664) be denoted by  $\Phi(\lambda, \mu, \nu) = 0$ , the equation of the asymptotes may be presented in the form

$$4685 \quad \Phi(a, b, c) \phi(a, \beta, \gamma) = (a\alpha + b\beta + c\gamma)^2 \Delta \dots\dots\dots (3).$$

PROOF.—(i.) The asymptotes are identical with a pair of tangents from the centre; therefore, put  $\alpha_0, \beta_0, \gamma_0$  for  $\alpha', \beta', \gamma'$  in (4680); thus

$$\phi(a, \beta, \gamma) \phi(\alpha_0, \beta_0, \gamma_0) = k^2 (a\alpha + b\beta + c\gamma)^2 = k^2 \Sigma^2 \dots\dots\dots (4),$$

since the polar becomes the line at infinity.

Now, multiplying the three equations in (4672) by  $\alpha, \beta, \gamma$  respectively, and adding, we obtain  $\phi(a, \beta, \gamma) = k(\lambda\alpha + \mu\beta + \nu\gamma)$ , and therefore

$$\phi(\alpha_0, \beta_0, \gamma_0) = k(a\alpha + b\beta + c\gamma) = k\Sigma \dots\dots\dots (5),$$

since the line at infinity (4612) is the pole of the centre.

From (4) and (5), by eliminating  $k$ , equation (1) is produced; and by dividing (4) by (5), we get equation (2).

Again, taking the values of  $\alpha, \beta, \gamma$  from (4673), we have

$$\frac{\lambda\alpha + \mu\beta + \nu\gamma}{k} = \frac{\Phi(\lambda, \mu, \nu)}{\Delta}, \text{ and therefore } \frac{a\alpha_0 + b\beta_0 + c\gamma_0}{k} = \frac{\Phi(a, b, c)}{\Delta}.$$

By the last equation, (2) is converted into (3). See also (4966).

COR.—Since the centre  $(\alpha_0, \beta_0, \gamma_0)$  is on the asymptotes, we have

$$4686 \quad \phi(\alpha_0, \beta_0, \gamma_0) = \Sigma^2 \Delta \div \Phi(a, b, c).$$

4687 The semi-axes of the general conic (4656) are the values of  $r$  obtained from the quadratic

$$\begin{vmatrix} \left(a + \frac{as \cos A}{r^2}\right), & h, & g, & a \\ h, & \left(b + \frac{bs \cos B}{r^2}\right), & f, & b \\ g, & f, & \left(c + \frac{cs \cos C}{r^2}\right), & c \\ a, & b, & c, & \end{vmatrix} = 0,$$

where  $a, b, c$  are the sides of the trigon, and

$$s = abc\Delta \div \Phi(abc).$$



PROOF.—The centre being  $\alpha_0\beta_0\gamma_0$ , put  $\alpha - \alpha_0 = x$ ,  $\beta - \beta_0 = y$ ,  $\gamma - \gamma_0 = z$ .  $\alpha\beta\gamma$  being a point on the conic, and  $r$  the radius to it from the centre, we have, by (4602),

$$r^2 = \frac{abc}{\Sigma^2} (x^2a \cos A + y^2b \cos B + z^2c \cos C) \dots\dots\dots (1).$$

Also (4656),  $\phi(\alpha, \beta, \gamma) = \phi(\alpha_0 + x, \beta_0 + y, \gamma_0 + z) = 0$ .

Expand and write  $l, m, n$  for  $aa_0 + h\beta_0 + g\gamma_0$ ,  $ha_0 + b\beta_0 + f\gamma_0$ ,  $ga_0 + f\beta_0 + c\gamma_0$ .

The terms in  $x, y, z$  become

$$lx + my + nz = l(\alpha - \alpha_0) + \&c. = \Sigma - \Sigma = 0 \quad (4007) \dots\dots\dots (2),$$

and we obtain  $\phi(x, y, z) = -\phi(\alpha_0, \beta_0, \gamma_0) = \Sigma^2\Delta \div \Phi(a, b, c)$  (4686)  $\dots\dots\dots (3)$ .

The maximum and minimum values of  $r^2$  and therefore of

$$x^2a \cos A + y^2b \cos B + z^2c \cos C \dots\dots\dots (4)$$

are required, subject to the equations (2) and (3). By the method of undetermined multipliers (1862), the quadratic above is found.—*Ferrers's Tril. Coord.*, Ch. 4, Art. 18.

**4688** The area of the conic  $= \frac{\pi \Sigma abc \Delta}{\{\Phi(a, b, c)\}^{\frac{3}{2}}}$ .

PROOF.—If the roots of the quadratic (4687) are  $\pm r_1^{-1}$ ,  $\pm r_2^{-1}$ , the area will be  $\pi r_1 r_2$ . The coefficient of  $r^{-4}$  reduces by trigonometry to  $-\Sigma^2 s^2$ , and the absolute term is  $-\Phi(a, b, c)$ . Hence the product of the roots is found.

**4689** The conic will be an ellipse, hyperbola, or parabola, according as  $\Phi(a, b, c)$  (4664) is positive, negative, or zero.

PROOF.—The squares of the semi-axes have opposite signs in the hyperbola. Therefore the product of the roots of the quadratic (4687) must for an hyperbola be negative, and therefore  $\Phi$  negative in (4688).

$\Phi(a, b, c) = 0$  makes the curve touch the line at infinity (4664), a property which distinguishes the parabola.

The condition that the general conic (4656) may be a rectangular hyperbola is

**4690**  $a + b + c = 2f \cos A + 2g \cos B + 2h \cos C$ .

PROOF.—Let the asymptotes be

$$la + m\beta + n\gamma = 0, \quad l'a + m'\beta + n'\gamma = 0.$$

Forming the product, equating coefficients with (4685), and denoting  $\phi(a, b, c)$  by  $\phi$ , we get the proportions

$$\begin{aligned} \frac{ll'}{a\phi - a^2\Delta} &= \frac{mm'}{b\phi - b^2\Delta} = \frac{nn'}{c\phi - c^2\Delta} = \frac{mn' + m'n}{2(f\phi - bc\Delta)} \\ &= \frac{nl' + n'l}{2(g\phi - ca\Delta)} = \frac{lm' + l'm}{2(h\phi - ab\Delta)}. \end{aligned}$$

We may therefore substitute these denominators in (4620) for the condition of perpendicularity of the asymptotes. The result reduces to the equation above, by (837).

For another method, see (5002).

**4691** The general conic (4656) will become a circle when the following relation exists between the coefficients :

$$\begin{aligned} & b \sin^2 C + c \sin^2 B - 2f \sin B \sin C \\ &= c \sin^2 A + a \sin^2 C - 2g \sin C \sin A \\ &= a \sin^2 B + b \sin^2 A - 2h \sin A \sin B. \end{aligned}$$

PROOF.—Equate coefficients of the equation of the conic (4656) with those of the circle in (4751).

**4692** The equation of the pair of lines drawn from a point  $\alpha'\beta'\gamma'$  to the points of intersection of the conic  $\phi$  and the line  $L \equiv \lambda\alpha + \mu\beta + \nu\gamma = 0$  is, writing  $L'$  for  $\lambda\alpha' + \mu\beta' + \nu\gamma'$ , with the notation of (4656-7),

$$L'^2\phi(\alpha, \beta, \gamma) - 2LL'(\phi_\alpha\alpha' + \phi_\beta\beta' + \phi_\gamma\gamma') + L^2\phi(\alpha', \beta', \gamma') = 0.$$

PROOF.—By the method of (4133).

**4693** The *Director-Circle* of the conic, that is, the locus of intersection of tangents at right angles, is, in Cartesians,

$$C(x^2 + y^2) - 2Gx - 2Fy + A + B = 0.$$

PROOF.—Let the equation of a tangent through  $xy$  be

$$m\xi - \eta + (y - mx) = 0.$$

Therefore in the tangential equation (4665) put  $\lambda = m$ ,  $\mu = -1$ ,  $\nu = y - mx$ , and apply the condition, *Product of roots of quadratic in  $m = -1$*  (4078).

The equation of the same circle in trilinears is

$$\begin{aligned} \mathbf{4694} \quad & (B + C + 2F \cos A) \alpha^2 + (C + A + 2G \cos B) \beta^2 + (A + B + 2H \cos C) \gamma^2 \\ & + 2(A \cos A - H \cos B - G \cos C - F) \beta\gamma \\ & + 2(-H \cos A + B \cos B - F \cos C - G) \gamma\alpha \\ & + 2(-G \cos A - F \cos B + C \cos C - H) \alpha\beta = 0; \end{aligned}$$

or, in the form of (4751),

**4695**

$$(a\alpha + b\beta + c\gamma) \left( \frac{B + C + 2F \cos A}{a} \alpha + \&c. \right) = \frac{\Phi(a, b, c)}{abc} (a\beta\gamma + b\gamma\alpha + c\alpha\beta).$$

4 M

PROOF.—The equation of a pair of tangents (4681) through a point  $\alpha\beta\gamma$  in trilinears, when the tangents are at right angles, represents the limiting case of a rectangular hyperbola. Therefore the equation referred to must have the coefficients of  $\alpha^2, \beta^2$ , &c. connected by the relation in (4690), which thus becomes the equation of the locus of the point  $\alpha\beta\gamma$ ; i.e., the director-circle.

**4696** When the general conic is a parabola,  $C=0$  in (4693) and  $\Phi(a, b, c) = 0$  in (4695), by (4430) and (4689), and these equations then represent the directrix.

### PARTICULAR CONICS.

**4697** A conic circumscribing the quadrilateral, the equations of whose sides are  $\alpha=0, \beta=0, \gamma=0, \delta=0$ , (Fig. 38)

$$\alpha\gamma = k\beta\delta.$$

PROOF.—This is a curve of the second degree, and it passes through the points where  $\alpha$  meets  $\beta$  and  $\delta$ , and also where  $\gamma$  meets  $\beta$  and  $\delta$ .

**4698** The circumscribing circle is  $\alpha\gamma = \pm\beta\delta$ ; + or —, as the origin of coordinates lies without or within the quadrilateral.

PROOF.—Transform (4697) into Cartesians (4009); equate coefficients of  $x$  and  $y$  and put the coefficients of  $xy$  equal to zero.

**4699** A conic having  $\alpha$  and  $\gamma$  for tangents and  $\beta$  for the chord of contact: (Fig. 39)

$$\alpha\gamma = k\beta^2.$$

PROOF.—Make  $\delta$  coincide with  $\beta$  in (4698).

**4700** A conic having two common chords  $\alpha$  and  $\beta$  with a given conic  $S$ : (Fig. 40)

$$S = k\alpha\beta.$$

**4701** A conic having a common chord of contact  $\alpha$  with a given conic  $S$ : (Fig. 41)

$$S = k\alpha^2.$$

**4702** COR.—If  $RPQ$  be drawn always parallel to a given line,  $PN^2 \propto RP.PQ$ , by (4317).

**4703** A conic having a common tangent  $T$  at a point  $x'y'$  and a common chord with the conic  $S$ : (Fig. 42)

$$S = T(lx + my + nz).$$

**4704** A conic osculating  $S$  at the point  $x'y'$  where  $T$  touches at one extremity of the common chord  $l(x - x') + m(y - y')$ : (Fig. 43)

$$S = T(lx + my - lx' - my').$$

**4705** A conic having common tangents  $T, T'$  at common points with the conic  $S$ : (Fig. 44)

$$S = kTT'.$$

**4706** A conic having four coincident points with the conic  $S$  at the point where  $T$  touches: (Fig. 45)

$$S = kT^2.$$

**4707** The conics  $S + L^2 = 0$ ,  $S + M^2 = 0$ ,  $S + N^2 = 0$ , (Fig 46) having respectively  $L, M, N$  for common chords of contact with the conic  $S$ , will have the six chords of intersection

$$L \pm M = 0, \quad M \pm N = 0, \quad N \pm L = 0,$$

passing three and three through the same points.

PROOF.—From  $(S + M^2) - (S + N^2) = (M + N)(M - N)$ , &c.

By supposing one or more of the conics to become right lines, various theorems may be obtained.

**4709** The diagonals of the inscribed and circumscribed quadrilaterals of a conic all pass through the same point and form a harmonic pencil.

PROOF.—(Fig. 47.) By (4707), or by taking  $LM = R^2$  and  $L'M' = R^2$  for the equations of the conic by (4784).

**4710** If three conics have a chord common to all, the other three chords common to pairs pass through the same point.

PROOF.—(Fig. 48.) Take  $S, S + LM, S + LN$  for the conics,  $L$  being the chord common to all; then  $M, N, M - N$  are the other common chords.

**4711** The hyperbola  $xy = (0x+0y+p)^2$

is of the form (4699), and has for a chord of contact at infinity  $0x+0y+p=0$ ,  $x, y$  being the tangents from the centre.

**4712** The parabola  $y^2 = (0x+0y+p)x$

has the tangent at infinity  $0x+0y+p=0$ .

**4713** So the general equation of a parabola may be put in the form of (4699). Thus

$$(ax+\beta y)^2 + (2gx+2fy+c)(0x+0y+1) = 0.$$

Here  $ax+\beta y$  is the chord of contact, that is, a diameter;  $2gx+2fy+c$  is the finite tangent at its extremity, and  $0x+0y+1$  the tangent at the other extremity, supposed at infinity.

**4714** The general conic may be written

$$(ax^2+2hxy+by^2) + (2gx+2fy+c)(0x+0y+1) = 0.$$

For this is of the form  $\alpha\gamma+k\beta\delta$ ,  $\delta$  being at infinity.

**4715** The conics  $S$  and  $S-k(0x+0y+1)^2$

have double contact at infinity, and are similar.

**4716** The parabolas  $S$  and  $S-k^2$

have a contact of the third order at infinity.

PROOF.—For  $S$  and  $S-(0x+0y+k)^2$  have the line at infinity for a chord of contact; and, by (4712), this chord of contact is also a tangent to both curves.

**4717** All circles are said to pass through the same two imaginary points at infinity (see 4918) and through two real or imaginary finite points.

PROOF.—The general equation of the circle (4144) may be written

$$(x+iy)(x-iy) + (2gx+2fy+c)(0x+0y+1) = 0;$$

and this is of the form (4697). Here the lines  $x \pm iy$  intersect  $0x+0y+1$  in two imaginary points which have been called the *circular points at infinity*, and  $2gx+2fy+c$  in two finite points  $P, Q$ ; and these points are all situated on the locus  $x^2+y^2+2gx+2fy+c=0$ .

**4718** Concentric circles touch in four imaginary points at infinity.

PROOF.—The centre being the origin, equation (4136) may be written  $(x+iy)(x-iy) = (0x+0y+r)^2$ , which, by (4699), shows that the lines  $x \pm iy$  have each double contact with the (*supplementary*) curve at infinity, and the variation of  $r$  does not affect this result. Compare (4711).

**4719** The equation of any conic may be put in the form

$$x^2 + y^2 = e^2 \gamma^2.$$

Here  $x=0$ ,  $y=0$  are two sides of the trigon intersecting at right angles in the focus;  $\gamma=0$ , the third side, is the directrix, and  $e$  is the eccentricity.

The conic becomes a circle when  $e=0$  and  $\gamma=\infty$ , so that  $e\gamma=r$ , the radius, (4718).

**4720** Two imaginary tangents drawn through the focus are, by (4699),

$$(x+iy)(x-iy) = 0.$$

These tangents are identical with the lines drawn through the two circular points at infinity (see 4717). Hence, if two tangents be drawn to the conic from each of the circular points at infinity, they will intersect in two imaginary points, and also in two real points which are the foci of the conic.

All confocal conics, therefore, have four imaginary common tangents, and two opposite vertices of the quadrilateral formed by the tangents are the foci of the conics.

**4721** If the axes are oblique, this universal form of the equation of the conic becomes

$$x^2 + 2xy \cos \omega + y^2 = e^2 \gamma^2.$$

The two imaginary tangents through the focus must now be written

$$\{x+y(\cos \omega + i \sin \omega)\} \{x+y(\cos \omega - i \sin \omega)\} = 0.$$

**4722** Any two lines including an angle  $\theta$  form, with the lines drawn from the two circular points at infinity to their point of intersection, a pencil of which the anharmonic ratio is  $e^{i(\pi-2\theta)}$ .

PROOF.—Take the two lines for sides  $\beta$ ,  $\gamma$  of the trigon. The equation of the other pair of lines to the circular points will be obtained by elimin-



ating  $a$  between the equations of the line at infinity and the circum-circle,

$$\text{viz.,} \quad aa + b\beta + c\gamma = 0 \quad \text{and} \quad \frac{a}{a} + \frac{b}{\beta} + \frac{c}{\gamma} = 0. \quad (4738)$$

The result is  $\beta^2 + 2\beta\gamma \cos \theta + \gamma^2 = 0$ ;

or, in factors,  $(\beta + e^{i\theta}\gamma)(\beta + e^{-i\theta}\gamma) = 0$ .

The anharmonic ratio of the pencil formed by the four lines  $\beta$ ,  $\beta + e^{i\theta}\gamma$ ,  $\gamma$ ,  $\beta + e^{-i\theta}\gamma$  is, by (4648, i.),

$$-e^{i\theta} : e^{-i\theta} = -e^{i2\theta} = e^{i(\pi-2\theta)}.$$

**4723** COR.—If  $\theta = \frac{1}{2}\pi$ , the lines are at right angles, and the four lines form a harmonic pencil. [*Ferrers' Tril. Coords.*, Ch. VIII.]

#### THE CIRCUMSCRIBING CONIC OF THE TRIGON.

**4724** The equation of this conic (Fig. 49) is

$$l\beta\gamma + m\gamma a + na\beta = 0 \quad \text{or} \quad \frac{l}{a} + \frac{m}{\beta} + \frac{n}{\gamma} = 0.$$

PROOF.—The equation is of the second degree, and it is satisfied by  $a=0$ ,  $\beta=0$  simultaneously. It therefore passes through the point  $a\beta$ . Similarly through  $\beta\gamma$  and  $\gamma a$ .

The tangents at  $A$ ,  $B$ , and  $C$  are

$$\textbf{4726} \quad \frac{m}{\beta} + \frac{n}{\gamma} = 0, \quad \frac{n}{\gamma} + \frac{l}{a} = 0, \quad \frac{l}{a} + \frac{m}{\beta} = 0.$$

PROOF.—By writing (4724) in the form

$$m\gamma a + \beta(l\gamma + na) = 0,$$

$l\gamma + na = 0$  is seen, by (4697), to be the tangent at  $a\gamma$ ; for the intersections of  $a$  and  $\gamma$ , with the curve, now coincide, and  $\delta$  (now  $l\gamma + na$ ) passes through the two coincident points.

**4729** The tangent, or polar, of the point  $a'\beta'\gamma'$  is, by (4659),

$$(m\gamma' + n\beta')a + (na' + l\gamma')\beta + (l\beta' + ma')\gamma = 0.$$

**4730** The tangents at  $A$ ,  $B$ ,  $C$  (Fig. 49) meet the opposite sides respectively in  $P$ ,  $Q$ ,  $R$  on the right line

$$\frac{a}{l} + \frac{\beta}{m} + \frac{\gamma}{n} = 0. \quad \text{By (4604).}$$

**4731** The line  $\frac{a}{l} - \frac{\beta}{m}$  passes through  $(D)$ , the intersection of the tangents at  $A$  and  $B$ .

**4732** The diameter through the intersection of the tangents at  $A$  and  $B$  is

$$naa - nb\beta + (la - mb)\gamma = 0.$$

PROOF.—The coordinates of the point of intersection are  $l : m : -n$ , by (4726–7), and the coordinates of the centre of  $AB$  are  $b : a : 0$ . The diameter passes through these points, and its equation is given by (4616).

**4733** The coordinates of the centre of the conic are as

$$l(-la + mb + n\epsilon) : m(la - mb + n\epsilon) : n(la + mb - n\epsilon).$$

PROOF.—By (4610), the point being the intersection of two diameters like (4732). Otherwise, by (4677).

**4734** The secant through  $(a_1\beta_1\gamma_1)$ ,  $(a_2\beta_2\gamma_2)$ , any two points on the conic, and the tangent at the first point are respectively,

$$\frac{la}{a_1a_2} + \frac{m\beta}{\beta_1\beta_2} + \frac{n\gamma}{\gamma_1\gamma_2} = 0 \quad \text{and} \quad \frac{la}{a_1^2} + \frac{m\beta}{\beta_1^2} + \frac{n\gamma}{\gamma_1^2} = 0.$$

PROOF.—The first is a right line, and it is satisfied by  $a = a_1$ , &c., and also by  $a = a_2$ , &c., by (4725). The second equation is what the first becomes when  $a_2 = a_1$ , &c. For the tangential equation, see (4893).

**4735** The conic is a parabola when

$$la^2 + m^2b^2 + n^2\epsilon^2 - 2mn\beta\epsilon - 2nl\epsilon a - 2lmab = 0,$$

**4736** or  $\sqrt{la} + \sqrt{mb} + \sqrt{n\epsilon} = 0$ .

PROOF.—Substitute the coordinates of the centre (4733) in  $aa + b\beta + \epsilon\gamma = 0$ , the equation of the line at infinity (4612).

Otherwise, the conic must touch the line at infinity; therefore put  $a, b, \epsilon$  for  $\lambda, \mu, \nu$  in (4893).

**4737** The conic is a rectangular hyperbola when

$$l \cos A + m \cos B + n \cos C = 0,$$

and in this case it passes through the orthocentre of the triangle.

PROOF.—By (4690), and the coordinates of the orthocentre (4634).

#### THE CIRCUMSCRIBING CIRCLE OF THE TRIGON.

**4738**  $\beta\gamma \sin A + \gamma a \sin B + a\beta \sin C = 0,$

or  $\frac{\sin A}{a} + \frac{\sin B}{\beta} + \frac{\sin C}{\gamma} = 0.$

PROOF.—The values of the ratios  $l : m : n$ , in (4724), may be found geometrically from the equations of the tangents (4726–8).

For the coordinates of the centre, see (4642).

#### THE INSCRIBED CONIC OF THE TRIGON.

$$4739 \quad l^2a^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma a - 2lm a\beta = 0.$$

$$4740 \quad \text{or} \quad \sqrt{(la)} + \sqrt{(m\beta)} + \sqrt{(n\gamma)} = 0.$$

PROOF.—(Fig. 50.) The first equation may be written

$$n\gamma(n\gamma - 2la - 2m\beta) + (la - m\beta)^2 = 0.$$

By (4699) this represents a conic of which the lines  $\gamma$  and  $n\gamma - 2la - m\beta$  are the tangents at  $F$  and  $f$ , and  $la - m\beta$  the chord of contact. Similarly, it may be written so as to shew that  $a$  and  $\beta$  touch the conic.

4741 The three pairs of tangents at  $F, f$ , &c., are

$$\left. \begin{array}{l} 2m\beta + 2n\gamma - la \\ \text{and } a \end{array} \right\}, \quad \left. \begin{array}{l} 2n\gamma + 2la - m\beta \\ \text{and } \beta \end{array} \right\}, \quad \left. \begin{array}{l} 2la + 2m\beta - n\gamma \\ \text{and } \gamma \end{array} \right\},$$

and they have their three points of intersection  $P, Q, R$  on the right line  $la + m\beta + n\gamma$ . By (4604).

4742 The coordinates of the centre of the conic are as

$$nb + mc : lc + na : ma + lb.$$

PROOF.—By putting  $a$  and  $\beta =$  zero alternately in (4739), we find, for the coordinates of the points of contact,

$$\text{at } D, \beta = \frac{2\Delta n}{nb + mc}; \quad \text{and at } E, a = \frac{2\Delta n}{na + lc};$$

therefore the equation of the diameter through  $C$  bisecting  $DE$  is, by (4603),

$$\frac{a}{nb + mc} = \frac{\beta}{lc + na}.$$

Similarly the diameter bisecting  $DF$  is  $\frac{\gamma}{ma + lb} = \frac{a}{nb + mc}$ .

Therefore the point of intersection, or centre, is defined by the ratios given above.

Otherwise, by (4677), and the values in (4665), writing for  $a, b, c, f, g, h$  the coefficients in (4739).

4743 The secant through  $a_1\beta_1\gamma_1, a_2\beta_2\gamma_2$  any two points on the curve.

$$\begin{aligned} & a\sqrt{l}(\sqrt{\beta_1\gamma_2} + \sqrt{\beta_2\gamma_1}) + \beta\sqrt{m}(\sqrt{\gamma_1a_2} + \sqrt{\gamma_2a_1}) \\ & + \gamma\sqrt{n}(\sqrt{a_1\beta_2} + \sqrt{a_2\beta_1}) = 0. \end{aligned}$$

PROOF.—Put  $\alpha_1\beta_1\gamma_1$  for  $a\beta\gamma$ , and shew that the expression vanishes by (4740).

**4744** The tangent at the point  $\alpha_1\beta_1\gamma_1$ :

$$\alpha\sqrt{\frac{l}{a_1}} + \beta\sqrt{\frac{m}{\beta_1}} + \gamma\sqrt{\frac{n}{\gamma_1}} = 0.$$

PROOF.—Put  $\alpha_1 = \alpha$ , &c., in (4743), and divide by  $2\sqrt{(\alpha_1\beta_1\gamma_1)}$ .

**4745** The equation of the polar must be obtained from (4739) by means of (4659).

**4746** The conic is a parabola when

$$\frac{l}{a} + \frac{m}{b} + \frac{n}{c} = 0.$$

PROOF.—Similar to that of (4736).

THE INSCRIBED CIRCLE OF THE TRIGON.

**4747** 
$$\alpha^2 \cos^4 \frac{A}{2} + \beta^2 \cos^4 \frac{B}{2} + \gamma^2 \cos^4 \frac{C}{2} \\ - 2\beta\gamma \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} - 2\gamma\alpha \cos^2 \frac{C}{2} \cos^2 \frac{A}{2} - 2\alpha\beta \cos^2 \frac{A}{2} \cos^2 \frac{B}{2}.$$

**4748** or  $\cos \frac{A}{2} \sqrt{\alpha} + \cos \frac{B}{2} \sqrt{\beta} + \cos \frac{C}{2} \sqrt{\gamma} = 0.$

**4749** The  $a$ -escribed circle: (4629)

$$\cos \frac{A}{2} \sqrt{-\alpha} + \sin \frac{B}{2} \sqrt{\beta} + \sin \frac{C}{2} \sqrt{\gamma}.$$

PROOF.—At the point of contact where  $\gamma = 0$ , we have, in (4740), geometrically,  $r$  being the radius of the circle,

$$l : m = \beta : \alpha = r \cot \frac{A}{2} \sin A : r \cot \frac{B}{2} \sin B = \pm \cos^2 \frac{1}{2} A : \cos^2 \frac{1}{2} B;$$

+ for the inscribed; — for the escribed circle and  $\pi - B$  instead of  $B$ .

**4750** The tangent at  $\alpha'\beta'\gamma'$ , by (4744), is

$$\cos \frac{A}{2} \frac{\alpha}{\sqrt{\alpha'}} + \cos \frac{B}{2} \frac{\beta}{\sqrt{\beta'}} + \cos \frac{C}{2} \frac{\gamma}{\sqrt{\gamma'}} = 0.$$

The polar is obtained as in (4745).

GENERAL EQUATION OF THE CIRCLE.

**4751** 
$$(la + m\beta + n\gamma)(\alpha \sin A + \beta \sin B + \gamma \sin C) \\ + k(\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C) = 0.$$

4 N

PROOF.—The second term is the circumscribing circle (4738), and the first is linear by (4609); therefore the whole represents a circle. By varying  $k$ , a system of circles is obtained whose radical axis (4161) is the line  $la + m\beta + n\gamma$ , the circumscribing circle being one of the system.

**4752** If  $l'a + m'\beta + n'\gamma$  be the radical axis of a second system of circles represented by a similar equation, the radical axis of any two circles of the two systems defined by  $k$  and  $k'$  will be

$$k'(la + m\beta + n\gamma) - k(l'a + m'\beta + n'\gamma) = 0.$$

PROOF.—By eliminating the term

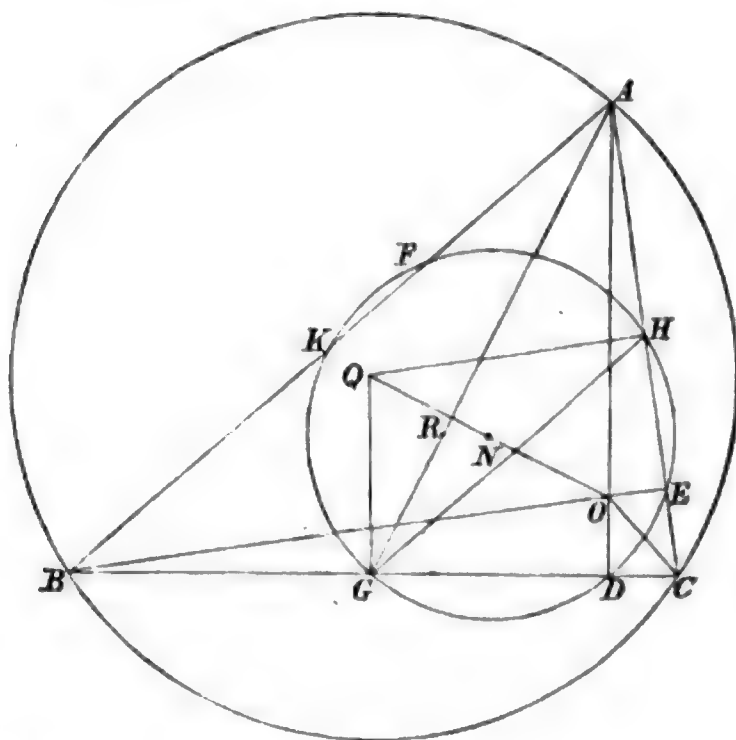
$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C.$$

**4753** To find the coefficient of  $x^2 + y^2$  in the circle when only the trilinear equation is given.

RULE.—Make  $\alpha, \beta, \gamma$  the coordinates of a point from which the length of the tangent is known, and divide by the square of that length; or, if the point be within the circle, substitute “half the shortest chord through the point” for “the tangent.”

PROOF.—If  $S = 0$  be the equation of the circle, and  $m$  the required coefficient; then, for a point not on the curve,  $S \div m = \text{square of tangent or semi-chord, by (4160).}$

#### THE NINE-POINT CIRCLE.



**4754**  $\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C$   
 $- 2(\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C) = 0.$

PROOF.—The equation represents a circle because it may be expressed in the form

$$(a \cos A + \beta \cos B + \gamma \cos C)(a \sin A + \beta \sin B + \gamma \sin C) - 2(\beta \gamma \sin A + \gamma a \sin B + a \beta \sin C) = 0.$$

See Proof of (4751). Now, when  $a = 0$ , the equation becomes

$$(\beta \sin B - \gamma \sin C)(\beta \cos B - \gamma \cos C) = 0,$$

which shews, by (4631, '3), that the circle bisects  $BC$  and passes through  $D$ , the foot of the perpendicular from  $A$ .

**4754a** The equation of the nine-point circle in Cartesian coordinates, with the side  $BC$  and perpendicular on it from  $A$  for  $x$  and  $y$  axes respectively, is

$$x^2 + y^2 - R \sin(B - C)x - R \cos(B - C)y = 0,$$

where  $R$  is the radius of the circum-circle.

#### THE TRIPLICATE-RATIO CIRCLE.

**4754b** \* Let the point  $S$  (Fig. 165) be chosen, so that its trilinear coordinates are proportional to the sides of the trigon. Draw lines through  $S$  parallel to the sides, then the circle in question passes through the six points of intersection, and the intercepted chords are in the *triplicate-ratio* of the sides.

[The following abbreviations are used,  $a, b, c$ , and not  $\alpha, \beta, \gamma$ , being in this article written for the sides of the trigon  $ABC$ .]

$$K \equiv a^2 + b^2 + c^2; \quad \lambda \equiv \sqrt{(b^2c^2 + c^2a^2 + a^2b^2)}; \quad \Delta \equiv ABC;$$

$$\mu \equiv \frac{\lambda}{K}; \quad \omega = \angle BFD = DE'F', \text{ \&c.}; \quad \theta = DFD' = DE'D', \text{ \&c.}$$

By hypothesis,  $\frac{a}{a} = \frac{\beta}{b} = \frac{\gamma}{c} = \frac{2\Delta}{a^2 + b^2 + c^2} \text{ (4007)} = \frac{2\Delta}{K} \dots\dots\dots (1),$

$$\frac{BD'}{BF} = \frac{a}{c} = \frac{a}{\gamma} \text{ (1)} = \frac{BF'}{BD},$$

therefore  $BF \cdot BF' = BD \cdot BD'$ , therefore  $F, F', D, D'$  are concyclic.

If  $AS, BS, CS$  produced meet the opposite sides in  $l, m, n$ ,

$$\frac{Bn}{An} = \frac{a \sin BCn}{b \sin ACn} = \frac{a\alpha}{b\beta} = \frac{a^2}{b^2}, \text{ by (1) } \dots\dots\dots (2).$$

\* The theorems of (1 to 36) are for the most part due to Mr. R. Tucker, M.A. The original articles will be found in *The Quarterly Journal of Pure and Applied Mathematics*, Vol. xix., No. 76, and Vol. xx., Nos. 77 and 78.

Other and similar investigations have been made by MM. Lemoine and Taylor and Prof. Neuberg, *Mathesis*, 1881, 1882, 1884.



$$SF' = BD = \frac{\gamma}{\sin B} = \frac{2c\Delta}{K \sin B} \quad (1) = \frac{ac^2}{K}. \text{ Similarly } BF' = \frac{ca^2}{K}, \&c. \dots (3).$$

$$DD' = DP \frac{\sin A}{\sin C} = \frac{ca^2}{K} \cdot \frac{a}{c} = \frac{a^3}{K}, \&c. \dots (4).$$

$$BD' = BD + DD' = \frac{a(c^2 + a^2)}{K}, \&c. \dots (5).$$

$$FD = \sqrt{(BD^2 + BF^2 - 2BD \cdot BF \cos B)} = \frac{c\lambda}{K}, \text{ by (2) and (5)} \dots (6).$$

Hence  $DEF$  and  $D'E'F'$  are triangles similar to  $ABC$ , and they are equal to each other because  $ESF = E'SF = E'SF'$ , &c. (Euc. I. 37.)

$$DF' = \sqrt{(BD^2 + BF'^2 - 2BD \cdot BF' \cos B)} = \frac{abc}{K} \dots (7).$$

Hence

$$DF' = FE' = ED'.$$

$$D'F = \frac{b}{a} BD' = \frac{b(c^2 + a^2)}{K} \&c. \dots (8).$$

$$\cos \omega = \frac{BF'^2 + FD^2 - BD^2}{2BF' \cdot FD} = \frac{a^2 + c^2 + b^2}{2\lambda} \quad (5 \& 6) = \frac{K}{2\lambda} \dots (9).$$

$$\sin \omega = \sqrt{\left(1 - \frac{K^2}{4\lambda^2}\right)} = \frac{2\Delta}{\lambda} \quad (708) \dots (10).$$

$$\cos \theta = \cos (A - \omega), \&c. = \frac{a^2 \cos A + bc}{\lambda} \dots (11).$$

$$AFE' + BDF' + CED' = \frac{AF \cdot AE' \sin A}{2} + \&c. = \mu^2 \Delta = DEF, \text{ by (6)} \dots (12).$$

Or, geometrically, by Euclid I. 37.

$$\text{Radius of T. R. circle, } \rho = \mu R, \text{ by (6) } (R = \text{circum-radius}) \dots (13).$$

The trilinear equation of the T. R. circle is

$$abc(a^3 + \beta^3 + \gamma^3) = \frac{abc}{K}(aa + b\beta + c\gamma)^3 + a^3\beta\gamma + b^3\gamma a + c^3a\beta \dots (14),$$

$$\begin{aligned} \text{or } (b^2 + c^2)a^2 + (c^2 + a^2)\beta^2 + (a^2 + b^2)\gamma^2 &= \left\{ (a^2 + b^2)(a^2 + c^2) + b^2c^2 \right\} \frac{\beta\gamma}{bc} \\ &+ \left\{ (b^2 + c^2)(b^2 + a^2) + c^2a^2 \right\} \frac{\gamma a}{ca} + \left\{ (c^2 + a^2)(c^2 + b^2) + a^2b^2 \right\} \frac{a\beta}{ab} \dots (15). \end{aligned}$$

Obtained by substituting the trilinear coordinates of  $D, E, F$ , through which points the circle passes, in (4751), to determine the ratios  $l : m : n$  and  $k$ . The coordinates of  $D$  are

$$0, \frac{a(a^2 + b^2) \sin C}{K}, \frac{ac^2 \sin B}{K}.$$

Similarly those of  $E$  and  $F$ .

## THE SEVEN-POINT CIRCLE.\*

**4754c** Let lines be drawn through  $A, B, C$  (Fig. 165) parallel to the sides of the triangles  $DEF, D'E'F'$ , as in the figure, intersecting each other in  $P, P', L, M, N$ . Let  $Q$  be the circum-centre; then the seven points  $P, P', L, M, N, Q, S$  all lie on the circumference of a circle concentric with the T. R. circle. (16)

The proof depends on Euclid III. 21, and the similar triangles  $DEF, D'E'F'$ .

The radius  $\rho'$  of the seven-point circle is

$$\rho' = \frac{\rho}{\lambda} \sqrt{K^2 - 3\lambda^2} = \frac{2PP' \sin 2\omega}{1 - 3 \tan^2 \omega} \dots\dots\dots \begin{cases} (17), \\ (18), \end{cases}$$

obtained from  $\rho'^2 = \rho^2 + SD^2 - 2\rho SD \cos (B - TDD')$ .

Expand and substitute  $\cos TDD' = \frac{DD'}{2\rho} = \frac{a^2}{2\rho K}$ , by (3) and (5),

$$\sin TDD' = \cos \theta \text{ (11), } \cos B = \frac{c^2 + a^2 - b^2}{2ca}, \sin B = \frac{2\Delta}{ac}, \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

$$3\rho + \rho' = R^2, \text{ by (17) and (13); } \frac{\rho'}{\rho} = \sqrt{\frac{\cos 3\omega}{\cos \omega}}, \text{ by (17) and (9) ... } \begin{cases} (19), \\ (20). \end{cases}$$

The trilinear equation of the seven-point circle is

$$abc(a^2 + \beta^2 + \gamma^2) = a^3\beta\gamma + b^3\gamma\alpha + c^3\alpha\beta \dots\dots\dots (21),$$

or  $a\beta\gamma + b\gamma\alpha + c\alpha\beta = \frac{1}{K}(bca + ca\beta + ab\gamma)(a\alpha + b\beta + c\gamma) \dots\dots\dots (22).$

If the coordinates of  $P$  are  $\alpha_1, \beta_1, \gamma_1$ , and those of  $P'$   $\alpha'_1, \beta'_1, \gamma'_1$ ; then

$$\alpha_1\alpha'_1 = \beta_1\beta'_1 = \gamma_1\gamma'_1 \dots\dots\dots (23).$$

The equation of  $STQ$  is, by (4615),

$$a \sin (B - C) + \beta \sin (C - A) + \gamma \sin (A - B) \dots\dots\dots (24).$$

And the equation of  $PP'$  is

$$\frac{a}{a} (a^4 - b^2c^2) + \frac{\beta}{b} (b^4 - c^2a^2) + \frac{\gamma}{c} (c^4 - a^2b^2) = 0 \dots\dots\dots (25).$$

The point  $S$  has been called the Symmedian point of the triangle. It has also this property. *The line joining the mid-*

\* This circle was discovered by M. H. Brocard, and has been called "The Brocard Circle," the points  $P, P'$  being called the Brocard points.

point of any side to the mid-point of the perpendicular on that side passes through  $S$ .

PROOF.—Let  $X, Y, Z$  (Fig. 166) be the feet of the perpendiculars;  $x, y, z$  the mid-points of the same, and  $X', Y', Z'$  the mid-points of the sides. Now the trilinear coordinates of  $X', S$ , and  $x$  in order are proportional to

$$\begin{vmatrix} 0, & c, & b \\ a, & b, & c \\ 1, & \cos C, & \cos B \end{vmatrix}. \quad \begin{array}{l} \text{This determinant vanishes;} \\ \text{therefore the three points are on} \\ \text{the same right line, by (4615).} \end{array}$$

That the three lines  $X'x, Y'y, Z'z$  are concurrent appears at once by (970), since  $CX = 2Y'x$ , &c.

The Symmedian point may also be defined as the intersection of the three lines drawn from  $A, B, C$  to the corresponding vertices of the triangle formed by tangents to the circum-circle at  $A, B, C$ .

Let  $Ba, C\beta, A\gamma$  be taken  $= CX, AY, BZ$  respectively. Then  $Aa, B\beta, C\gamma$  meet in a point  $\Sigma$ , by (976), and this point by similarity of figure is the Symmedian point of the triangle formed by lines through  $A, B, C$  parallel to the sides  $BC, CA, AB$ .

If the sides of  $X'Y'Z'$  be bisected, similar reasoning shews that  $\sigma$ , the Symmedian point of the triangle  $X'Y'Z'$ , lies on  $S\Sigma$ .

It can also be shewn that, if  $A'B'C'$  be any triangle having its sides parallel to those of  $ABC$  and its vertices on  $SA, SB, SC$ , the sides of the two triangles intersect in six points on a circle whose centre lies midway between the circum-centres of the same triangles. When  $A'B'C'$  shrinks to the point  $S$ , the circle becomes the T. R. circle.

A more general theorem respecting the triangle and circle is the following—

Take  $ABC$  any triangle, and let  $DD'EE'FF'$  be the points in order, in which any circle cuts the sides.

$$\text{Let } \left. \begin{array}{lll} BD = pc, & CE = qa, & AF = rb \\ CD' = p'b, & AE' = q'c, & BF' = r'a \end{array} \right\} \dots\dots\dots (26).$$

From  $BD.BD' = BF.BF'$ , &c., Euclid III. 35, we can write three equations which are satisfied by the values

$$p = r' = tac, \quad q = p' = tab, \quad r = q' = tbc \dots\dots\dots (27),$$

and from these equations it appears that

$$DF = \sigma c; \quad D'F' = \sigma a, \text{ \&c., where } \sigma = \sqrt{(t^2\lambda^2 - tK + 1)} \dots\dots (28),$$

so that  $DEF$  and  $D'E'F'$  are both similar to  $ABC$ .

Also  $DF' = tabc$ , therefore  $DF' = FE' = ED' \dots\dots\dots (29)$ .

From  $\sin BFD = \frac{tac \sin B}{\sigma}$  we can obtain

$$\cot BFD = \cot \phi = \mp \frac{tK-2}{4t\Delta} \dots\dots\dots (30).$$

The radius of the circle  $= \sigma R \dots\dots\dots (31)$ ,  
and the coordinates of its centre are

$$\alpha = R \left\{ \cos A + \frac{t(Ka^2 - a^4 - b^4 - c^4)}{2bc} \right\}. \text{ Similarly } \beta \text{ and } \gamma \dots\dots (32).$$

The equation of the circle is

$$a\beta\gamma + b\gamma\alpha + ca\beta = t(aa + b\beta + c\gamma) \{ abc(1 - ta^2) + \&c. \} \dots\dots (33),$$

$$\text{or } tabc \{ a^2(1 - ta^2) + \beta^2(1 - tb^2) + \gamma^2(1 - tc^2) \} \\ = a\beta\gamma \{ (1 - tb^2)(1 - tc^2) + t^2b^2c^2 \} + \&c. \dots\dots (34).$$

When  $t = 0$ ,  $\sigma = 1$  and the circle is the circum-circle  $\dots\dots\dots (35)$ .

When  $tK = 1$ ,  $\sigma = t\lambda = \frac{\lambda}{K}$  and the circle is the T. R. circle  $\dots\dots (36)$ .

## CONIC AND SELF-CONJUGATE TRIANGLE.

When the sides of the trigon are the polars of the opposite vertices, the general equation of the conic takes the form

$$4755 \quad l^2\alpha^2 + m^2\beta^2 - n^2\gamma^2 = 0.$$

PROOF.—(Fig. 51.) The equation may be written in any one of the three ways,

$$l^2\alpha^2 = (n\gamma + m\beta)(n\gamma - m\beta), \quad m^2\beta^2 = (n\gamma + l\alpha)(n\gamma - l\alpha), \\ n^2\gamma^2 = (l\alpha + m\beta)(l\alpha - m\beta).$$

Hence, by (4699),  $\alpha$  or  $BC$  is the chord of contact of the tangents  $n\gamma \pm m\beta$  ( $AQ$ ,  $AS$ ) drawn from  $A$ , and  $\beta$  is the chord of contact of the tangents  $n\gamma \pm l\alpha$  ( $BR$ ,  $BP$ ) drawn from  $B$ . Hence  $\alpha$ ,  $\beta$  are the polars of  $A$ ,  $B$  respectively; and therefore  $\gamma$  or  $AB$  is the polar of  $C$  (4130). Also  $\gamma$  may be considered to be the chord of contact of the imaginary tangents  $l\alpha \pm m\beta$  drawn from  $C$ .

4756 If the points of intersection of  $\alpha$  and  $\beta$  with the conic be joined, the equations of the sides of the quadrilateral so formed are

$$QR, \quad l\alpha + m\beta + n\gamma = 0, \quad SP, \quad l\alpha + m\beta - n\gamma = 0, \\ PQ, \quad -l\alpha + m\beta + n\gamma = 0, \quad RS, \quad l\alpha - m\beta + n\gamma = 0.$$

Hence  $QR$ ,  $SP$  and  $PQ$ ,  $RS$  intersect on the line  $\gamma$  in  $A'$  and  $B'$ .

**4757** Each pencil of four lines in the diagram is a harmonic pencil, by the test in (4649).

**4758** The triangle  $A'B'C$  is also self-conjugate with regard to the conic.

PROOF.—The equations of its sides  $CB'$ ,  $CA'$ ,  $A'B'$  are

$$la - m\beta = 0, \quad la + m\beta = 0, \quad \gamma = 0.$$

Denote these by  $\alpha'$ ,  $\beta'$ ,  $\gamma$ , and put  $\alpha$ ,  $\beta$  in (4755) in terms of  $\alpha'$ ,  $\beta'$ . The equation referred to  $A'B'C$  thus becomes  $\alpha'^2 + \beta'^2 - 2n^2\gamma^2 = 0$ , which is of the same form as (4755).

**4759** It is clear that the triangles  $AQS$  and  $BPR$ , formed by a pair of tangents and the chord of contact in each case, are also self-conjugate.

**4760** Taking  $A'B'C$  for the trigon, and denoting the sides by  $\alpha$ ,  $\beta$ ,  $\gamma$ , the equations of the sides  $RS$ ,  $PQ$ ,  $QR$ ,  $SP$  of the quadrilateral become respectively

$$n\gamma \pm la = 0, \quad m\beta \pm n\gamma = 0.$$

Ex.—As an example of (4611), we may find the coordinates of  $P$  from the equations

$$\left. \begin{aligned} \alpha\alpha + \beta\beta + \gamma\gamma &= \Sigma \\ 0 + m\beta - n\gamma &= 0 \\ -la + 0 + n\gamma &= 0 \end{aligned} \right\} \text{ from which } \begin{cases} \alpha = \Sigma lm \div (amn + bnl + clm) \\ \beta = \Sigma mn \div (amn + bnl + clm) \\ \gamma = \Sigma nl \div (amn + bnl + clm). \end{cases}$$

To obtain the coordinates of  $Q$ ,  $R$ , and  $S$ , change the signs of  $m$ ,  $n$ , and  $l$  respectively.

#### ON LINES PASSING THROUGH IMAGINARY POINTS.

**4761** LEMMA I.—The right line passing through two conjugate imaginary points is real, and is identical with the line passing through the points obtained by substituting unity for  $\sqrt{-1}$  in the given coordinates.

PROOF.—Let  $(a + ia', b + ib')$  be one of the imaginary points, and therefore  $(a - ia', b - ib')$  the conjugate point. The equation of the line passing through them is, by (4083) and reducing,  $b'x - a'y + a'b - ab' = 0$ , which is real.

But this is also the line obtained by taking for the coordinates of the points  $(a + a', b + b')$  and  $(a - a', b - b')$ .

LEMMA II.—If  $P$ ,  $S$  and  $Q$ ,  $R$  are two pairs of conjugate imaginary points, the lines  $PS$  and  $QR$  are real, as has just been shown, and, therefore, also their point of intersection is

real. The other pairs of lines  $PQ$ ,  $RS$  and  $PR$ ,  $QS$  are imaginary. But the points of intersection of each pair are real, and are identical with the points which are obtained by substituting unity for  $\sqrt{-1}$  in the given coordinates, and drawing the six lines accordingly.

PROOF.—Let the coordinates of the four points be as under—

$$\begin{array}{ll} P \dots a + ia', & b + ib', & Q \dots a + ia', & \beta + i\beta', \\ S \dots a - ia', & b - ib', & R \dots a - ia', & \beta - i\beta'. \end{array}$$

The equations of  $PR$  and  $QS$ , by (4083), are  $L + iM$  and  $L - iM$ , where

$$\begin{aligned} L &= (b - \beta) x - (a - a) y + a\beta - ab + a'\beta' - a'b', \\ M &= (b' + \beta') x - (a' - a') y + a'\beta - a'b - a\beta' + ab'. \end{aligned}$$

Now the lines  $L \pm iM = 0$  intersect in the same real point as the lines  $L \pm M = 0$ , because the values  $L = 0$ ,  $M = 0$  satisfy both equations simultaneously. Hence, to determine this point, we have only to take  $i$  as unity in the given coordinates.

LEMMA III.—If  $P$ ,  $S$  are real points, and  $Q$ ,  $R$  a pair of conjugate imaginary points, the lines  $PS$  and  $QR$  are both real, by Lemma I., and consequently their point of intersection is real. The remaining pairs of lines  $PQ$ ,  $RS$  and  $PR$ ,  $QS$  and their points of intersection are all imaginary. But the line joining these two imaginary points of intersection is real, and is identical with the line obtained by substituting unity for  $\sqrt{-1}$  in the given coordinates and drawing the six lines accordingly.

PROOF.—Let the coordinates of the four points be as under—

$$\begin{array}{ll} P \dots x_1 y_1, & Q \dots a + ia', & \beta + i\beta', \\ S \dots x_2 y_2, & R \dots a - ia', & \beta - i\beta'. \end{array}$$

Since the coordinates of  $R$  are obtained from those of  $Q$  by merely changing the sign of  $i$ , the equations of the four imaginary lines will take the forms

$$\begin{array}{ll} PQ \dots A - iB, & SQ \dots C - iD, \\ PR \dots A + iB, & SR \dots C + iD. \end{array}$$

Now let the coordinates of the point of intersection of  $PQ$  and  $SR$  be  $L + iM$ ,  $L' + iM'$ , then will  $L - iM$ ,  $L' - iM'$  be the coordinates of the intersection of  $PR$  and  $SQ$ , for the equations of this pair of lines are got from those of  $PQ$  and  $SR$  by merely changing the sign of  $i$ . The points of intersection are therefore conjugate imaginary points, and the line joining them is real, by Lemma I. Also, since that line is obtained by writing 1 for  $i$  in the coordinates of those points, it will also be obtained by writing 1 for  $i$  in the original coordinates of  $Q$  and  $R$  and constructing the figure as before.



**4762** To find a common pole and polar of two given conics :

(i.) If the conics intersect in four real points  $P, Q, R, S$ , construct the complete quadrilateral (4652). Then  $A'B'C$  (Fig. 51) is a self-conjugate triangle for each conic, by (4758), and therefore each vertex and the opposite side form a common pole and polar to the conics.

(ii.) If the conics do not intersect at all in real points, the triangle  $A'B'C$  is still real, by Lemma II. (4761), and can be constructed in the manner shown.

(iii.) If two of the points ( $P, S$ ) are real, and two ( $Q, R$ ) imaginary, then, by Lemma III., the vertex  $A'$  and the side  $B'C$  are real, and may be constructed, and they form a common pole and polar of the given conics.

Returning to the triangle of reference  $ABC$ ,

**4763** Let  $la = n\gamma \cos \phi$ ,  $m\beta = n\gamma \sin \phi$ ; then the chord joining two points  $\phi_1, \phi_2$  is

$$la \cos \frac{1}{2}(\phi_1 + \phi_2) + m\beta \sin \frac{1}{2}(\phi_1 + \phi_2) = n\gamma \cos \frac{1}{2}(\phi_1 - \phi_2),$$

and therefore the tangent at the point  $\phi'$  is

**4764**  $la \cos \phi' + m\beta \sin \phi' = n\gamma.$

**4765** Putting  $l^2 = L$ ,  $m^2 = M$ ,  $n^2 = -N$ , the conic (4755) becomes

$$La^2 + M\beta^2 + N\gamma^2 = 0 \dots\dots\dots(1).$$

**4766** The tangent or polar of  $a'\beta'\gamma'$  is

$$La a' + M\beta \beta' + N\gamma \gamma' = 0 \dots\dots\dots(2).$$

**4767** Hence the pole of  $\lambda a + \mu \beta + \nu \gamma = 0$

is  $\left(\frac{\lambda}{L}, \frac{\mu}{M}, \frac{\nu}{N}\right) \dots\dots\dots(3).$

**4768** The tangential equation is

$$\frac{\lambda^2}{L} + \frac{\mu^2}{M} + \frac{\nu^2}{N} = 0 \dots\dots\dots(4),$$

and this is the condition that the conic (1) may be touched by the four lines

$$\lambda a \pm \mu \beta \pm \nu \gamma = 0.$$

**4769** In like manner,

$$La'^2 + M\beta'^2 + N\gamma'^2 = 0 \dots\dots\dots (5)$$

is the condition that (1) may pass through the four points

$$(\alpha', \pm\beta', \pm\gamma').$$

**4770** The locus of the pole of the line  $\lambda\alpha + \mu\beta + \nu\gamma$  with respect to such conics is

$$\frac{\lambda\alpha'^2}{a} + \frac{\mu\beta'^2}{\beta} + \frac{\nu\gamma'^2}{\gamma} = 0.$$

PROOF.—By (3), if  $(\alpha, \beta, \gamma)$  be the pole,  $a = \frac{\lambda}{L}$  &c.,  $\therefore L = \frac{\lambda}{a}$ , in (5), the equation of condition.

**4771** The locus of the pole of the line  $l\alpha + m\beta + n\gamma$ , with respect to the conics which touch the four lines  $\lambda\alpha \pm \mu\beta \pm \nu\gamma$

is 
$$\frac{\lambda^2\alpha}{l} + \frac{\mu^2\beta}{m} + \frac{\nu^2\gamma}{n} = 0.$$

PROOF.—By (3), if  $(\alpha, \beta, \gamma)$  be the pole,  $a = \frac{l}{L}$  &c.,  $\therefore L = \frac{l}{a}$ , &c., in (4), the equation of condition.

**4772** The locus of the centre of the conic is given in each case (4770, '1) by taking the line at infinity

$$a \sin A + \beta \sin B + \gamma \sin C$$

for the fixed line, since its pole is the centre.

**4773** Thus the locus of the centre of the conic passing through the four points  $(\alpha' \pm \beta' \pm \gamma')$  is

$$\frac{\alpha'^2 \sin A}{a} + \frac{\beta'^2 \sin B}{\beta} + \frac{\gamma'^2 \sin C}{\gamma} = 0.$$

**4774** The coordinates of the centre of the conic (1) are given by

$$\frac{La}{a} = \frac{M\beta}{b} = \frac{N\gamma}{c}.$$

PROOF.—Let the conic cut the side  $a$  in the points  $(0\beta_1\gamma_1)$ ,  $(0\beta_2\gamma_2)$ . The right line from  $A$  bisecting the chord will pass through the centre of the conic, and its equation will be  $\beta : \gamma = \beta_1 + \beta_2 : \gamma_1 + \gamma_2$ . Now  $\beta_1 + \beta_2$  is the sum of the roots of the quadratic in  $\beta$  obtained by eliminating  $\gamma$  and  $a$  from the equations  $La^2 + M\beta^2 + N\gamma^2 = 0$ ,  $a = 0$ , and  $aa + b\beta + c\gamma = \Sigma$ . Similarly for  $\gamma_1 + \gamma_2$  eliminate  $a$  and  $\beta$ . The equation of the diameter through  $A$  being found, those through  $B$  and  $C$  are symmetrical with it.

**4775** The condition that the conic (1) may be a parabola is

$$\frac{a^2}{L} + \frac{b^2}{M} + \frac{c^2}{N} = 0.$$

PROOF.—This is, by (4), the condition of touching the line at infinity

$$aa + b\beta + c\gamma = 0.$$

**4776** The condition that (1) may be a rectangular hyperbola is  $L + M + N = 0$ , and in this case the curve passes through the centres of the inscribed and escribed circles of the trigon.

PROOF.—By (4690), ( $a, b, c$  are now  $L, M, N$ ). (1) is now satisfied by  $a = \pm\beta = \pm\gamma$ , the four centres in question.

**4777** Circle referred to a self-conjugate triangle :

$$a^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C = 0.$$

PROOF.—The line joining  $A$  to the centre is  $\frac{M\beta}{b} = \frac{N\gamma}{c}$  (4774). Therefore  $\frac{M}{b \cos B} = \frac{N}{c \cos C}$ , the condition of perpendicularity to  $a$  by (4622). Similarly  $\frac{N}{c \cos C} = \frac{L}{a \cos A}$ , therefore (1) takes the form above.

## IMPORTANT THEOREMS.

### CARNOT'S THEOREM.

**4778** If  $A, B, C$  (Fig. 52) are the angles of a triangle, and if the opposite sides intersect a conic in the pairs of points  $a, a'; b, b'; c, c'$ ; then

$$Ac.Ac'.Ba.Ba'.Cb.Cb' = Ab.Ab'.Bc.Bc'.Ca.Ca'.$$

PROOF.—Let  $\alpha, \beta, \gamma$  be the semi-diameters parallel to  $BC, CA, AB$ ; then, by (4317),  $Ab.Ab' : Ac : Ac' = \beta^2 : \gamma^2$ . Compound this with two similar ratios.

**4779** COR.—If the conic touches the sides in  $a, b, c$ , then

$$Ac^2.Ba^2.Cb^2 = Ab^3.Bc^3.Ca^3.$$

**4780** The reciprocal of Carnot's theorem is: If  $A, B, C$  (Fig. 52) are the sides of a triangle, and if pairs of tangents from the opposite angles are  $a, a'$ ;  $b, b'$ ;  $c, c'$ ; then

$$\begin{aligned} & \sin (Ac) \sin (Ac') \sin (Ba) \sin (Ba') \sin (Cb) \sin (Cb') \\ &= \sin (Ab) \sin (Ab') \sin (Bc) \sin (Bc') \sin (Ca) \sin (Ca'), \end{aligned}$$

where  $(Ac)$  signifies the angle between the lines  $A$  and  $c$ .

PROOF.—Reciprocating the former figure with respect to any origin  $O$ , let  $A, B, C$  (i.e.,  $RQ, QP, PR$ ) be the polars of the vertices  $A, B, C$ . Then, by (4130),  $Q, R$  will be the poles of  $AB, AC$ ; and  $b, b'$ , the polars of the points  $b, b'$ , will intersect in  $R$  and touch the reciprocal conic. Similarly,  $c, c'$  will intersect in  $Q$ .  $A, b'$  are perpendicular to  $OA, Ob'$ , and therefore  $\angle Ab' = \angle AOb'$ , and so of the rest.

#### PASCAL'S THEOREM.

**4781** The opposite sides of a hexagon inscribed to a conic meet in three points on the same right line.

PROOF.—(Fig. 53.) Let  $\alpha, \beta, \gamma, \gamma', \beta', \alpha'$  be the consecutive sides of the hexagon, and let  $u$  be the diagonal joining the points  $\alpha\alpha'$  and  $\gamma\gamma'$ . The equation of the conic is either  $\alpha\gamma - k\beta u = 0$  or  $\alpha'\gamma' - k'\beta'u = 0$ , and, since these expressions vanish for all points on the curve, we must have  $\alpha\gamma - k\beta u = \alpha'\gamma' - k'\beta'u$  for any values of the coordinates. Therefore  $\alpha\gamma - \alpha'\gamma' = u(k\beta - k'\beta')$ . Therefore the lines  $\alpha, \alpha'$  and also  $\gamma, \gamma'$  meet on the line  $k\beta - k'\beta'$ ; and  $\beta, \beta'$  evidently meet on that line.

Otherwise, by projecting a hexagon inscribed in a circle with its opposite sides parallel upon any plane not parallel to that of the circle. The line at infinity, in which the pairs of parallel sides meet, becomes a line in which the corresponding sides of a hexagon inscribed in a conic meet at a finite distance (1075 *et seq.*).

**4782** With the same vertices there are sixty different hexagons inscribable in any conic, and therefore sixty different Pascal lines corresponding to any six points on a conic.

PROOF.—Half the number of ways of taking in order five vertices  $B, C, D, E, F$  after  $A$  is the number of different hexagons that can be drawn, and the demonstration in (4781) applies equally to all.

#### BRIANCHON'S THEOREM.

**4783** The three diagonals of a hexagon circumscribed to a conic pass through the same point (Fig. 54).

PROOF.—Let the three conics  $S+L^2$ ,  $S+M^2$ ,  $S+N^2$ , in (4707), become three pairs of right lines, then the three lines  $L-M$ ,  $M-N$ ,  $N-L$  become the diagonals of a circumscribing hexagon.

Pascal's and Brianchon's theorems may be obtained, the one from the other, by reciprocation (4840).

## THE CONIC REFERRED TO TWO TANGENTS AND THE CHORD OF CONTACT.

Let  $L=0$ ,  $M=0$ ,  $R=0$  (Fig. 55) be the sides of the trigon;  $L$ ,  $M$  being tangents and  $R$  the chord of contact.

**4784** The equation of the conic is  $LM=R^2$ . (4699)

**4785** The lines  $AP$ ,  $BP$ , and  $CP$  are respectively

$$\mu L = R, \quad \mu R = M, \quad \mu^2 L = M. \quad [\text{By (4604).}]$$

Since the point  $P$  on the curve is determined by the value of  $\mu$ , it is convenient to call it the point  $\mu$ .

**4788** The points  $\mu$  and  $-\mu$  ( $P$  and  $Q$ ) are both on the line  $\mu^2 L = M$  drawn through  $C$ .

**4789** The secant through the points  $\mu$ ,  $\mu'$  ( $P$ ,  $P'$ ) is

$$\mu\mu' L - (\mu + \mu') R + M = 0.$$

PROOF.—Write it  $\mu(\mu' L - R) - (\mu' R - M)$ , and, by (4604), it passes through the point  $\mu'$ . Similarly through  $\mu$ . Otherwise, determine the coordinates of the intersection of  $\mu L - R$  and  $\mu R - M$ , and of  $\mu' L - R$  and  $\mu' R - M$  by (4610), and the equation of the secant by (4616).

**4790** COR.—The tangents at the points  $\mu$  and  $-\mu$  ( $P$ ,  $Q$ ) are therefore

$$\mu^2 L \mp 2\mu R + M = 0.$$

**4791** These tangents intersect on  $R$ . [Proof by subtraction.]

**4792** Theorem.—If the equation of a right line contains an indeterminate  $\mu$  in the second degree, it may be written as above, and the line must therefore touch the conic  $LM=R^2$ .

**4793** The polar of the point ( $L'$ ,  $M'$ ,  $R'$ ) is

$$LM' - 2RR' + L'M = 0.$$

PROOF.—For  $\mu + \mu'$  and  $\mu\mu'$ , in (4789), put the values of the sum and product of the roots of  $\mu^2 L' - 2\mu R' + M' = 0$  (4790).

**4794** Similarly the polar of the point of intersection of  $aL - R$  and  $bR - M$  is

$$abL - 2aR + M = 0.$$

**4795** The line  $CE$  joining the vertex  $C$  to the intersection of two tangents at  $\mu$  and  $\mu'$ , or at  $-\mu$  and  $-\mu'$ , is

$$\mu\mu' L - M = 0.$$

Otherwise, if two tangents meet on any line  $aL - M$ , drawn through  $C$ , the product of their  $\mu$ 's is equal to  $a$ .

PROOF.—Eliminate  $R$  from the equations of the two tangents (4790).

**4796** The chords  $PQ'$ ,  $P'Q$  and the line  $CE$  all intersect in the same point on  $R$ .

PROOF.—The equations of  $PQ'$ ,  $P'Q$  are, by (4789),

$$\mu\mu' L \pm (\mu - \mu') R - M = 0,$$

and, by addition and subtraction, we obtain  $\mu\mu' L - M = 0$  (4795), or  $R = 0$ .

**4797** The lines  $\mu\mu' L + M$  ( $CD$ ) and  $R$  intersect on the chord  $PP'$  which joins the points  $\mu$ ,  $\mu'$ ; or—The extremities of any chord passing through the intersection of  $aL + M$  and  $R$  have the product of their  $\mu$ 's equal to  $a$ .

**4798** The chord joining the points  $\mu \tan \phi$ ,  $\mu \cot \phi$  touches a conic having the same tangents  $L$ ,  $M$  and chord of contact  $R$ .

PROOF.—The equation of the chord is, by (4789),

$$\mu^2 L - \mu R (\tan \phi + \cot \phi) + M = 0,$$

and this touches the conic  $LM \sin^2 2\phi = R^2$  at the point  $\mu$ , by (4792).

**4799** The tangents at the points  $\mu \tan \phi$ ,  $\mu \cot \phi$  intersect on the conic  $LM = R^2 \sin^2 2\phi$ .

PROOF.—Write the equations of the two tangents, by (4790), and then eliminate  $\mu$ .

**4800** Ex. 1.—To find the locus of the vertex of a triangle circumscribing a fixed conic and having its other vertices on two fixed right lines.

Take  $LM = R^2$  for the conic (Fig. 56),  $aL + M$ ,  $bL + M$  for the lines  $CD$ ,  $CE$ . Let one tangent,  $DE$ , touch at the point  $\mu$ ; then, by (4795), the others,



$PD, PE$ , will touch at the points  $\frac{a}{\mu}, \frac{b}{\mu}$ , and therefore, by (4790), their equations will be

$$\frac{a^2}{\mu^2}L - \frac{2a}{\mu}R + M, \quad \frac{b^2}{\mu^2}L - \frac{2b}{\mu}R + M.$$

Eliminate  $\mu$ , and the locus of  $P$  is found to be  $(a+b)^2LM = 4abR^2$ .

[Salmon, Art. 272.]

**4801** Ex. 2.—To find the envelope of the base of a triangle inscribed in a conic, and whose sides pass through fixed points  $P, Q$ .

(Fig. 57.) Take the line through  $P, Q$  for  $R$ ;  $LM - R^2$  for the conic;  $aL - M$ ,  $bL - M$  for the lines joining  $P$  and  $Q$  to the vertex  $C$ . Let the sides through  $P$  and  $Q$  meet in the point  $\mu$  on the conic; then, by (4797), the other extremities will be at the points  $-\frac{a}{\mu}$  and  $-\frac{b}{\mu}$ , and therefore, by (4789), the equation of the base will be  $abL + (a+b)\mu R + \mu^2M = 0$ . By (4792), this line always touches the conic  $4abLM = (a+b)^2R^2$ . [Ibid.]

**4802** Ex. 3.—To inscribe a triangle in a conic so that its sides may pass through three fixed points. (See also 4823.)

We have to make the base  $abL + (a+b)\mu R + \mu^2M$  (4801) pass through a third fixed point. Let this point be given by  $cL = R$ ,  $dR = M$ . Eliminating  $L, M, R$ , we get  $ab + (a+b)\mu c + \mu^2cd = 0$ , and since, at the point  $\mu$ ,  $\mu L = R$ ,  $\mu^2L = M$ , that point must be on the line  $abL + (a+b)cR + cdM$ . The intersections of this line with the conic give two solutions by two positions of the vertex. [Ibid.]

#### RELATED CONICS.

**4803** A conic having double contact with the conics  $S$  and  $S'$  (Fig. 58) is

$$\mu^2E^2 - 2\mu(S + S') + F^2 = 0,$$

where  $E, F$  are common chords of  $S$  and  $S'$ , so that  $S - S' = EF$ .

PROOF.—The equation may be written in either of the ways

$$(\mu E + F)^2 = 4\mu S \quad \text{or} \quad (\mu E - F)^2 = 4\mu S',$$

showing that  $\mu E \pm F$  are the chords of contact  $AB, CD$ . There are three such systems, since there are three pairs of common chords.

**4804** COR. 1.—A conic touching four given lines  $A, B, C, D$ , the diagonals being  $E, F$  (Fig. 59):

$$\mu^2E^2 - 2\mu(AC + BD) + F^2 = 0.$$

Here  $S = AC$  and  $S' = BD$ , two pairs of right lines.

Otherwise, if  $L, M, N$  be the diagonals and  $L \pm M \pm N$  the sides, the conic becomes

$$4805 \quad \mu^2 L^2 - \mu(L^2 + M^2 - N^2) + M^2 = 0.$$

For this always touches  
 $(L^2 + M^2 - N^2)^2 - 4L^2 M^2$  or  $(L + M + N)(M + N - L)(N + L - M)(L + M - N)$ .  
 [Salmon, Art. 287.]

4806 COR. 2.—A conic having double contact with two circles  $C, C'$  is

$$\mu^2 - 2\mu(C + C') + (C - C')^2 = 0.$$

4807 The chords of contact become

$$\mu + C - C' = 0 \quad \text{and} \quad \mu - C + C' = 0.$$

4808 The equation may also be written

$$\sqrt{C} \pm \sqrt{C'} = \sqrt{\mu},$$

which signifies that the sum or difference of the tangents drawn from any point on the conic to the circles is constant.

## ANHARMONIC PENCILS OF CONICS.

4809 The anharmonic ratio of the pencil drawn from any point on a conic through four fixed points upon it is constant.

PROOF.—Let the vertices of the quadrilateral in Fig. (38) be denoted by  $A, B, C, D$ , and let  $P$  be the fifth point. Multiplying the equation of the conic (4697) by the constants  $AB, CD, BC, DA$ , we have

$$\begin{aligned} k \frac{AB \cdot CD}{BC \cdot DA} &= \frac{AB\alpha \cdot CD\gamma}{BC\beta \cdot DA\delta} = \frac{PA \cdot PB \sin APB \cdot PC \cdot PD \sin CPD}{PB \cdot PC \sin BPC \cdot PD \cdot PA \sin DPA} \\ &= \frac{\sin APB \cdot \sin CPD}{\sin BPC \cdot \sin DPA}. \end{aligned}$$

Compare (1056).

4810 If the fifth point be taken for origin in the system (4784, Fig. 55), and if the four lines through it be

$$L - \mu_1 R, \quad L - \mu_2 R, \quad L - \mu_3 R, \quad L - \mu_4 R,$$

4 P

the anharmonic ratio of the pencil is, by (4650),

$$= \frac{(\mu_1 - \mu_2)(\mu_3 - \mu_4)}{(\mu_1 - \mu_4)(\mu_2 - \mu_3)}.$$

**4811** COR. 1.—If four lines through any point, taken for the vertex  $LM$ , meet the conic in the points  $\mu_1, \mu_2, \mu_3, \mu_4$ , the anharmonic ratio of these points, with any fifth point on the conic, is equal to that of the points  $-\mu_1, -\mu_2, -\mu_3, -\mu_4$ , in which the same lines again meet the conic.

**4812** COR. 2.—The reciprocal theorem is—If from four points upon any right line four tangents be drawn to a conic, the anharmonic ratio of the points of section with any fifth tangent is equal to the corresponding ratio for the other four tangents from the same points.

**4813** The anharmonic ratio of the segments of any tangent to a conic made by four fixed tangents is constant.

PROOF.—Let  $\mu, \mu_1, \mu_2, \mu_3, \mu_4$  (Fig. 60) be the points of contact. The anharmonic ratio of the segments is the same as that of the pencil of four lines from  $LM$  to the points of section; that is, of  $\mu\mu_1L-M, \mu\mu_2L-M, \mu\mu_3L-M, \mu\mu_4L-M$ , a pencil homographic (4651) with that in (4810).

**4814** If  $P, P'$  are the polars of a point with respect to the conics  $S, S'$ , then  $P+kP'$  will be the polar of the same point with respect to the conic  $S+kS'$ .

**4815** Hence the polar of a given point with regard to a conic passing through four given points (the intersections of  $S$  and  $S'$ ) always passes through a fixed point, by (4101).

If  $Q, Q'$  are the polars of another point with respect to the same conics,  $Q+kQ'$  is the polar with respect to  $S+kS'$ .

**4816** Hence the polars of two points with regard to a system of conics through four points form two homographic pencils (4651).

**4817** The locus of intersections of corresponding lines of two homographic pencils having fixed vertices (Fig. 61) is a conic passing through the vertices; and, conversely, if the conic be given, the pencils will be homographic.

PROOF.—For eliminating  $k$  from  $P+kP'=0, Q+kQ'=0$ , we get  $PQ'=P'Q$ .

**4818** COR.—The locus of the pole of the line joining the two points in (4816) is a conic.

PROOF.—For the pole is the intersection of  $P+kP'$  and  $Q+kQ'$ .

**4819** The right lines joining corresponding points  $AA'$ , &c. (Fig. 62) of two homographic systems of points lying on two right lines, envelope a conic.

PROOF.—This is the reciprocal theorem to (4817); or it follows from (4813).

**4820** If two conics have double contact (Fig. 63), the anharmonic ratio of the points of contact  $A, B, C, D$  of any four tangents to the inner conic is the same as that of each set of four points  $(a, b, c, d)$  or  $(a', b', c', d')$  in which the tangents meet the other conic.

PROOF.—By (4798). The  $\mu$ 's for the points on the latter conic will be equal to the  $\mu$ 's of the points of contact multiplied by  $\tan \phi$  for one set, and by  $\cot \phi$  for the other, and therefore the ratio (4810) will be unaltered.

**4821** Conversely, if three chords of a conic  $aa', bb', cc'$  be fixed, and a fourth  $dd'$  moves so that  $\{abcd\} = \{a'b'c'd'\}$ , then  $dd'$  envelopes a conic having double contact with the given one.

For theorems on a right line cut in involution by a conic, see (4824-8).

## CONSTRUCTION OF CONICS.

### THEOREMS AND PROBLEMS.

**4822** If a polygon inscribed to a conic (Fig. 64) has all its sides but one passing through fixed points  $A, B, \dots Y$ , the remaining side  $az$  will envelope a conic having double contact with the given one.

PROOF.—Let  $a, b, \dots z$  be the vertices of the polygon, and  $a, a', a'', a'''$  four successive positions of  $a$ . Then, by (4811),

$$\{a, a', a'', a'''\} = \{b, b', b'', b'''\} = \&c. = \{z, z', z'', z'''\}.$$

Therefore, by (4821), the side  $az$  envelopes a conic, &c.

**4823** Poncelet's construction for inscribing in a conic a polygon having its  $n$  sides passing through  $n$  given points.

*Inscribe three polygons, each of  $n+1$  sides, so that  $n$  of each may pass through the fixed points, and let the remaining sides be  $a'z'$ ,  $a''z''$ ,  $a'''z'''$ , denoted in figure (65) by AD, CF, EB. Let MLN, the line joining the intersections of opposite sides of the hexagon ABCDEF (4781), meet the conic in K; then K will be a vertex of the required polygon.*

PROOF.—  $\{D.KACE\} = \{A.KDFB\}$ , each pencil passing through K, P, N, L; therefore the anharmonic ratio  $\{KACE\} = \{KDFB\}$  for any vertex on the conic, by (4809); i.e.,  $\{Ka'a''a'''\} = \{Kz'z''z'''\}$ . But, if  $az$  be the remaining side of a fourth polygon inscribed like the others, we have by (4811), as in (4822),  $\{aa'a''a'''\} = \{zz'z''z'''\}$ . Hence K is the point where  $a$  and  $z$  coincide.

**4824** Lemma.—A system of conics passing through four fixed points meets any transversal in a system of points in involution (1066).

PROOF.—Let  $u, u'$  be two conics passing through the four points; then  $u + ku'$  will be any other. Take the transversal for  $x$  axis, and put  $y = 0$  in each conic, and let their equations thus become  $ax^2 + 2gx + c = 0$  and  $a'x^2 + 2g'x + c' = 0$ . These determine the points where the transversal meets  $u$  and  $u'$ . It will then meet  $u + ku'$  in two points given by  $ax^2 + 2gx + c + k(a'x^2 + 2g'x + c') = 0$ , and these points are in involution with the former, by (1065).

Geometrically (Fig. 66),

$$\{a.AdbA'\} = \{c.AdbA'\} \quad (4809),$$

therefore  $\{ACBA'\} = \{AB'C'A'\} = \{A'C'B'A'\}$ , therefore by (1069).

**4825** COR. 1.—One of the conics of the system resolves itself into the two diagonals  $ac, bd$ . Hence the points  $B, B', C, C'$  are in involution with  $D, D'$ , where the transversal cuts the diagonals.

**4826** COR. 2.—A transversal meets a conic and two tangents in four points in involution, so as to meet the chord of contact in one of the foci of the system.

For, in (Fig. 66), if  $b$  coincides with  $c$ , and  $a$  with  $d$ , the transversal meets the tangents in  $C, C'$ , while  $B, B', D, D'$ , all coincide in  $F$  (Fig. 67), one of the foci on the chord of contact.



**4827** The reciprocal theorem to (4824) is—Pairs of tangents from any point to a system of conics touching four fixed lines, form a system in involution (4850).

**4828** The condition that  $\lambda x + \mu y + \nu z$  may be cut in involution by three conics is the vanishing of the determinant

$$\begin{vmatrix} A_1 & H_1 & B_1 \\ A_2 & H_2 & B_2 \\ A_3 & H_3 & B_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 & 2f_1 & 2g_1 & 2h_1 \\ a_2 & b_2 & c_2 & 2f_2 & 2g_2 & 2h_2 \\ a_3 & b_3 & c_3 & 2f_3 & 2g_3 & 2h_3 \\ \lambda & 0 & 0 & 0 & \nu & \mu \\ 0 & \mu & 0 & \nu & 0 & \lambda \\ 0 & 0 & \nu & \mu & \lambda & 0 \end{vmatrix}.$$

where  $A_1, H_1, B_1$  belong to the first conic and have the values in (4988).

PROOF.—The quadratic  $A_1x^2 + 2H_1xy + B_1y^2 = 0$ , obtained in (4987), determines the pair of points of intersection with the first conic. The similar equation for the third conic will have  $A_3 = A_1 + \lambda A_2$ , &c., if the points are all in involution (1065). The third equation is therefore derived from the other two; therefore the determinant vanishes, by (583).

By expanding and dividing by  $\nu^3$ , the second determinant above of the sixth order is obtained.

### *Newton's Method of Generating a Conic.*

**4829** Two constant angles  $aPb, aQb$  (Fig. 68) move about fixed vertices  $P, Q$ . If  $a$  moves on a fixed right line,  $b$  describes a conic which passes through  $P$  and  $Q$ .

PROOF.—Taking four positions of  $a$ , we have (see 1054),

$$\{P.bb'b''b'''\} = \{P.aa'a''a'''\} = \{Q.aa'a''a'''\} = \{Q.bb'b''b'''\}.$$

Therefore, by (4817), the locus of  $b$  is a conic.

### *Maclaurin's Method of Generating a Conic.*

**4830** The vertex  $V$  of a triangle (Fig. 69), whose sides pass through fixed points  $A, B, C$ , and whose base angles move on fixed lines  $Oa, Ob$ , describes a conic passing through  $A$  and  $B$ .

PROOF.—The pencils of lines through  $A$  and  $B$  in the figure are both homographic with the pencil through  $C$ , and are therefore homographic with each other. Therefore the locus of  $V$  is a conic, by (4817).





Otherwise, let  $\alpha, \beta, \gamma$  be the sides of  $ABC$ ;  $l\alpha + m\beta + n\gamma, l'\alpha + m'\beta + n'\gamma$  the fixed lines  $Oa, Ob$ ; and  $a = \mu\beta$  the moving base  $ab$ .

Then the equations of the sides will be

$$(l\mu + m)\beta + n\gamma = 0, \quad (l'\mu + m')\alpha + n'\mu\gamma = 0.$$

Eliminate  $\mu$ ; then  $lm'\alpha\beta = (m\beta + n\gamma)(l'\alpha + n'\gamma)$ , the conic in question, by (4697).

**4831** Given five points, to find geometrically any number of points on the circumscribing conic, and to find the centre.

Let  $A, B, C, D, E$  (Fig. 70) be the five points. Draw any line through  $A$  meeting  $CD$  in  $P$ . Draw  $PQ$  through the intersection of  $AB$  and  $DE$  meeting  $BC$  in  $Q$ ; then  $QE$  will meet  $PA$  in  $F$ , a sixth point on the curve, as is evident from Pascal's theorem (4781).

To find the centre, choose  $AP$  in the above construction parallel to  $CD$ , and find two diameters, as in (1252).

**4832** To find the points of contact of a conic with five right lines.

Let  $ABCDE$  (Fig. 71) be the pentagon. Join  $D$  to the intersection of  $AC$  and  $BE$ . This line will pass through the point of contact of  $AB$ , and so on.

PROOF.—By (4783), supposing two sides of the hexagon to become one straight line.

**4833** To describe a conic, given four points upon it and a tangent.

Let  $a, a', b, b'$  (exterior letters in Fig. 52) be the four points. Then, if  $AB$  is a tangent,  $c, c'$  coincide, and Carnot's theorem (4778) gives the ratio  $Ac^2 : Bc^2$ . Then by (4831). Since there are two values of this ratio,  $\pm (Ac : Bc)$ , two conics may be drawn as required.

**4834** To describe a conic, given four tangents and a point.

Let  $a, a', b, b'$  (interior letters in Fig. 52) be the four tangents. Then, if  $Q$  be the given point on the curve, the lines  $c, c'$  must coincide in direction, and (4780) gives the ratio  $\sin^2 (Ac) : \sin^2 (Bc)$ , by which the direction of a fifth tangent through  $Q$  is determined. Then by (4832). The two values  $\pm (\sin Ac : \sin Bc)$  furnish two solutions.

Otherwise by (4804), determining  $\mu$  by the coordinates of the given point.

**4835** To describe a conic, given three points and two tangents.

Let  $A, A', A''$  be the points (Fig. 67, supplying obvious letters). Let the two tangents meet  $AA'$  in the points  $C, C'$ . Find  $F, F'$ , the foci of the system  $AA', CC'$  in involution (1066) determining the centre by (985). Similarly, find  $G, G'$ , the foci of a system on the line  $AA''$ . Then, by (4826), the chord of contact of the tangents may be any of the lines  $FG, FG', F'G, F'G'$ . There are accordingly four solutions, and the construction of (4831) determines the conic.

**4836** To describe a conic, given two points and three tangents.

Let  $AB, BC, CA$  (Fig. 167) be the tangents, and  $P, P'$  the points. Draw a transversal through  $PP'$  meeting the three tangents in  $Q, Q', Q''$ . Find  $F$ , a focus of the system  $PP', QQ'$  in involution (1066, 985);  $G$  a focus for  $PP', QQ''$ , and  $H$  for  $PP', Q'Q''$ . Construct a triangle with its sides passing through  $F, G, H$ , and with its vertices  $L, M, N$  on  $BC, CA, AB$ , by the method of (4823), which is equally applicable to a rectilinear figure as to a conic.  $L, M, N$  will be the points of contact. The reason for the construction is contained in (4826). There will, in general, be four solutions.

If the conic be a parabola, the foregoing constructions can be adapted by considering one tangent at infinity always to be given.

**4837** To draw a parabola through four given points  $a, a', b, b'$ .

This is problem 4833 with the tangent at infinity.

In figure (52), suppose  $cc'$  to coincide and  $AB$  to remove to infinity so as to become the tangent at  $c$ , the opposite vertex at infinity of a parabola, and therefore to be perpendicular to the axis.  $Cc$  then becomes a diameter of the parabola, and Carnot's theorem (4778) shows that

$$\frac{Ca.Ca'}{Cb.Cb'} = \frac{Ac^2}{Ab^2} \cdot \frac{Ba^2}{Bc^2} = \frac{\sin^2 ACc}{\sin^2 BCc},$$

since the points  $C, a, a', b, b'$  are all on the axis of the parabola relatively to the infinite distance of  $AcB$ . This result, however, is at once obtained from equation (4221),  $Ca.Ca' : Cb.Cb'$  being the ratio of the products of the roots of two similar quadratics. Thus a diameter of the parabola can be drawn through  $C$  by the known ratio of the sines of  $ACc$  and  $BCc$ .

Next, describe a circle round three of the given points  $a, a', b$ . By the property (1263) and the known direction of the axis, the other point in which the circle cuts the parabola can be found.

Five points being known, we can, by Pascal's theorem, as in (4831),

obtain two parallel chords, and then find  $P$ , the extremity of their diameter, by the proportion, square of ordinate  $\propto$  abscissa (1239).

Lastly, draw the diameter and tangent at  $P$ , and then, by equality of angles (1224), draw a line from  $P$  which passes through the focus. By obtaining in the same way another pair of parallel chords, a second line through the focus is found, thus determining its position.

**4838** To draw a parabola when four tangents are given.

This is effected by the construction of (4832, Fig. 71). Let  $AB, BC, AE, ED$  be the four tangents, and  $CD$  the tangent at infinity. Then any line drawn to  $C$  will be parallel to  $BC$ , and any line to  $D$  will be parallel to  $ED$ .

**4839** To draw a parabola, given three points and one tangent.

This is effected by the construction of (4835, Fig. 67). Let  $bC'$  be the tangent at  $\infty$ ; then the centre of involution  $O$  must be at  $C$ , so that  $CC.CO' = 0.\infty = CA.CA' = CF^2$ , determining  $F$ .  $F'$ , another point on the chord of contact, being found by joining  $AA''$  or  $A'A''$ ,  $FF'$  will be the diameter through  $a$ , since the other point of contact  $b$  is at infinity.

**4840** To draw a parabola, given one point and three tangents.

This is the case of (4834), in which one of the given tangents  $b'$  is at infinity.  $R$  must therefore be at infinity, and  $QR, PR$  and the tangent  $b$ , since they all join  $R$  to finite points, must be parallel. The ratio found determines another tangent, and the case is reduced to that of (4838).

**4841** To draw a parabola, given two points and two tangents.

This is problem (4836). Suppose  $AC$  in that construction to be the tangent at infinity.  $F, G, H$  will be determined as in (4839) by mean proportionals. The chords  $LM, NM$  will become parallel, since  $M$  is at infinity; and we have to draw  $LN$  and the parallel lines from  $L$  and  $N$  to pass through  $F, G, H$  in their new positions, so that the vertices  $L, N$  may lie on  $BC$  and  $AB$ .

Otherwise by (4509), the intercepts  $s$  and  $t$  can readily be found from the two equations furnished by the given points.

**4842** To describe a conic touching three right lines and touching a given conic twice.

Let  $AD, CF, EB$  (Fig. 65) be the three lines as they cut the given conic. Join  $AB, AF, BC, BE$ , and determine  $K$  by the Pascal line  $MLN$ .  $K$  will be one point of contact of the two conics, by (4822) and the proof in (4823), since  $AD, CF, EB$ , and the tangent at  $K$  are four positions of the "remaining side" in that proposition. The problem is thus reduced to

(4834), since four tangents and  $K$  the point of contact of one of them are now known.

**4843** To describe a conic touching each of two given conics twice, and passing through a given point or touching a given line.

*Proceed by (4803), determining  $\mu$  by the last condition.*

To describe a conic touching the conics  $S+L^2$ ,  $S+M^2$ ,  $S+N^2$  (4707) and touching  $S$  twice. [Salmon, Art. 387.]

## THE METHOD OF RECIPROCAL POLARS.

DEF.—The polar reciprocal of a curve is the envelope of the polars of all the points on the curve, or it is the locus of the poles of all tangents to the curve, taken in each case with respect to an arbitrary fixed origin and circle of reciprocation.

**4844** Thus, in figure (72), to the points  $P, Q, R$  on one curve correspond the tangents  $qr, rp$ , and chord of contact  $pq$  on the reciprocal curve; and to the points  $p, q, r$  correspond the tangents  $QR, RP$ , and chord  $PQ$ .

The angle between the tangents at  $P$  and  $Q$  is evidently equal to the angle  $pOq$ , since  $Op, Oq, Or$  are respectively perpendicular to  $QR, RP, PQ$ .

**4845** THEOREM.—*The distance of a point from a line is to its distance from the origin as the distance of the pole of the line from the polar of the point is to its distance from the origin.*

PROOF.—(Fig. 73.) Take  $O$  for origin and centre of auxiliary circle,  $PT$  the polar of  $c$ ,  $pt$  the polar of  $C$ ,  $CP$  perpendicular on polar of  $c$ ,  $cp$  perpendicular on polar of  $C$ . Then

$r^2 = OC \cdot Ot = Oc \cdot OT$  } Therefore, by subtraction,  $OC \cdot mt = Oc \cdot MT$ ,  
 and  $OC \cdot Om = Oc \cdot OM$  } or  $OC \cdot cp = Oc \cdot CP$ ;  
 that is,  $CP : CO :: cp : cO$ . Q. E. D.

COR.—By making  $CP$  constant, we see that the reciprocal of a circle is a conic having its focus at the origin and its directrix the polar of the circle's centre.

## GENERAL RULES FOR RECIPROCATING.

- 4846** *A point becomes the polar of the point, and a right line becomes the pole of the line.\**
- 4847** *A line through a fixed point becomes a point on a fixed line.*
- 4848** *The intersection of two lines becomes the line which joins their poles.*
- 4849** *Lines passing through a fixed point become the same number of points on a fixed line, the polar of the point.*
- 4850** *A right line intersecting a curve in  $n$  points becomes  $n$  tangents to the reciprocal curve passing through a fixed point.*
- 4851** *Two lines intersecting on a curve become two points whose joining line touches the reciprocal curve.*
- 4852** *Two tangents and the chord of contact become two points on the reciprocal curve and the intersection of the tangents at those points.*
- 4853** *A pole and polar of any curve become respectively a polar and pole of the reciprocal curve; and a point of contact and tangent become respectively a tangent and its point of contact.*
- 4854** *The locus of a point becomes the envelope of a line.*
- 4855** *An inscribed figure becomes a circumscribed figure.*
- 4856** *Four points connected by six lines or a quadrangle become four lines intersecting in six points or a quadrilateral.*
- 4857** *The angle between two lines is equal to the angle subtended at the origin by the corresponding points. (4844)*
- 4858** *The origin becomes a line at infinity, the polar of the origin.*
- 4859** *Two lines through the origin become two points at infinity on the polar of the origin.*
- 4860** *Two tangents through the origin to a curve become two points at infinity on the reciprocal curve.*
- 4861** *The points of contact of such tangents become asymptotes of the reciprocal curve.*
- 4862** *The angle between the same tangents is equal to the angle between the asymptotes. (4857)*

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\* That is, with respect to the circle of reciprocation, and so throughout with the exception of (4853).



**4863** According as the tangents from the origin to a conic are real or imaginary, the reciprocal curve is an hyperbola or ellipse.

**4864** If the origin be taken on the conic, the reciprocal curve is a parabola.

For, by (4860, '1), the asymptotes are parallel and at infinity.

**4865** A trilinear equation is converted by reciprocation into a tangential equation.

Thus  $\alpha\gamma = k\beta\delta$  is a conic passing through four of the intersections of the lines  $\alpha, \beta, \gamma, \delta$ . Reciprocating, we get a tangential equation of the same form  $AC = kBD$ , and this is a conic touching four of the lines which join the points whose tangential equations are  $A = 0, B = 0, C = 0, D = 0$ . See (4907).

**4866** The equation of the reciprocal of the conic  $a^2y^2 + b^2x^2 = a^2b^2$  with the same origin and axes is

$$a^2x^2 + b^2y^2 = k^4,$$

where  $k$  is the radius of the auxiliary circle whose centre is the centre of the conic.

PROOF.—Let  $p$  be the perpendicular on the tangent,  $\theta$  its inclination; then  $k^4r^{-2} = p^2 = a^2\cos^2\theta + b^2\sin^2\theta$  (4732).

**4867** The same when the origin of reciprocation is the point  $x'y'$ ,

$$(xx' + yy' + k^2)^2 = a^2x^2 + b^2y^2.$$

PROOF:  $k^2r^{-1} = p = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} - (x'\cos\theta + y'\sin\theta)$ .

**4868** The reciprocal curve of the general conic (4656), the auxiliary circle being  $x^2 + y^2 = k^2$  or  $x^2 + y^2 + z^2 = 0$  in trilinears, will be symmetrically

$$A\xi^2 + B\eta^2 + C\zeta^2 + 2F\eta\zeta + 2G\zeta\xi + 2H\xi\eta = 0,$$

replacing  $\zeta$  by  $-k^2$ .

PROOF.—Let  $\xi\eta$  be a point on the reciprocal curve, then the polar of  $\xi\eta$ , namely,  $x\xi + y\eta - k^2 = 0$ , must touch the conic, by (4853). Therefore, by (4665), we must substitute  $\xi, \eta, -k^2$  for  $\lambda, \mu, \nu$  in the tangential equation  $A\lambda^2 + \&c. = 0$ .

**4869** From the reciprocal of a curve with respect to the origin of coordinates, to deduce the reciprocal with respect to an origin  $x'y'$ , substitute in the given reciprocal equation

$$\frac{k^2x}{xx' + yy' + k^2} \text{ for } x \quad \text{and} \quad \frac{k^2y}{xx' + yy' + k^2} \text{ for } y.$$



PROOF.—Let  $P$  be the perpendicular from the origin on the tangent and  $PR = k^2$ . The perpendicular from  $x'y'$  is  $P - x' \cos \theta - y' \sin \theta$ ,

$$\therefore \frac{k^2}{\rho} = \frac{k^2}{R} - x' \cos \theta - y' \sin \theta, \quad \therefore \frac{k^2}{R} = \frac{xx' + yy' + k^2}{\rho};$$

$$\therefore R \cos \theta = \frac{k^2 \rho \cos \theta}{xx' + yy' + k^2}.$$

## TANGENTIAL COORDINATES.

**4870** By employing these coordinates, theorems which are merely the reciprocals of those already deduced in trilinears may be proved independently. See (4019) for a description of this system.

The following proposition serves to transform by reciprocation the whole system of trilinear coordinates of points and equations of right lines and curves, into tangential coordinates of right lines and equations of points and curves.

### THEOREM OF TRANSFORMATION.

**4871** Given the trilinear equation of a conic (4656), the tangential equation of the reciprocal conic in terms of  $\lambda, \mu, \nu$ , the perpendiculars from three fixed points  $A', B', C'$  upon the tangent (Fig. 74) will be as follows,  $O$  being the origin of reciprocation and  $OA', OB', OC' \equiv p, q, r$  :—

$$\mathbf{4872} \quad \frac{a\lambda^2}{p^2} + \frac{b\mu^2}{q^2} + \frac{c\nu^2}{r^2} + \frac{2f\mu\nu}{qr} + \frac{2g\nu\lambda}{rp} + \frac{2h\lambda\mu}{pq} = 0.$$

PROOF.—Let  $\alpha = 0, \beta = 0, \gamma = 0$  be the sides of the original trigon  $ABC$ . The poles of these lines will be  $A', B', C'$ , the vertices of the trigon for the reciprocal curve. Let  $RS$  be the polar of a point  $P$  on the given conic;  $\alpha, \beta, \gamma$  the perpendiculars from  $P$  upon  $BC, CA, AB$ ; i.e., the trilinear coordinates of  $P$ . Let  $\lambda, \mu, \nu$  be the perpendiculars from  $A', B', C'$  upon  $RS$ ; i.e., the tangential coordinates of the polar of  $P$  referred to  $A', B', C'$ . Then, by (4845),  $\frac{\alpha}{OP} = \frac{\lambda}{OA'}, \frac{\beta}{OP} = \frac{\mu}{OB'}, \frac{\gamma}{OP} = \frac{\nu}{OC'}$ . Substitute these values of  $\alpha, \beta, \gamma$  in (4656) and divide by  $OP^2$ .

**4873** The angular relation between the trigons  $ABC$  and  $A'B'C'$  is

$$B'OC' = \pi - A, \quad C'OB' = \pi - B, \quad A'OB' = \pi - C.$$

**4874** If  $ABC$  be self-conjugate with regard to the circle of reciprocation, it will coincide with  $A'B'C'$ .

**4875** Now let  $O$  be the circum-centre (4629) of  $A'B'C'$  (Fig. 74), then it will be the in-centre of  $ABC$ , and, by (4873),

$$2A' = \pi - A, \quad 2B' = \pi - B, \quad 2C' = \pi - C.$$

Also  $p = q = r$  in (4872), which becomes  $\phi(\lambda, \mu, \nu) = 0$ , so that the conic and its reciprocal are represented by the same equation. Consequently any relation in trilinear coordinates has its interpretation in tangential coordinates. We have then the following rule:—

**4876** RULE.—To convert any expression in trilinears into tangentials, consider the origin of the former as the in-centre of the trigon, change  $\alpha, \beta, \gamma$  into  $\lambda, \mu, \nu$ , and interpret the result by the rules for reciprocating (4846–65). If the angles of the original trigon are involved, change these by (4875) into the angles of the reciprocal trigon, of which the origin will now be the circum-centre.

**4877** Referring trilinears and tangentials to the same trigon  $ABC$ , the equation of a point, as shown in (4021), becomes

$$\frac{\alpha}{p_1} \lambda + \frac{\beta}{p_2} \mu + \frac{\gamma}{p_3} \nu = 0;$$

**4878** or, by multiplying by  $\frac{1}{2}\Sigma$ ,

$$BOC\lambda + COA\mu + AOB\nu = 0. \quad (\text{Fig. 3})$$

The equation of a point can generally be obtained directly from the figure by means of this formula.

#### EQUATIONS IN TANGENTIAL COORDINATES.

For direct demonstrations of the following theorems, the reader may consult *Ferrers' Trilinear Coordinates*, Chap. VII.

**4879** The point dividing  $AB$  in the ratio  $\alpha : \beta$ , that is, the intersection with the internal or external bisector of  $C$ , is

$$a\lambda \pm b\mu = 0. \quad \text{Centre of } AB \quad \lambda + \mu = 0.$$

The point  $O$  in (4878) is now on the side  $AB$ .

**4881** Mass-centre,  $\lambda + \mu + \nu = 0$ . [For  $BOC = COA = AOB$ .

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**4882** In-centre,  $a\lambda + b\mu + c\nu = 0$ . By (4878), for  
**4883**  $a$  ex-centre,  $-a\lambda + b\mu + c\nu = 0$ . [  $\frac{\pm BOC}{a} = \frac{COA}{b} = \frac{AOB}{c}$ .

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**4884** Circum-centre  $\lambda \sin 2A + \mu \sin 2B + \nu \sin 2C = 0$ .

PROOF.—For  $BOC = \frac{1}{2}R^2 \sin 2A$ , &c. in (4878). Otherwise.—By reciprocation (4876),  $\alpha \sin A + \beta \sin B + \gamma \sin C = 0$  is the line at infinity referred to the trigon  $ABC$ ; therefore

$$\lambda \sin A + \mu \sin B + \nu \sin C = 0$$

is the equation of the pole of that line referred to  $A'B'C'$ ; that is,

$$\lambda \sin 2A' + \mu \sin 2B' + \nu \sin 2C', \text{ by (4875).}$$


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**4885** Foot of perpendicular from  $C$  upon  $AB$ ,

$$\lambda \tan A + \mu \tan B = 0.$$

**4886** Orthocentre  $\lambda \tan A + \mu \tan B + \nu \tan C = 0$ .

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**4887** Inscribed conic of  $ABC$ ,

[Proof below.

$$L\mu\nu + M\nu\lambda + N\lambda\mu = 0.$$

**4888** Point of contact with  $AB$ ,

$$M\lambda + L\mu = 0.$$


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**4889** In-circle (4629),

$$(\mathfrak{s}-a)\mu\nu + (\mathfrak{s}-b)\nu\lambda + (\mathfrak{s}-c)\lambda\mu = 0.$$

**4890** Point of contact with  $AB$ ,  $(\mathfrak{s}-b)\lambda + (\mathfrak{s}-a)\mu = 0$ .

**4891**  $a$  ex-circle,  $(\mathfrak{s}-b)\lambda\mu + (\mathfrak{s}-c)\nu\lambda - \mathfrak{s}\mu\nu = 0$ .

PROOF.—Since the coordinates of  $AB$  of the trigon are  $0, 0, \nu$ , the equation of the inscribed conic must be satisfied when any two of the coordinates  $\lambda, \mu, \nu$  vanish, therefore it must be of the form (4887). Otherwise by reciprocating (4724).

If the circle touches  $AB$  in  $D$  (Fig. 3),  $\lambda : -\mu = AD : BD = \mathfrak{s}-a : \mathfrak{s}-b$  (Fig. of 709), which proves (4890).

(4888) is the equation of the point of contact, because the line  $(0, 0, \nu)$  passes through it and also touches the conic (4887).

(4889) is the in-circle by (4887) and (4890) and what precedes.

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**4892** Circumscribed conic, [By (4876) applied to (4739, '40).

$$L^2\lambda^2 + M^2\mu^2 + N^2\nu^2 - 2MN\mu\nu - 2NL\nu\lambda - 2LM\lambda\mu = 0, \text{ (4740)}$$

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**4893** or  $\sqrt{L\lambda} + \sqrt{M\mu} + \sqrt{N\nu} = 0.$

**4894** Tangent at  $A$ ,  $M\mu = N\nu.$

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**4895** Circum-circle

$$a^4\lambda^3 + b^4\mu^3 + c^4\nu^3 - 2b^2c^2\mu\nu - 2c^2a^2\nu\lambda - 2a^2b^2\lambda\mu = 0;$$

**4896** or  $a\sqrt{\lambda} + b\sqrt{\mu} + c\sqrt{\nu} = 0.$

PROOF.—By (4876) applied to (4747, '8), and by  $\cos \frac{A}{2} = \sin A'$  (4875)

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**4897** Relation between the coordinates of any right line:

$$a^2(\lambda - \mu)(\lambda - \nu) + b^2(\mu - \nu)(\mu - \lambda) + c^2(\nu - \lambda)(\nu - \mu) = \Sigma^2.$$

**4898** Coordinates of the line at infinity:

$$\lambda = \mu = \nu.$$

PROOF.—The trilinear coordinates of the origin and centre of the reciprocal conic are  $\alpha = \beta = \gamma$ , (4876). It is also self-evident.

**4899** The point  $l\lambda + m\mu + n\nu = 0$  will be at infinity when  $l + m + n = 0.$

PROOF.—By (4876), for the line  $la + m\beta + n\gamma = 0$  will pass through the origin  $\alpha = \beta = \gamma$  when  $l + m + n = 0.$

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**4900** A curve will be touched by the line at infinity when the sum of the coefficients vanishes.

PROOF.—By (4876), for this is the condition that the origin in trilinears,  $\alpha = \beta = \gamma$  shall be on the curve.

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**4901** The equation of the centre of the conic  $\phi(\lambda, \mu, \nu)$  is

$$\phi_\lambda + \phi_\mu + \phi_\nu = 0,$$

**4902** or  $(a + h + g)\lambda + (h + b + f)\mu + (g + f + c)\nu = 0.$

PROOF.—The coordinates of the in-centre of  $ABC$  (4876) are  $\alpha' = \beta' = \gamma'$ , therefore the polar of this point with regard to the conic  $\phi(\alpha, \beta, \gamma)$  is  $\phi_\alpha + \phi_\beta + \phi_\gamma = 0$  (4658). This point and polar reciprocate into a polar and point, of which the former, being the reciprocal of the in-centre, or origin, is the line at infinity, and therefore the latter is the centre of  $\phi(\lambda, \mu, \nu)$ , while its equation is as stated.

**4903** The equation of the two points in which the line  $(\lambda', \mu', \nu')$  cuts the conic is

$$\phi(\lambda', \mu', \nu') \phi(\lambda, \mu, \nu) = (\phi_\lambda \lambda' + \phi_\mu \mu' + \phi_\nu \nu')^2. \quad (4680)$$

**4904** The coordinates of the asymptotes are found from the equations

$$\phi(\lambda, \mu, \nu) = 0 \quad \text{and} \quad \phi_\lambda + \phi_\mu + \phi_\nu = 0.$$

PROOF.—These are the conditions that the line  $(\lambda, \mu, \nu)$  should touch the curve and also pass through the centre (4901).

**4905** The equation of the two circular points at infinity is

$$a^2(\lambda - \mu)(\lambda - \nu) + b^2(\mu - \nu)(\mu - \lambda) + c^2(\nu - \lambda)(\nu - \mu) = 0.$$

PROOF.—Put  $\lambda' = \mu' = \nu'$  in (4903) to make the line at infinity, and for the conic take the in-circle (4889).

**4906** The general equation of a circle is

$$a^2(\lambda - \mu)(\lambda - \nu) + b^2(\mu - \nu)(\mu - \lambda) + c^2(\nu - \lambda)(\nu - \mu) \\ = (l\lambda + m\mu + n\nu)^2 \dots \dots (1),$$

where  $l\lambda + m\mu + n\nu = 0$  is the equation of the centre.

PROOF.—The general equation of a conic in trilinears may, by (4601), be put in the form

$a(\beta - \beta_0)(\gamma - \gamma_0) + b(\gamma - \gamma_0)(\alpha - \alpha_0) + c(\alpha - \alpha_0)(\beta - \beta_0) = (l\alpha + m\beta + n\gamma)^2$ , where  $l\alpha + m\beta + n\gamma = 0$  is the directrix, and  $\alpha_0\beta_0\gamma_0$  the focus. Now let the focus be the in-centre of the trigon, and therefore  $\alpha_0 = \beta_0 = \gamma_0 = \frac{1}{2}\Sigma\delta^{-1}$  (709). By this relation and  $a\alpha + b\beta + c\gamma = \Sigma$ , the equation is expressed as

$$a(\delta - \alpha)(\alpha - \beta)(\alpha - \gamma) + \&c. = (l'\alpha + m'\beta + n'\gamma)^2,$$

$$\text{or} \quad (\alpha - \beta)(\alpha - \gamma) \cos^2 \frac{1}{2}A + \&c. = (l'\alpha + m'\beta + n'\gamma)^2.$$

Reciprocating by (4876), this becomes

$$(\lambda - \mu)(\lambda - \nu) \sin^2 A' + \&c. = (l\lambda + m\mu + n\nu)^2,$$

the constant factor introduced on the right being involved in  $l, m, n$ ; and  $\sin A' = \cos \frac{1}{2}A$ , by (4875). And we know that this is a circle by (4845 Cor.), and that the directrix of the conic reciprocates into the centre of the circle.

*Otherwise.*—The left side of (1) represents the two circular points at infinity (4905), and, if for the right we take the equation of a point, the whole represents a conic, as in (4909), of the form  $AC = B^2$ . In this case,  $A, C$ , the points of contact of tangents from  $B$ , being the circular points, the conic must be a circle with  $B = 0$  for its centre.

*Abridged Notation.*

**4907** Let  $A = 0$ ,  $B = 0$ ,  $C = 0$ ,  $D = 0$  (Fig. 75) be the tangential equations of the four points of a quadrangle, where  $A \equiv a_1\lambda + b_1\mu + c_1\nu$ ,  $B \equiv a_2\lambda + b_2\mu + c_2\nu$ , and so on. Then the equation of the inscribed conic will be  $AC = kBD$ .

PROOF.—The equation is of the second degree in  $\lambda, \mu, \nu$ ; therefore the line  $(\lambda, \mu, \nu)$  touches a conic. The coordinates of one line that touches this conic are determined by the equations  $A = 0$ ,  $B = 0$ . That is, the line joining the two points  $A, B$  touches the conic, and so of the rest.

**4908** If the points  $B, D$  coincide (Fig. 76), the equation becomes  $AC = kB^2$ ; and  $A = 0$ ,  $C = 0$  are the points of contact of tangents from the point  $B = 0$ .

**4909** Referring the conic to the trigon  $ABC$  (Fig. 78), and taking  $AC = k^2B^2$  for its equation, let a tangent  $ef$  be drawn, and let  $Ae : eB = k : m$ . The equations of the points  $e$  and  $f$  will be

$$mA + kB = 0, \quad mkB + C = 0.$$

PROOF.—The first equation corresponds to (4879). For the equation of  $f$ , eliminate  $A$  from  $mA + kB = 0$  and  $AC = k^2B^2$ .

**4910** Let  $e, h$  (Fig. 77) be two points on  $AB$  whose equations are  $mA + kB = 0$ ,  $m'A + kB = 0$ . The equation of the point  $p$ , in which tangents from  $e$  and  $h$  intersect, is

$$mm'A + (m + m')kB + C = 0.$$

PROOF.—The equation may be put in the form

$$(mA + kB)(m'A + kB) = 0,$$

because  $k^2B^2 = AC$  if the line touches the conic. The equation being of the first degree in  $A, B, C$ , must represent *some* point. That is, the relation between  $\lambda, \mu, \nu$  involved in it makes the straight line  $\lambda\alpha + \mu\beta + \nu\gamma$  pass through a certain point. But the equation is satisfied when  $mA + kB = 0$ , a relation which makes the straight line pass through  $e$ . Hence a tangent through  $e$  passes through a certain fixed point. Similarly, by  $m'A + kB = 0$ , another tangent passes through  $h$  and the same fixed point.

**4911** COR.—Let  $m' = m$ , then the equation of the point of contact of the tangent joining the points  $mA + kB$  and  $mkB + C$  (4909) ( $e$  and  $f$ , Fig. 78) will be

$$m^2A + 2mkB + C = 0.$$

**4912** If in Fig. (78) the trilinear coordinates of the points



$A, B, C$  are  $x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3$ , the coordinates of the point of contact  $p$  of the tangent defined by  $m$  will be

$$m^2x_1 + 2mkx_2 + x_3, \quad m^2y_1 + 2mky_2 + y_3, \quad m^2z_1 + 2mkz_2 + z_3,$$

and the tangent at  $p$  divides the two fixed tangents in the ratios  $k : m$  and  $mk : 1$ , by (4909).

**4913** NOTE.—The equation  $U$  or  $\Phi(\lambda, \mu, \nu) = 0$  (4665) expresses the condition that  $\lambda\alpha + \mu\beta + \nu\gamma$  shall touch a certain conic. When  $U$  is about to break up into two factors, the minor axis of the conic diminishes (Fig. 79). Every tangent that can now be drawn to the conic passes very nearly through one end or other of the major axis. Ultimately, when the minor axis vanishes, the condition of the line touching the conic becomes the condition of its passing through one or other of two fixed points  $A, B$ . In this case,  $U$  consists of two factors, which, put equal to zero, are the equations of those points. The conic has become a straight line, and this line is touched at every point by a single tangent.

**4914** If  $U$  and  $U'$  (Fig. 80) be two conics in tangential coordinates,  $kU + U'$  is then a conic having for a tangent every tangent common to  $U$  and  $U'$ ; and  $kU + AB$  is a conic having in common with  $U$  the two pairs of tangents drawn from the points  $A, B$ .

The conic  $U'$  in this case merges into the line  $AB$ , or, more strictly, the two points  $A, B$ , as explained in (4913).

**4915** If either  $kU + U'$  or  $kU + AB$  breaks up into two factors, it represents two points which are the opposite vertices of the quadrilateral formed by the four tangents.

## ON THE INTERSECTION OF TWO CONICS.

### INTRODUCTORY THEOREM.

#### *Geometrical meaning of $\sqrt{-1}$ .\**

**4916** In a system of rectangular or oblique plane coordinates, let the operator  $\sqrt{-1}$  prefixed to an ordinate  $y$  denote the turning of the ordinate about its foot as a centre through a right angle in a plane perpendicular to the plane of  $xy$ . The repetition of this operation will turn the ordinate

\* [The fiction of imaginary lines and points is not ineradicable from Geometry. The theory of Quaternions removes all imaginariness from the symbol  $\sqrt{-1}$ , and, as it appears that a partial application of that theory presents the subject of Projection in a much clearer light, I have here introduced the notion of the multiplication of vectors at right angles to each other.]

through another right angle in the same plane so as to bring it again into the plane of  $xy$ . The double operation has converted  $y$  into  $-y$ . But the two operations are indicated algebraically by  $\sqrt{-1} \cdot \sqrt{-1} \cdot y$  or  $(\sqrt{-1})^2 y = -y$ , which justifies the definition.

It may be remarked, in passing, that *any operation* which, being performed twice in succession upon a quantity, changes its sign, offers a consistent interpretation of the multiplier  $\sqrt{-1}$ .

**4917** With this additional operator, borrowed from the Theory of Quaternions, equations of plane curves may be made to represent more extended loci than formerly. Consider the equation  $x^2 + y^2 = a^2$ . For values of  $x < a$ , we have  $y = \pm \sqrt{a^2 - x^2}$ , and a circle is traced out. For values of  $x > a$ , we may write  $y = \pm i \sqrt{x^2 - a^2}$ , where  $i \equiv \sqrt{-1}$ . The ordinate  $\sqrt{x^2 - a^2}$  is turned through a right angle by the vector  $i$ , and this part of the locus is consequently an equilateral hyperbola having a common axis with the circle and a common parameter, but having its plane at right angles to that of the circle. Since the foot of each ordinate remains unaltered in position, we may, for convenience, leave the operation indicated by  $i$  unperformed and draw the hyperbola in the original plane. In such a case, the circle may be called the *principal*, and the hyperbola the *supplementary*, curve, after Poncelet. When the coordinate axes are rectangular, the supplementary curve is not altered in any other respect than in that of position by the transformation of all its ordinates through a right angle; but, if the coordinate axes are oblique, there is likewise a change of figure precisely the same as that which would be produced by setting each ordinate at right angles to its abscissa in the  $xy$  plane.

In the diagrams, the supplementary curve will be shown by a dotted line, and the unperformed operation indicated by  $i$  must always be borne in mind. For, on account of it, there can be no geometrical relations between the principal and supplementary curves excepting those which arise from the possession of one common axis of coordinates. This law is in agreement with the algebraic one which applies to the real and imaginary parts of the equation  $x^2 - (iy)^2 = a^2$ . When  $y$  vanishes,  $x = a$  in both curves.

If either the ellipse  $b^2x^2 + a^2y^2 = a^2b^2$  or the hyperbola  $b^2x^2 - a^2y^2 = a^2b^2$  be taken for the principal curve, the other will be the supplementary curve.

It is evident that, by taking different conjugate diameters for coordinate axes, the same conic will have corresponding different supplementary curves. The phrase, "supplementary conic on the diameter  $DD$ ," for example, will refer to that diameter which forms the common axis of the principal and supplementary conic in question.

**4918** Let us now take the circle  $x^2 + y^2 = a^2$  and the right line  $x = b$ . When  $b$  is  $> a$ , the line intersects the supplementary right hyperbola in two points whose ordinates are  $\pm i \sqrt{b^2 - a^2}$ . By increasing  $b$  without limit, we get a pair of, so-called, *imaginary points at infinity*. These lie on the asymptotes of the hyperbola, and the equation of the asymptotes is  $(x + iy)(x - iy) = 0$ .

We can now give a geometrical interpretation to the statements in (4720). The two lines drawn from the focus of the conic  $b^2x^2 + a^2y^2 = a^2b^2$  to the "circular points at infinity" make angles of  $45^\circ$  with the major axis, and they touch the conic in its supplementary hyperbola  $b^2x^2 - a^2(iy)^2 = a^2b^2$ . An independent proof of this is as follows.

Draw a tangent from  $S$  (Fig. 81) to the supplementary hyperbola, and let  $x, y$  be the coordinates of the point of contact  $P$ . Then

$$x = \frac{a^2}{CS} \text{ (1170)} = \frac{a^2}{\sqrt{(a^2-b^2)}}; \text{ and } y = \frac{b}{a} \sqrt{(x^2-a^2)} = \frac{b^2}{\sqrt{(a^2-b^2)}},$$

by the value of  $x$ . Also

$$SN = x - CS = \frac{a^2}{\sqrt{(a^2-b^2)}} - \sqrt{a^2-b^2} = \frac{b^2}{\sqrt{(a^2-b^2)}}.$$

Therefore  $y = SN$ , therefore  $SP$  makes an angle of  $45^\circ$  with  $CN$ .

The following results are required in the theory of projection, and are illustrated in figures (82) to (86). Two ellipses are taken in each case for principal curves, and the supplementary hyperbolas are shown by dotted lines. As the planes of the principal and supplementary curves are really at right angles, the intersections of the solid lines with the dotted are only apparent. The intersections of the solid lines are *real* points, while the intersections of the dotted lines represent the *imaginary* points.

#### 4919 Two conics may intersect—

- (i.) *in four real points* (Fig. 82);
- (ii.) *in two real and two imaginary points* (Fig. 83);
- (iii.) *in four imaginary points* (Fig. 84).

[When the two hyperbolas in figures (83) and (84) are similar and similarly situated, two of their points of intersection recede to infinity (Figs. 85 and 86). Hence, and by taking the dotted lines for principal, and the solid for supplementary, curves, we also have the cases]

- (iv.) *in two real finite points and two imaginary points at infinity*;
- (v.) *in two imaginary finite points and two imaginary points at infinity*;
- (vi.) *in two imaginary finite points and two real points at infinity*;
- (vii.) *in two real finite points and two real points at infinity*.

#### 4920 Given two conics not intersecting, or intersecting in but two points, to draw the two supplementary curves which have a common chord of intersection conjugate to the

diameters upon which they are described, or in other words, to find the imaginary common chord of the conics.

Poncelet has shewn by geometrical reasoning (*Propriétés des Projectives*, p. 31) that such a chord must exist. The following is a method of determining its position—

Let  $(abcfgh\chi_{xy}1)^2 = 0$  and  $(a'b'c'f'g'h'\chi_{xy}1)^2 = 0$  .....(i.) be the equations of the conics  $C, C'$  (Fig. 89), the coordinate axes being rectangular. Suppose  $PQ$  to be the common chord sought. Then the diameters  $AB, A'B'$  conjugate to  $PQ$  bisect it in  $D$ , and the supplementary curves on those diameters intersect in the points  $P, Q$ . Now, let the coordinate axes be turned through an angle  $\theta$ , so that the  $y$  axis may become parallel to  $PQ$ , and therefore also to the tangents at  $A, B, A', B'$ . This is accomplished by substituting for  $x$  and  $y$ , in equations (i.), the values

$$x \cos \theta - y \sin \theta \quad \text{and} \quad y \cos \theta + x \sin \theta.$$

Let the transformed equations be denoted by  $(ABCFGH\chi_{xy}1)^2 = 0$  and  $(A'B'C'F'G'H'\chi_{xy}1)^2 = 0$ , in which the coefficients are all functions of  $\theta$ , excepting  $c$ , which is unaltered. Solving each of these equations as a quadratic in  $y$ , the solutions take the forms

$$y = ax + \beta \pm \sqrt{\mu(x^2 - 2px + q)}, \quad y = a'x + \beta' \pm \sqrt{\mu'(x^2 - 2p'x + q')} \dots \text{(ii.)},$$

with the values of  $a, \beta, \mu, p, q$  given in (4449-53), if for small letters we substitute capitals. Thus,  $a, \beta, \mu, p, q$  are obtained in terms of  $\theta$  and the original coefficients  $a, h, b, f, g, h$ .

Now, the coordinates of  $D$  being  $\xi = ON, \eta = DN$ , we have  $\eta = a\xi + \beta$  and  $\eta = a'\xi + \beta'$ , therefore  $a\xi + \beta = a'\xi + \beta'$  ..... (iii.).

The surd in equations (ii.) represents the ordinate of the conic conjugate to the diameter  $AB$  or  $A'B'$ . For values of  $x$  in the diagram  $> OM$  and  $< OR$ , the factor  $\sqrt{-1}$  appears in this surd, indicating an ordinate of the supplementary curve on  $AB$  or  $A'B'$ . Hence, equating the values of the common ordinate  $PD$ , we have

$$\mu(\xi^2 - 2p\xi + q) = \mu'(\xi^2 - 2p'\xi + q') \dots \text{(iv.)}.$$

Eliminating  $\xi$  between equations (iii.) and (iv.), we obtain an equation for determining  $\theta$ ; which angle being found, we can at once draw the diameters  $AB, A'B'$ .

## THE METHOD OF PROJECTION.

**4921** PROBLEM.—Given any conic and a right line in its plane and any plane of projection, to find a vertex of projection such that the line may pass to infinity while the conic is projected into a hyperbola or ellipse according as the right line does or does not intersect the given conic; and at the same time to give any assigned proportion and direction to the axes of the projected conic.

*Analysis.*—Let  $HCKD$  be the given conic, and  $BB$  the right line, in Fig. (87) not intersecting, and in Fig. (88) intersecting the conic. Draw  $HK$  the diameter of the conic conjugate to  $BB$ . Suppose  $O$  to be the required vertex of projection. Draw any plane  $ECGD$  parallel to  $OBB$ , intersecting the given conic in  $CD$  and the line  $HK$  in  $F$ , and draw the plane  $OHK$  cutting the former plane in  $E, F, G$  and the line  $BB$  in  $A$ ; and let the curve  $ECGD$  be the conical projection of  $HCKD$  on the plane parallel to  $OBB$ .

By similar triangles,

$$\frac{EF}{HF} = \frac{OA}{HA} \text{ and } \frac{FG}{FK} = \frac{OA}{AK}, \therefore \frac{EF.FG}{HF.FK} = \frac{OA^2}{HA.AK} \dots\dots\dots(1).$$

Let  $a, \beta$  be the semi-diameters of the given conic parallel to  $HK$  and  $CD$ ;

then 
$$\frac{CF^2}{HF.FK} = \frac{\beta^2}{a^2}, \therefore \frac{CF^2}{EF.FG} = \frac{\beta^2.HA.AK}{a^2.OA^2} \dots\dots\dots(2).$$

Now, since parallel sections of the cone are similar, if the plane of  $HCKD$  moves parallel to itself, the ratio on the right remains constant; therefore, by (1193), the section  $ECGK$  is an ellipse in Fig. (87) and an hyperbola in Fig. (88). Let  $a, b$  be the semi-diameters of this ellipse or hyperbola parallel to  $EG$  and  $CD$ , that is, to  $OA$  and  $BB$ ; then, by (2),

$$\frac{b^2}{a^2} = \frac{\beta^2}{a^2} \frac{HA.AK}{OA^2}, \therefore OA^2 = \frac{a^2\beta^2}{b^2a^2} HA.AK \dots\dots\dots(3).$$

But  $\frac{\beta^2}{a^2} HA.AK = AB^2$ , where  $AB$  in Fig. (88) is the ordinate at  $A$  of the given conic, but in Fig. (87) the ordinate of the conic supplementary to the given one on the diameter conjugate to  $BB$ . Therefore

$$AO^2 = \frac{a^2}{b^2} AB^2 \dots\dots\dots(4).$$

Hence  $AO, AB$  are parallel and proportional to  $a$  and  $b$ . And, since  $AB$  is given in magnitude and direction, we have two constants at our disposal, namely, the ratio of the semi-conjugate diameters  $a$  and  $b$  and the angle between them, or, which is the same thing, the eccentricity and the direction of the axes of the ellipse or hyperbola on the plane of projection.

**4922** The construction will be as follows:—

*Determine the point A as the intersection of BB with the diameter HK conjugate to it. Choose any plane of projection, and in a plane through BB, parallel to it, measure AO of the length given by equation (3) or (4), making the angle BAO equal to the required angle between a and b. O will be the vertex of projection, and any plane LMN parallel to OBB will serve for the plane of projection.*

**4923** COR. I.—If  $AO = AB$ , the projected curve in Fig. (88) will in every case be a right hyperbola.



**4924** COR. 2.—If  $BAO$  is a right angle, the axes of the projected ellipse or hyperbola are parallel and proportional to  $AO$  and  $AB$ . Hence, in this case, the eccentricity of the hyperbola will be  $e = OB : OA$ .

**4925** COR. 3.—If  $AO = AB$  and  $BAO =$  a right angle, the ellipse becomes a circle and the right hyperbola in Cor. 1 has its axes parallel to  $AO$  and  $AB$ .

**4926** To project a conic so that a given point in its plane may become the centre of the projected curve.

*Take for the line  $BB$  the polar of the given point, and construct as in (4922). For, if  $P$  be the given point, and  $BB$  its polar (Fig. 87 or 88),  $p$  the projection of  $P$  will have its polar at infinity, and will therefore be the centre of the projected ellipse or hyperbola, according as  $P$  is within or without the original conic.*

**4927** To project two intersecting conics into two similar and similarly situated hyperbolas of given eccentricity.

*Take the common chord of the conics for the line  $BB$  (Fig. 88), and project each conic as in (4922), employing the same vertex and plane of projection. Then, since the point  $A$  and the lines  $AB$  and  $AO$  are the same for each projection, corresponding conjugate diameters of the hyperbolas are parallel and proportional to  $AO$  and  $AB$ ; therefore, &c.*

**4928** To project two non-intersecting conics into similar and similarly situated ellipses of given eccentricity.

*Take the common chord of a certain two of the supplementary curves of the conics (4920), in other words, the imaginary common chord of the conics, for the line  $BB$ , and proceed as in (4927).*

**4929** To project two conics having a common chord of contact into two concentric, similar and similarly situated hyperbolas.

*Take the common chord for the line  $BB$ , and construct as in (4922). The common pole of the conics projects into a common centre and the common tangents into common asymptotes.*



**4930** To project any two conics into concentric conics.

*Find the common pole and polar of the given conics by (4762), and take the common polar for the line  $BB$  in the construction of (4922). The common pole projects into a common centre.*

**4931** Ex. 1. — Given two conics having double contact with each other, any chord of one which touches the other is cut harmonically at the point of contact and where it meets the common chord of contact of the conics.

[*Salmon's Conic Sections*, Art. 354.]

Let  $AB$  be the common chord of contact,  $PQ$  the other chord touching the inner conic at  $C$  and meeting  $AB$  produced in  $D$ . By (4929), project  $AB$ , and therefore the point  $D$ , to infinity. The conics become similar and similarly situated hyperbolas, and  $C$  becomes the middle point of  $PQ$  (1189). The theorem is therefore true in this case. Hence, by a converse projection, the more general theorem is inferred.

**4932** Ex. 2.—Given four points on a conic, the locus of the pole of any fixed line is a conic passing through the fourth harmonic to the point in which this line meets each side of the given quadrilateral. [*Ibid.*, Art. 354.]

Let the fixed line meet a side  $AB$  of the quadrilateral in  $D$ , and let  $ACBD$  be in harmonic ratio. Project the fixed line, and therefore the point  $D$ , to infinity.  $C$  becomes the middle point of  $AB$  (1055), and the pole of the fixed line becomes the centre of the projected conic. Now, it is known that the locus of the centre is a conic passing through the middle points of the sides of the quadrilateral. Hence, projecting back again, the more general theorem is inferred.

**4933** Ex. 3.—If a variable ellipse be described touching two given ellipses, while the supplementary hyperbolas of all three have a common chord  $AB$  conjugate to the diameters upon which they are described; the locus of the pole of  $AB$  with respect to the variable ellipse is an hyperbola whose supplementary ellipse touches the four lines  $CA, CB, C'A, C'B$ , where  $C, C'$  are the poles of  $AB$  with respect to the fixed ellipses.

(*Salmon*, Art. 355.)

PROOF.—Project  $AB$  to infinity and the three ellipses into circles. The poles  $P, C, C'$  become the centres  $p, c, c'$  of the circles. The locus of  $p$  is a hyperbola whose foci are  $c, c'$ . But the lines  $Ac, Bc$  now touch the supplementary ellipse of this hyperbola (4918). Therefore, projecting back again, we get  $AC, BC$  touching the supplementary ellipse of the conic which is the locus of  $P$ . Similarly,  $AC', BC'$  touch the same ellipse.

**4934** Any two lines at right angles project into lines which cut harmonically the line joining the two fixed points which are the projections of the circular points at infinity.

PROOF.—This follows from (4723).

**4935** The converse of the above proposition (4931), which is the theorem in Art. 356 of *Salmon*, is not universally true in any real sense. If the lines drawn through a given point to the two circular points at infinity form a harmonic pencil with two other lines through that point, the latter two are not necessarily at right angles, as the theorem assumes.

The following example from the same article is an illustration of this—

Ex.—Any chord  $BB$  (Fig. 88) of a conic  $HCKD$  is cut harmonically by any line  $PKAH$  through  $P$ , the pole of the chord, and the tangent at  $K$ .

The ellipse  $BKB$  here projects into a right hyperbola;  $B, B$  project to infinity. The harmonic pencil formed by  $PK$  and the tangent at  $K$ ,  $KB$  and  $KB$  projects into a harmonic pencil formed by  $pk$  and the tangent at  $k$ ,  $kb$  and  $kb$ , where  $b, b$  are the circular points at infinity: but  $pk$  is not at right angles to the tangent at  $k$  of the right hyperbola. The harmonic ratio of the latter pencil can, however, be independently demonstrated, and that of the former can then be inferred. (Note that  $k$  is  $G$  in figure 88.)

If we may suppose the ellipse to project into an imaginary circle having points at infinity, the imaginary radius of that circle may be supposed to be at right angles to the imaginary tangent. The right hyperbola, however, is the real projection which takes the place of the circle in this and all similar instances; and it is only in the case of principal axes that the radius is at right angles to the tangent.

## INVARIANTS AND COVARIANTS.

**4936** Let  $u \equiv (abcfgh)(xyz)^2$ ,  $u' \equiv (a'b'c'f'g'h')(xyz)^2$  be two conics as in (4401) with the notation of (1620).

The three values of  $k$ , for which  $ku + u' = 0$  represents two right lines, are the roots of the cubic equation

$$\mathbf{4937} \quad \Delta k^3 + \Theta k^2 + \Theta' k + \Delta' = 0,$$

$$\mathbf{4938} \quad \text{where } \Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2,$$

$$\mathbf{4939} \quad \Theta \equiv Aa' + Bb' + Cc' + 2Ff' + 2Gg' + 2Hh',$$

$$\text{and} \quad A \equiv bc - f^2, \quad F \equiv gh - af, \quad \&c. \quad (4665)$$

For the values of  $\Delta'$  and  $\Theta'$  interchange  $a$  with  $a'$ ,  $b$  with  $b'$ , &c.

PROOF.—The discriminant of  $ku + u'$ , which must vanish (4661), is evidently the determinant here written, and it is equivalent to the cubic in question.

$$\begin{vmatrix} ka + a', & kh + h', & kg + g' \\ kh + h', & kb + b', & kf + f' \\ kg + g', & kf + f', & kc + c' \end{vmatrix}$$

**4940**  $\Delta, \Theta, \Theta'$ , and  $\Delta'$  are invariants of the conic  $ku + u'$ .

That is, if the axes of coordinates be transformed in any manner, the ratios of the four coefficients in (4937) are unaltered.

PROOF.—The transformation is effected by a linear substitution, as in (1794). Let  $u, u'$  thus become  $v, v'$ . Then  $ku + u'$  becomes  $kv + v'$ , and  $k$  is unaltered. If the equation  $ku + u' = 0$  represents two right lines, it will continue to do so after transformation; but the condition for this is the vanishing of the cubic in  $k$ ; and  $k$  being constant, the ratios of the coefficients must be unalterable.

**4941** The equation of the six lines which join the four points of intersection of the conics  $u$  and  $u'$  is

$$\Delta u'^3 - \Theta u'^2 u + \Theta' u' u^2 - \Delta' u^3 = 0.$$

PROOF.—Eliminate  $k$  from (4937) by  $ku + u' = 0$ .

**4942** The condition that the conics  $u$  and  $u'$  may touch is

$$(\Theta\Theta' - 9\Delta\Delta')^2 = 4(\Theta^3 - 3\Delta\Theta')(\Theta'^2 - 3\Delta'\Theta),$$

**4943** or  $4\Delta\Theta^3 + 4\Delta'\Theta'^3 + 27\Delta^2\Delta'^2 - 18\Delta\Delta'\Theta\Theta' - \Theta^2\Theta'^2$ .

PROOF.—Two of the four points in (3941) must coincide. Hence two out of the three pairs of lines must coincide. The cubic (4937) must therefore have two equal roots. Let  $\alpha, \alpha, \beta$  be the roots; then the condition is the result of eliminating  $\alpha$  and  $\beta$  from the equations

$$\Delta(2\alpha + \beta) = -\Theta, \quad \Delta(\alpha^2 + 2\alpha\beta) = \Theta', \quad \Delta\alpha^2\beta = -\Delta' \quad (406).$$

**4944** The expression (4943) is the last term of the equation whose roots are the squares of the differences of the roots of the cubic in  $k$ , and when it is positive, the cubic in  $k$  has two imaginary roots; when it is negative, three real roots; and when it vanishes, two equal roots.

PROOF.—By (543) or (579). The last term of  $f(x)$  in (543) is now  $= 27F(\alpha)F(\beta)$ ,  $\alpha, \beta$  being the roots of  $3\Delta x^2 + 2\Theta x + \Theta' = 0$ . When this term is positive,  $f(x)$  has a real negative root (409), and therefore  $F(x)$  has then two imaginary roots; for, if  $(a-b)^2 = -c$ ,  $a-b = ic$ , and  $a$  and  $b$  are both imaginary. When the last term of  $f(x)$  is negative, all the roots of  $f(x)$  are positive, and therefore the roots of  $F(x)$  are all real.

#### INVARIANTS OF PARTICULAR CONICS.

**4945** When  $u = ax^2 + by^2 + cz^2$  and  $u' = x^2 + y^2 + z^2$ ,

$$\Delta = abc, \quad \Theta = bc + ca + ab, \quad \Theta' = a + b + c, \quad \Delta' = 1.$$

**4946** When  $u = (abc fgh)(xyz)^2$  and  $u' = x^2 + y^2 + z^2$ ,

$$\Theta = A + B + C, \quad \Theta' = a + b + c, \quad \Delta' = 1.$$

**4947** When  $u = x^2 + y^2 - r^2$  and  $u' = (x - \alpha)^2 + (y - \beta)^2 - s^2$ ,

$$\Delta = -r^2, \quad \Delta' = -s^2,$$

$$\Theta = \alpha^2 + \beta^2 - 2r^2 - s^2, \quad \Theta' = \alpha^2 + \beta^2 - r^2 - 2s^2.$$

**4948** The cubic for  $k$  reduces to

$$(k+1) \{s^2 k^2 + (r^2 + s^2 - \alpha^2 - \beta^2) k + r^2\} = 0.$$

**4949** When

$$u = b^2 x^2 + a^2 y^2 - a^2 b^2 \quad \text{and} \quad u' = (x - \alpha)^2 + (y - \beta)^2 - r^2,$$

$$\Delta = -a^4 b^4, \quad \Theta = a^2 b^2 \{ \alpha^2 + \beta^2 - a^2 - b^2 - r^2 \},$$

$$\Theta' = a^2 \beta^2 + b^2 \alpha^2 - a^2 b^2 - r^2 (a^2 + b^2), \quad \Delta' = -r^2.$$

**4950** When  $u = y^2 - 4mx$  and  $u' = (x - \alpha)^2 + (y - \beta)^2 - r^2$ ,

$$\Delta = -4m^2, \quad \Theta = -4m(a + m), \quad \Theta' = \beta^2 - 4ma - r^2, \quad \Delta' = -r^2.$$

**4951** When  $u = (abc fgh)(xyz)^2$  and  $u' = x^2 + 2xy \cos \omega + y^2$ ,

$$\Delta, \quad \Delta' = 0, \quad \Theta = c(a + b) - f^2 - g^2 + 2(fg - ch) \cos \omega,$$

$$\Theta' = c \sin^2 \omega.$$

Hence the following are invariants of the general conic, the inclination of the coordinate axes being  $\omega$ .

**4952** 
$$\frac{abc + 2fgh - af^2 - bg^2 - ch^2}{c \sin^2 \omega} = \frac{\Delta}{\Theta'} \dots\dots\dots (1),$$

**4953** 
$$\frac{c(a + b) - f^2 - g^2 + 2(fg - ch) \cos \omega}{c \sin^2 \omega} = \frac{\Theta}{\Theta'} \dots\dots\dots (2),$$

**4954** 
$$\frac{ab - h^2}{\sin^2 \omega} \dots\dots\dots (3), \quad \text{and} \quad \frac{a + b - 2h \cos \omega}{\sin^2 \omega} \dots\dots\dots (4).$$

For these are what (1) and (2) become when the axes are transformed so as to remove  $f$  and  $g$ .

If the origin be unaltered,  $c$  is invariable, and transformation of the axes will then leave invariable

$$4956 \quad \frac{2fgh - af^2 - bg^2}{\sin^2 \omega} \quad \text{and} \quad \frac{f^2 + g^2 - 2fg \cos \omega}{\sin^2 \omega},$$

as appears by subtracting (3) from (1) and (2) from (4).

**4958** Ex. (i.)—To find the evolute of the conic  $b^2x^2 + a^2y^2 = a^2b^2$ . See also (4547).

PROOF.—Denote the conic by  $u$ , and by  $u'$  the hyperbola  $c^2xy + b^2y'x - a^2x'y$  (4335), which intersects  $u$  in the feet of the normals drawn from  $x'y'$ . Two of these normals must always coincide if  $x'y'$  is to be on the evolute.  $u$  and  $u'$  must therefore touch. We have

$$\Delta = -a^4b^4, \quad \Theta = 0, \quad \Theta' = -a^2b^3(a^2x^2 + b^2y^2 - c^4), \quad \Delta' = -2a^2b^3c^2xy.$$

Substitute in (4942), and the equation of the evolute is found to be

$$(a^2x^2 + b^2y^2 - c^4)^3 + 27a^2b^2c^4x^2y^2 = 0.$$

**4959** Ex. (ii.)—Similarly the evolute of the parabola is obtained from

$$u = y^2 - 4mx, \quad u' = 2xy + 2(2m - x')y - 4my',$$

$$\Delta = -4m^3, \quad \Theta = 0, \quad \Theta' = -4(2m - x), \quad \Delta' = 4my,$$

producing the equation  $27my^2 = 4(x - 2m)^3$ . See also (4549).

**4960** Ex. (iii.)—The locus of the centre of a circle of radius  $R$ , touching the conic  $b^2x^2 + a^2y^2 - a^2b^2$ , is called a *parallel* to the conic. Its equation is

$$\begin{aligned} & R^3c^4 - 2R^2c^3 \{ c^2(a^2 + b^2) + (a^2 - 2b^2)x^2 + (2a^2 - b^2)y^2 \} \\ & + R^4 \{ c^4(a^4 + 4a^2b^2 + b^4) - 2c^3(a^4 - a^2b^2 + 3b^4)x^2 + 2c^2(3a^4 - a^2b^2 + b^4)y^2 \\ & \quad + (a^4 - 6a^2b^2 + 6b^4)x^4 + (6a^4 - 6a^2b^2 + b^4)y^4 + (6a^4 - 10a^2b^2 + 6b^4)x^2y^2 \} \\ & + R^3 \{ -2a^2b^2c^4(a^2 + b^2) + 2c^3(3a^4 - a^2b^2 + b^4)x^2 - 2c^2(a^4 - a^2b^2 + 3b^4)y^2 \\ & \quad - (6a^4 - 10a^2b^2 + 6b^4)(b^2x^4 + a^2y^4) + (4a^6 - 6a^4b^2 - 6a^2b^4 + 4b^6)x^2y^2 \\ & \quad + 2(a^2 - 2b^2)b^2x^6 + 2(b^2 - 2a^2)a^2y^6 - 2(a^4 - a^2b^2 + 3b^4)x^4y^2 \\ & \quad - 2(3a^4 - a^2b^2 + b^4)x^2y^4 \} \\ & + (b^2x^2 + a^2y^2 - a^2b^2)^3 \{ (x - c)^2 + y^2 \} \{ (x + c)^2 + y^2 \} = 0, \end{aligned}$$

PROOF.—If the curves in (4949) be made to touch,  $\alpha\beta$  will be a point on the curve parallel to  $u$  at a distance  $r$ . Therefore put the values of  $\Delta$ ,  $\Theta$ ,  $\Theta'$ , and  $\Delta'$  in equation (4942).

[Salmon, p. 325.]



**4961** When  $u'$  of (4936) represents two right lines,  $\Delta'$  vanishes, and

**4962**  $\Theta' = 0$  is the condition that the two lines should intersect on  $u$ ;

**4963**  $\Theta = 0$  is the condition that the two lines should be conjugate with regard to  $u$ .

PROOF.—Transform  $u' = 0$  into  $2xy = 0$ , so that the axes  $x, y$  are the right lines. This will not affect the invariants (4940). We now have, by (4937),

$$\Delta' = 0, \quad \Theta = 2(fg - ch), \quad \Theta' = -c.$$

$c = 0$  makes  $u$  pass through the origin  $xy$ ;  $fg = ch$  makes  $x$  and  $y$  conjugate. For in (4671), if  $\lambda x + \mu y + \nu$  becomes  $y = 0$ , then  $\lambda = \nu = 0$ , and the pole is given by  $H : B : F$ . But  $x \equiv a = 0$  at the pole, therefore  $H \equiv fg - ch = 0$ .

**4964** The condition that either of the lines in  $u'$  should touch  $u$  is, by (4943),

$$\Theta^3 = 4\Delta\Theta' \quad \text{or} \quad AB = 0,$$

with the above values of  $\Theta$  and  $\Theta'$ .

**4965** The equation of the two tangents to  $u$ , when  $\lambda x + \mu y + \nu$  is the chord of contact, is, with the notation of (4665),

$$u\Phi(\lambda, \mu, \nu) = (\lambda x + \mu y + \nu z)^2 \Delta.$$

PROOF.—The conic of double contact with  $u$ ,  $ku + (\lambda x + \mu y + \nu)^2$  (4699), must now become two right lines. In (4937)  $\Delta' = 0$  and  $\Theta' = 0$ , therefore  $k\Delta + \Theta = 0$ . But  $\Theta = \Phi(\lambda, \mu, \nu)$ . Hence eliminate  $k$ .

**4966** COR.—Taking the line at infinity  $ax + by + cz$ , we obtain the equation of the asymptotes (4685).

The invariant  $\Theta$  of the conic  $ku + u'$  vanishes—

**4967** (i.) Whenever an inscribed triangle of  $u'$  is self-conjugate to  $u$ .

**4968** (ii.) Whenever a circumscribed triangle of  $u$  is self-conjugate to  $u'$ .

**4969**  $\Theta'$  vanishes under similar conditions, transposing  $u$  and  $u'$  in (i.) and (ii.)

PROOF.—(i.)  $u$  becomes  $ax^2 + by^2 + cz^2$  (4765), and  $f = g = h = 0$ . Therefore  $\Theta$  in (4937) vanishes if  $a' = b' = c' = 0$ ; i.e., if  $u'$  is of the form  $f'yz + g'zx + h'xy$  (4724).



(ii.) In this case,  $f' = g' = h' = 0$  and  $\Theta$  vanishes if  $bc = f^2$ , &c., i.e., if the line  $x = 0$  touches  $u$ , &c.

**4970** If  $u, u'$  be two conics, and if  $\Theta^2 = 4\Delta\Theta'$ , any triangle inscribed in  $u'$  will circumscribe  $u$ , and conversely.

PROOF.—Let  $u = x^2 + y^2 + z^2 - 2yz - 2zx - 2xy$  and  $u' = 2fyz + 2gzx + 2hxy$ , both referred to the same triangle, (4739) and (4724). Then

$\Delta = -4$ ,  $\Theta = 4(f + g + h)$ ,  $\Theta' = -(f + g + h)^2$ ,  $\Delta' = 2fgh$ ;  
therefore  $\Theta^2 = 4\Delta\Theta'$ , a relation independent of the axes of reference (4940).

**4971** Ex. (i.)—The locus of the centre of a circle of radius  $r$ , circumscribing a triangle which circumscribes the conic  $b^2x^2 + a^2y^2 = a^2b^2$ , is

$$(x^2 + y^2 - a^2 - b^2 + r^2)^2 + 4\{b^2x^2 + a^2y^2 - a^2b^2 - r^2(a^2 + b^2)\} = 0,$$

from  $\Theta^2 = 4\Delta\Theta'$  and the values in (4949).

**4972** Ex. (ii.)—The distance between the centres of the inscribed and circumscribed circles of a triangle is thus found, by employing the values of  $\Theta, \Theta'$ , and  $\Delta$  in (4947), to be  $D = \sqrt{(r^2 \pm 2rr')}$ , as in (936).

**4973** The tangential equation of the four points of intersection of the two conics  $u = 0, u' = 0$  is

$$\mathbf{V}^2 = 4UU',$$

with the meanings

$$\mathbf{4974} \quad U \equiv (A B C F G H \chi \lambda \mu \nu)^2; \quad (4664)$$

$$U' \equiv (A' B' C' F' G' H' \chi \lambda \mu \nu)^2.$$

$$\mathbf{4976} \quad \mathbf{V} \equiv (A'' B'' C'' F'' G'' H'' \chi \lambda \mu \nu)^2.$$

$$\mathbf{4977} \quad A \equiv bc - f^2, \text{ \&c.}; \quad A' \equiv b'c' - f'^2, \text{ \&c.},$$

as in (4665), and

$$\mathbf{4978} \quad A'' \equiv bc' + b'c - 2ff', \quad F'' \equiv gh' + g'h - af' - a'f,$$

$$\mathbf{4979} \quad B'' \equiv ca' + c'a - 2gg', \quad G'' \equiv hf' + h'f - bg' - b'g,$$

$$\mathbf{4980} \quad C'' \equiv ab' + a'b - 2hh', \quad H'' \equiv fg' + f'g - ch' - c'h.$$

PROOF.—The tangential equation is the condition that  $\lambda a + \mu \beta + \nu \gamma$  may pass through one of the four points of intersection of  $u$  and  $u'$ . The tangential equation of the conic  $u + ku'$  is obtained by putting  $a + ka'$  for  $a$ , &c. in  $U$  (4665), and is  $U + k\mathbf{V} + k^2U' = 0$ . The tangential equation of the envelope of the system is  $\mathbf{V}^2 = 4UU'$  (4911). This is the condition that the line  $(\lambda, \mu, \nu)$  may pass through the consecutive intersections of the conics obtained by varying  $k$ . But these conics always intersect in the same four points. The above is therefore the tangential equation of the four points.

**4981** The equation of the four common tangents of two conics  $u, u'$  is

$$\mathbf{F}^2 = 4\Delta\Delta'uu',$$

where  $\mathbf{F} \equiv (a''b''c''f''g''h'' \chi \alpha\beta\gamma)^2,$

and  $a'' \equiv BC' + B'C - 2FF', \text{ \&c.},$

$$f'' \equiv GH' + G'H - AF' - A'F, \text{ \&c.},$$

as in (4978-81).

PROOF.—This is the reciprocal of the last theorem.  $U + kU'$  is a conic touching the four common tangents of the conics  $U$  and  $U'$ . The trilinear equation formed from this will, by (4667), be  $u\Delta + k\mathbf{F} + k^2u'\Delta' = 0$ . The envelope of this system of conics is the equation above, which must therefore represent the four common tangents.

The curve  $\mathbf{F}$  passes through the points of contact of  $u$  and  $u'$  with the locus represented by (4981).

**4982** Hence the eight points of contact of the two conics with their common tangents lie on the curve  $\mathbf{F}$ .

**4983** The reciprocal theorem from equation (4973) is—The eight tangents at the intersections of the conics envelope the conic  $\mathbf{V}$ .

**4984**  $\mathbf{F} = 0$  is the locus of a point from which the tangents to the two given conics  $u, u'$  form a harmonic pencil.

PROOF.—Putting  $\gamma = 0$  in (4681), we get a quadratic of the form  $aa^2 + 2ha\beta + b\beta^2 = 0$ , which determines the two points in which the line  $\gamma$  is cut by tangents from  $a', \beta', \gamma'$ . Let the similar quadratic for the second conic be  $a'a^2 + 2h'a\beta + b'\beta^2 = 0$ . Then, by (1064),  $ab' + a'b = 2hh'$  is the condition that the four points may be in harmonic relation. This equation will be found to produce  $\mathbf{F} = 0$ .

**4985** The actual values of  $a, h, b$ , suppressing the accents on  $a', \beta', \gamma'$ , are

$$\begin{aligned} C\beta^2 + B\gamma^2 - 2F\beta\gamma, \quad G\beta\gamma + F\gamma a - Ca\beta - H\gamma^2, \\ A\gamma^2 + Ca^2 - 2G\gamma a; \end{aligned}$$

and similarly for  $a', h', b'$ , with  $A'$  written for  $A$ , &c.

**4986** If the *anharmonic* ratio of the pencil of four tangents be given, the locus of the vertex will be  $\mathbf{F}^2 = kuu'$ . If the given ratio be infinity or zero, the locus becomes the four common tangents in (4981).

**4987**  $V = 0$  is the envelope of a conic every tangent of which is cut harmonically by the two conics  $u, u'$ ; i.e., the equation is the condition that  $\lambda a + \mu \beta + \nu \gamma$  should be cut harmonically by the two conics.

PROOF.—Eliminate  $\gamma$  between the line  $(\lambda, \mu, \nu)$ , and the conics  $u$  and  $u'$  separately, and let  $Aa^2 + 2Ha\beta + B\beta^2 = 0$  and  $A'a^2 + 2H'a\beta + B'\beta^2 = 0$  stand for the resulting equations. Then, by (1064),  $AB' + A'B = 2HH'$  produces the equation  $V = 0$ , which, by (4666), is the envelope of a conic.

**4988** The actual values of  $A, H, B$  are respectively

$$a\nu^2 + c\lambda^2 - 2g\nu\lambda, \quad h\nu^2 - g\mu\nu - f\nu\lambda + c\lambda\mu, \quad c\mu^2 + b\nu^2 - 2f\mu\nu;$$

and similarly for  $A', H', B'$ , with  $a'$  for  $a$ , &c.

**4989**  $F^2 = 4\Delta\Delta'uu'$  is a *covariant* (1629) of the conics  $u, u'$ .

For the four common tangents are independent of the axes of reference.

**4990**  $U = 0$  and  $V = 0$  (4973) are both *contravariants* (1814) of  $u$  and  $u'$ .

PROOF.—For  $U = 0$  is the condition that  $\lambda a + \mu \beta + \nu \gamma = 0$  shall touch the conic  $u$ ; and  $V = 0$  is the condition that the same line shall be cut harmonically by  $u$  and  $u'$ ; and if all the equations be transformed by a reciprocal substitution (1813, '14), the right line and the conditions remain unaltered.

**4991** Any conic covariant with  $u$  and  $u'$  can be expressed in terms of  $u, u'$ , and  $F$ ; and the tangential equation can be expressed in terms of  $U, U'$  and  $V$ .

**4992** Ex. (1).—The polar reciprocal of  $u$  with respect to  $u'$  is

$$\Theta u' = F.$$

PROOF.—Referring  $u, u'$  to their common self-conjugate triangle,

$$u = ax^2 + by^2 + cz^2, \quad u' = x^2 + y^2 + z^2,$$

$$F = a(b+c)x^2 + b(c+a)y^2 + c(a+b)z^2.$$

The polar of  $\xi, \eta, \zeta$  with respect to  $u'$  is  $\xi x + \eta y + \zeta z$ , and the condition that this may touch  $u$  is  $bc\xi^2 + ca\eta^2 + ab\zeta^2 = 0$  (4664), or, which is the same thing,  $(bc+ca+ab)(x^2+y^2+z^2) = F$  or  $\Theta u' = F$  (4945).

**4993** Ex. (2).—The enveloping conic  $V$  in (4987) may also be written

$$\Theta u' + \Theta' u = F.$$

PROOF.—With the same assumptions as in Ex. (1),  $\mathbf{V}$  in (4973) becomes  $(b+c)\lambda^2 + (c+a)\mu^2 + (a+b)\nu^2 = 0$ . The trilinear equation is, therefore, by (4667),

$$(c+a)(a+b)x^2 + (a+b)(b+c)y^2 + (b+c)(c+a)z^2 = 0,$$

or  $(bc+ca+ab)(x^2+y^2+z^2) + (a+b+c)(ax^2+by^2+cz^2) = \mathbf{F}.$

**4994** Ex. (3).—The condition that  $\mathbf{F}$  may become two right lines is

$$\Delta\Delta'(\Theta\Theta' - \Delta\Delta') = 0.$$

PROOF.—Referring to Ex. (1),  $A = bc$ ,  $B = ca$ ,  $C = ab$ ,  $F = G = H = 0$ ,  $A' = B' = C' = 1$ ; therefore, in (4981),  $a'' = B + C = a(b+c)$ , &c. Hence

$$\text{the discriminant } \Delta \text{ of } \mathbf{F} = abc(b+c)(c+a)(a+b),$$

or  $abc \{ (a+b+c)(bc+ca+ab) - abc \} = \text{the above, by (4945).}$

**4995** To reduce the two conics  $u, u'$  to the forms

$$x^2 + y^2 + z^2 = 0, \quad ax^2 + \beta y^2 + \gamma z^2 = 0.$$

By (4945),  $a, \beta, \gamma$  will be the roots of the cubic

$$\Delta k^3 - \Theta k^2 + \Theta' k - \Delta' = 0 \dots\dots\dots (1),$$

and  $x^2, y^2, z^2$  will be found in terms of  $u, u'$  and  $\mathbf{F}$ , by solving the three equations  $x^2 + y^2 + z^2 = u$ ,  $ax^2 + \beta y^2 + \gamma z^2 = u'$  and (by

4994),  $a(\beta + \gamma)x^2 + \beta(\gamma + a)y^2 + \gamma(a + \beta)z^2 = \mathbf{F} \dots\dots (2).$

**4996** Ex. (1): Given  $x^3 + y^3 + 2y + 2x + 3 = 0$ ;  $x^3 + 2y^3 + 4y + 2x + 6 = 0$ ; to be reduced as above. To compute the invariants, we take

$$\begin{array}{cccccc} a & b & c & f & g & h \\ = 1 & 1 & 3 & 1 & 1 & 0 \text{ in the first equation.} \\ \text{and } = 1 & 2 & 6 & 2 & 1 & 0 \text{ in the second.} \end{array}$$

therefore

$$\begin{array}{cccccc} A & B & C & F & G & H \\ = 2 & 2 & 1 & -1 & -1 & 1 \text{ in the first equation.} \\ \text{and } = 8 & 5 & 2 & -2 & -2 & 2 \text{ in the second.} \end{array}$$

Therefore (4938, '9)  $\Delta = 1$ ,  $\Theta = 6$ ,  $\Theta' = 11$ ,  $\Delta' = 6$ . The roots of equation (1) are now 1, 2, 3. Therefore (2) becomes  $5X^2 + 8Y^2 + 9Z^2 = \mathbf{F}$ . Computing  $\mathbf{F}$  also by (4981) with the above values of  $A, B$ , &c., we get the three equations as under, introducing  $z$  for the sake of symmetry,

$$\begin{array}{l} X^2 + Y^2 + Z^2 = x^2 + y^2 + 3z^2 + 2yz + 2zx, \\ X^2 + 2Y^2 + 3Z^2 = x^2 + 2y^2 + 6z^2 + 4yz + 2zx, \\ 5X^2 + 8Y^2 + 9Z^2 = 5x^2 + 8y^2 + 22z^2 + 16yz + 10zx, \end{array}$$

The solution gives  $X = x+1$ ,  $Y = y+1$ ,  $Z = 1$ , and the equations in the forms required are  $(x+1)^2 + (y+1)^2 + 1 = 0$ ,  $(x+1)^2 + 2(y+1)^2 + 3 = 0$ .

**4997** Ex. (2).—To find the envelope of the base of a triangle inscribed in a conic  $u'$  so that two of its sides touch  $u$ .

Let  $u = x^2 + y^2 + z^2 - 2yz - 2zx - 2xy - 2hxyz$ ,  
 and  $u' = 2fyz + 2gzx + 2hxy$ ,  
 $x$  and  $y$  being the sides touched by  $u$ . Then  $u + ku'$  will be a conic touched  
 by the third side  $z$ . By finding the invariants, it appears that  $\Theta^2 - 4\Delta\Theta' = 4\Delta\Delta'k$ , whence  $k$  is determined, and the envelope becomes

$$(\Theta^2 - 4\Delta\Theta') u' + 4\Delta\Delta' u = 0.$$

Compare (4970).

**4998** The tangential equation of the two circular points at infinity (4717) is

$$\lambda^2 + \mu^2 = 0.$$

PROOF.—This is the condition that  $\lambda x + \mu y + \nu$  should pass through either of those points, since  $x \pm iy = c$  is the general form of such a line.

**4999**  $U = 0$  being the tangential equation of a conic, the discriminant of  $kU + U'$  is

$$\Delta^2 k^3 + \Delta\Theta' k^2 + \Delta'\Theta k + \Delta'^2.$$

PROOF.—The discriminant of  $kU + U'$  is identical in form with (4937), but the capitals and small letters must be interchanged. Let then the discriminant be  $\bar{\Delta}k^3 + \bar{\Theta}k^2 + \bar{\Theta}'k + \bar{\Delta}' = 0$ . We have

$$\bar{\Delta} = \Delta^2 \text{ (4670), } \bar{\Theta} = (BC - F^2) A' + \&c. = A'a\Delta + \&c. \text{ (4668) } = \Delta\Theta'.$$

Similarly  $\bar{\Theta}' = \Delta'\Theta$ ,  $\bar{\Delta}' = \Delta'^2$ .

**5000** If  $\Theta, \Theta'$  be the invariants of any conic  $U$  and the pair of circular points  $\lambda^2 + \mu^2$  (4998); then  $\Theta = 0$  makes the conic a parabola, and  $\Theta' = 0$  makes it an equilateral hyperbola.

PROOF.—The discriminant of  $kU + \lambda^2 + \mu^2$  is  $k^2\Delta^2 + k(a+b)\Delta + ab - h^2$ . For, as above,  $\bar{\Delta} = \Delta^2$ ;  $\bar{\Theta} = A'a\Delta + B'b\Delta = (a+b)\Delta$  since  $A' = B' = 1$ ,  $C' \&c. = 0$ ;  $\bar{\Theta}' = (A'B' - H^2)C = C = ab - h^2$ ; and  $\bar{\Delta}' = 0$ . The rest follows from the conditions (4471) and (4474).

**5001** The tangential equation of the circular points is, in trilinear notation (see the note at 5030),

$$\lambda^2 + \mu^2 + \nu^2 - 2\mu\nu \cos A - 2\nu\lambda \cos B - 2\lambda\mu \cos C.$$

PROOF:  $\lambda^2 + \mu^2 = 0$ , in Cartesians, shows that the perpendicular let fall from any point whatever upon any line passing through one of the points is infinite. Therefore, by (4624).

**5002** The conditions in (4689) and (4690), which make the general conic a parabola or equilateral hyperbola, may be obtained by forming  $\Theta$  and  $\Theta'$  for the conic and equation (5001) and applying (5000).



**5003** If  $\Theta^2 = 4\Theta$ , the conic passes through one of the circular points.

**5004** When  $u'$  in (4984) reduces to  $\lambda^2 + \mu^2$ , that is, to the circular points at infinity,  $\mathbf{F}$  becomes the locus of intersection of tangents to  $u$  at right angles, and produces the equations of the director-circle (4693) and (4694).

**5005** The tangential equation of a conic confocal with  $U$  is

$$kU + \lambda^2 + \mu^2 = 0;$$

**5006** And if the left side, by varying  $k$ , be resolved into two factors, it becomes the equation of the foci of the system.

PROOF.—Since  $\lambda^2 + \mu^2$  represents the two circular points at infinity (4998),  $kU + \lambda^2 + \mu^2 = 0$ , by (4914), is the tangential equation of a conic touched by the four imaginary tangents of  $U$  from those points. But these tangents intersect in two pairs in the foci of  $U$  (4720); and, for the same reason, in the foci of  $kU + \lambda^2 + \mu^2$ , which must therefore have the same foci.

If  $kU + \lambda^2 + \mu^2$  consists of two factors, it represents two points which, by (4913), are the intersections of the pairs of tangents just named, and are therefore the foci.

**5007** The general Cartesian equation of a conic confocal with  $u = 0$  (4656) is

$$k^2 \Delta u + k \{ C(x^2 + y^2) - 2Gx - 2Fy + A + B \} + 1 = 0.$$

PROOF.—(5005) must be transformed. Written in full, by (4664), it becomes  $(kA+1)\lambda^2 + (kB+1)\mu^2 + kC\nu^2$ . Hence, by (4667), the trilinear equation will be

$$\begin{aligned} \{ (kB+1)kC - k^2F^2 \} a^2 + \&c. &= k^2(BC - F^2) a^2 + kCa^2 + \&c. \\ &= k^2a\Delta a^2 + kCa^2 + \&c., \end{aligned} \quad (4668)$$

and so on, finally writing  $x, y, 1$  for  $a, \beta, \gamma$ .

#### TO FIND THE FOCI OF THE GENERAL CONIC (4656).

(First Method.)

**5008** Substitute in  $kU + \lambda^2 + \mu^2$  either root of its discriminant  $k^2\Delta^2 + k(a+b)\Delta + ab - h^2 = 0$  (5000), and it becomes resolvable into two factors  $(\lambda x'_1 + \mu y'_1 + \nu)(\lambda x_2 + \mu y_2 + \nu)$ . The foci are  $x_1y_1$  and  $x_2y_2$ , real for one value of  $k$  and imaginary for the other.

PROOF.—By (5006) the two factors represent the two foci, consequently the coordinates of the foci are the coefficients of  $\lambda, \mu, \nu$  in those factors.



(Second Method.)

**5009** Let  $xy$  be a focus; then, by (4720), the equation of an imaginary tangent through that point is  $(\xi-x)+i(\eta-y)=0$  or  $\xi+i\eta-(x+iy)=0$ . Therefore substitute, in the tangential equation (4665), the coefficients  $\lambda=1$ ,  $\mu=i$ ,  $\nu=-(x+iy)$ , and equate real and imaginary parts to zero. The resulting equations for finding  $x$  and  $y$  are, with the notation of (4665),

$$\mathbf{5010} \quad 2(Cx-G)^2 = \Delta \left[ a-b + \sqrt{\{4h^2+(a-b)^2\}} \right].$$

$$\mathbf{5011} \quad 2(Cy-F)^2 = \Delta \left[ b-a + \sqrt{\{4h^2+(a-b)^2\}} \right].$$

**5012** If the conic is a parabola,  $C=0$ , and the coordinates of the focus are given by

$$(F^2+G^2)x = FH + \frac{1}{2}(A-B)G,$$

$$(F^2+G^2)y = GH - \frac{1}{2}(A-B)F.$$

**5013** Ex. — To find the foci of  $2x^2+2xy+2y^2+2x=0$ . By the first method, we have

$$\begin{array}{l} \left. \begin{array}{l} a, \quad b, \quad c, \quad f, \quad g, \quad h \\ = 2, \quad 2, \quad 0, \quad 0, \quad 1, \quad 1 \end{array} \right\} \begin{array}{l} \text{from which } \Delta = -2. \text{ The quad-} \\ \text{ratic for } k \text{ is} \\ k^2\Delta^2 + 4k\Delta + 3 = (2k-3)(2k-1) = 0, \\ \text{therefore } k = \frac{3}{2} \text{ or } \frac{1}{2}. \end{array} \\ \text{and } \left. \begin{array}{l} A, \quad B, \quad C, \quad F, \quad G, \quad H \\ = 0, \quad -1, \quad 3, \quad 1, \quad -2, \quad 0 \end{array} \right\} \end{array}$$

$$\text{Taking } \frac{3}{2}, \quad kU + \lambda^2 + \mu^2 = \frac{3}{2}(-\mu^2 + 3\nu^2 + 2\mu\nu - 4\nu\lambda) + \lambda^2 + \mu^2 = 0,$$

$$\text{or} \quad 2\lambda^2 - 12\nu\lambda - \mu^2 + 9\nu^2 + 6\mu\nu = 0.$$

Solving for  $\lambda$ , this is thrown into the factors

$$\{2\lambda + \mu\sqrt{2} - 3(2 + \sqrt{2})\nu\} \{2\lambda - \mu\sqrt{2} - 3(2 - \sqrt{2})\nu\}.$$

Therefore the coordinates of the foci, after rationalizing the fractions, are

$$-\frac{2-\sqrt{2}}{3}, \quad -\frac{\sqrt{2}-1}{3} \quad \text{and} \quad -\frac{2+\sqrt{2}}{3}, \quad \frac{\sqrt{2}+1}{3}.$$

**5014** Otherwise, by the second method, equations (5010, '1) become, in this instance,  $(3x+2)^2 = \pm 2$ ,  $(3y-1)^2 = \pm 2$ , the solution of which produces the same values of  $x$  and  $y$ .

**5015** When the axes are oblique, the coordinates  $x, y$  of a focus are found from the equations

$$\{C(x+y\cos\omega) - F\cos\omega - G\}^2 = \frac{1}{2}\Delta(\sqrt{I^2-4J}+2a-I)$$

$$(Cy-F)^2\sin^2\omega = \frac{1}{2}\Delta(\sqrt{I^2-4J}-2a+I),$$

where  $I$  and  $J$  are the invariants (4955) and (4954) respec-

tively. The equations may be solved for  $x' = x + y \cos \omega$  and  $y' = y \sin \omega$ , which are the rectangular coordinates of the focus with the same origin and  $x$  axis.

PROOF.—Following the method of (5009), the imaginary tangent through the focus is, by (4721),  $\xi - x + (\eta - y)(\cos \omega + i \sin \omega)$ . The two equations obtained from the tangential equation are, writing  $\Delta a$  for  $BC - F^2$ , &c. (4668),

$X^2 - Y^2 = -\Delta(a + b - 2h \cos \omega - 2a \sin^2 \omega)$ ,  $XY = \Delta(h \sin \omega - a \sin \omega \cos \omega)$ ; where  $X = C(x + y \cos \omega) - F \cos \omega - G$  and  $Y = (Cy - F) \sin \omega$ .

**5016** If the equation of the conic to oblique axes be

$$ax^2 + 2hxy + by^2 + c = 0,$$

the equations for determining the foci reduce to

$$\frac{y(x + y \cos \omega)}{a \cos \omega - h} = \frac{x(y + x \cos \omega)}{b \cos \omega - h} = \frac{c}{ab - h^2}.$$

**5017** The condition that the line  $\lambda x + \mu y + \nu z$  may touch the conic  $u + (\lambda'x + \mu'y + \nu'z)^2$  is

$$U + \phi(\mu\nu' - \mu'\nu, \nu\lambda' - \nu'\lambda, \lambda\mu' - \lambda'\mu) = 0. \quad (4656, 4936, '74)$$

**5018** or  $(\Delta + U')U = \Pi^2$ , (4938)

where  $2\Pi \equiv \lambda'U_\lambda + \mu'U_\mu + \nu'U_\nu$ . (4674)

PROOF.—Put  $a + \lambda'^2$  for  $a$ , &c. in  $U$  of (4664). The second form follows from the first through the identity

$$\Delta\phi(\mu\nu' - \mu'\nu, \&c.) = UU' - \Pi^2.$$

**5019** Otherwise, let  $P' \equiv u_x x + u_y y + u_z z$ , the polar of  $x', y', z'$  (4659), then the condition that  $P'$  may touch  $u + P''^2$  becomes, in terms of the coordinates of the poles,

**5020**  $(1 + u'')u' = \phi_x x'' + \phi_y y'' + \phi_z z''$ . (See 4657).

PROOF.—If we put  $u_x, u_y, u_z$ , from (4659), for  $\lambda, \mu, \nu$  in  $U$  to obtain the condition of touching, the result is  $\Delta u'$ ; and similar substitutions made in  $\Pi$  give  $\Delta(\phi_x x'' + \&c.)$ , therefore (5018) becomes  $(1 + u'')u' = (\phi_x x'' + \&c.)$ .

**5021** The condition that the conics

$$u + (\lambda'x + \mu'y + \nu'z)^2, \quad u + (\lambda''x + \mu''y + \nu''z)^2$$

may touch each other is

$$(\Delta + U')(\Delta + U'') = (\Delta \pm \Pi)^2. \quad (4938-74)$$

PROOF.—Make one of the common chords

$$(\lambda'x + \mu'y + \nu'z) \pm (\lambda''x + \mu''y + \nu''z)$$

touch either conic by substituting  $\lambda' \pm \lambda''$  for  $\lambda$ , &c. in (5018). The result is  $(\Delta + U')(U' \pm 2\Pi + U'') = (U' \pm \Pi)^2$ , which reduces to the form above.

**5022** The condition, in terms of the coordinates of the poles of the two lines, is found from the last, as in (5019), and is

$$(1+u')(1+u'') = \{1 \pm (\phi_x x'' + \phi_y y'' + \phi_z z'')\}^2.$$

**5023** The Jacobian,  $J$ , of three conics  $u, v, w$ , is the locus of a point whose polars with respect to the conics all meet in a point. Its equation is

$$\begin{vmatrix} a_1x+h_1y+g_1z & a_2x+h_2y+g_2z & a_3x+h_3y+g_3z \\ h_1x+b_1y+f_1z & h_2x+b_2y+f_2z & h_3x+b_3y+f_3z \\ g_1x+f_1y+c_1z & g_2x+f_2y+c_2z & g_3x+f_3y+c_3z \end{vmatrix} = 0.$$

PROOF.—The equation is the eliminant of the equations of the three polars passing through a point  $\xi\eta\zeta$ , viz.,  $u_x\xi + u_y\eta + u_z\zeta = 0$ ,  $v_x\xi + v_y\eta + v_z\zeta = 0$ ,  $w_x\xi + w_y\eta + w_z\zeta = 0$ . See (4657) and (1600).

**5024** The equation of a conic passing through five points  $a_1\beta_1\gamma_1, a_2\beta_2\gamma_2$ , &c. is the determinant equation annexed; and the equation of a conic touching five right lines  $\lambda_1\mu_1\nu_1, \lambda_2\mu_2\nu_2$ , &c. is the same in form,  $\lambda, \mu, \nu$  taking the place of  $a, \beta, \gamma$ .

$$\begin{vmatrix} a^2 & \beta^2 & \gamma^2 & \beta\gamma & \gamma a & a\beta \\ a_1^2 & \beta_1^2 & \gamma_1^2 & \beta_1\gamma_1 & \gamma_1 a_1 & a_1\beta_1 \\ a_2^2 & \beta_2^2 & \gamma_2^2 & \beta_2\gamma_2 & \gamma_2 a_2 & a_2\beta_2 \\ a_3^2 & \beta_3^2 & \gamma_3^2 & \beta_3\gamma_3 & \gamma_3 a_3 & a_3\beta_3 \\ a_4^2 & \beta_4^2 & \gamma_4^2 & \beta_4\gamma_4 & \gamma_4 a_4 & a_4\beta_4 \\ a_5^2 & \beta_5^2 & \gamma_5^2 & \beta_5\gamma_5 & \gamma_5 a_5 & a_5\beta_5 \end{vmatrix} = 0.$$

PROOF.—The determinant is the eliminant of six equations of the type (4656) in the one case and (4665) in the other. By (583).

**5025** If three conics have a common self-conjugate triangle, their Jacobian is three right lines.

PROOF.—The Jacobian of  $a_1x^2 + b_1y^2 + c_1z^2, a_2x^2 + b_2y^2 + c_2z^2, a_3x^2 + b_3y^2 + c_3z^2$  is, by (5023),  $xyz = 0$ .

For the condition that three conics may have a common point, see *Salmon's Conics*, 6th edit., Art. 389a, and *Proc. Lond. Math. Soc.*, Vol. iv., p. 404, *J. J. Walker, M.A.*

**5026** A system of two conics has four covariant forms  $u, u', \mathbf{F}, \mathbf{J}$ , connected by the equation

$$\begin{aligned} J^2 = & \mathbf{F}^3 - \mathbf{F}^2 (\Theta u' - \Theta' u) + \mathbf{F} (\Delta' \Theta u^2 + \Delta \Theta' u'^2) \\ & + \mathbf{F} u u' (\Theta \Theta' - 3 \Delta \Delta') - \Delta \Delta'^2 u^3 - \Delta' \Delta^2 u'^3 \\ & + \Delta' u^2 u' (2 \Delta \Theta' - \Theta^2) + \Delta u'^2 u (2 \Delta' \Theta - \Theta'^2). \end{aligned}$$

PROOF.—Form the Jacobian of  $u, u'$ , and  $\mathbf{F}$ . This will be the equation of the sides of the common self-conjugate triangle (4992, 5025). Compare the result with that obtained by the method of (4995).

**5027** By parity of reasoning, there are four contravariant forms  $U, U', \mathbf{V}, \mathbf{\Gamma}$  where  $\mathbf{\Gamma}$  is the tangential equivalent of  $\mathbf{J}$ , and represents the vertices of the self-conjugate triangle. Its square is expressed in terms of  $U, U', \mathbf{V}$  and the invariants precisely as  $\mathbf{J}^2$  is expressed in (5026).

**5028** The locus of the centre of a conic which always touches four given lines is a right line.

PROOF.—Let  $U = 0, U' = 0$  be the tangential equations of two fixed conics, each touching the four lines; then, by (4914),  $U + kU' = 0$  is another conic also touching the four lines. The coordinates of its centre will be  $\frac{G + kG'}{C + kC'}$  and  $\frac{F + kF'}{C + kC'}$ , by (4402). The point is thus seen, by (4032), to lie on the line joining the centres of the two fixed conics and to divide that line in the ratio  $kC' : C$ .

**5029** To find the locus of the focus of a conic touching four given lines.

In the equations (5010, '1) for determining the coordinates of the focus, write  $A + kA'$  for  $A$ , &c., and eliminate  $k$ . The result in general is a cubic curve. If  $\Sigma, \Sigma'$  be parabolas,  $\Sigma + k\Sigma'$  is a parabola having three tangents in common with  $\Sigma$  and  $\Sigma'$ . If  $C = C' = 0$  the locus becomes a circle. If the conics be concentric, they touch four sides of a parallelogram, and the locus is a rectangular hyperbola.

*Note on Tangential Coordinates.*

**5030** It must be borne in mind that a tangential equation in trilinear notation (that is, when the variables are the coefficients of  $\alpha, \beta, \gamma$  in the tangent line  $l\alpha + m\beta + n\gamma$ ) will not agree with the equation of the same locus expressed in the tangential coordinates  $\lambda, \mu, \nu$  of (4019). Thus, to convert equation (5001), which, for distinctness, will now be written

$$l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C = 0$$

into tangential coordinates, we must substitute, by (4023),  $a\lambda, b\mu, c\nu$  for  $l, m, n$ . The equation then becomes

$$a^2\lambda^2 + b^2\mu^2 + c^2\nu^2 - 2bc \cos A \mu\nu - 2ca \cos B \nu\lambda - 2ab \cos C \lambda\mu = 0.$$

Put  $2bc \cos A = b^2 + c^2 - a^2$ , &c., and the result is the equation as presented in (4905).

*Corrigenda.*—In (4678) and (4692) erase the coefficient 2; and in (4680) and (4903) supply the factor 4 on the left of the equation.

# THEORY OF PLANE CURVES.

## TANGENT AND NORMAL.

**5100** Let  $P$  (Fig. 90) be a point on the curve  $AP$ ;  $PT$ ,  $PN$ ,  $PG$ , the tangent, ordinate, and normal intercepted by the  $x$  axis of coordinates. See definitions in (1160). Let  $\angle PTX = \psi$ .

$$\mathbf{5101} \quad \tan \psi = \frac{dy}{dx}, \text{ by (1403); } \sin \psi = \frac{dy}{ds}; \quad \cos \psi = \frac{dx}{ds}.$$

$$\mathbf{5104} \quad \text{Sub-tangent } NT = yx_x, \quad \text{Sub-normal } NG = yy_x.$$

$$\mathbf{5106} \quad PT = y \sqrt{(1+x_x^2)}, \quad PT' = x \sqrt{(1+y_x^2)}.$$

$$\mathbf{5108} \quad PG = y \sqrt{(1+y_x^2)}, \quad PG' = x \sqrt{(1+x_x^2)}.$$

Let  $OP = r$  (Fig. 91),  $u = r^{-1}$ ,  $AOP = \theta$ ,  $OPT = \phi$ ; Arc  $AP = s$ . Then, by infinitesimals,

$$\mathbf{5110} \quad \sin \phi = r \frac{d\theta}{ds}, \quad \cos \phi = \frac{dr}{ds}, \quad \tan \phi = r \frac{d\theta}{dr}.$$

$$\mathbf{5113} \quad (dx)^2 + (dy)^2 = (ds)^2, \quad s_x = \sqrt{(1+y_x^2)}.$$

$$\mathbf{5114} \quad \tan \psi = \frac{r_x \sin \theta + r \cos \theta}{r_x \cos \theta - r \sin \theta} \quad (1768)$$

$$\mathbf{5115} \quad \text{Intercepts of Normal } OG = r \frac{dr}{dx}, \quad OG' = r \frac{dr}{dy}. \text{ (Fig. 90)}$$

PROOF:  $OG = \frac{r \sin OPG}{\sin PGN} = \frac{r \cos \phi}{\sin NPT} = r \frac{r_z}{x_z} = r \frac{dr}{dx}.$

**5116**  $s_\theta = \sqrt{(r^2 + r_\theta^2)}, \quad s_r = \sqrt{(1 + r^2 \theta_r^2)}.$

PROOF.—By  $r\theta_z = \sin \phi$  and  $\tan \phi = r\theta_r$  (5110).

## EQUATIONS OF THE TANGENT AND NORMAL.

The equation of the curve being  $y = f(x)$  or  $u \equiv \phi(x, y) = 0$ , the equation of the tangent at  $xy$  is

**5118**  $\eta - y = \frac{dy}{dx} (\xi - x), \quad (4120)$

**5119** or  $\xi y_x - \eta = xy_x - y,$

**5120** or  $\xi u_x + \eta u_y = xu_x + yu_y. \quad (1708)$

**5121** If  $\phi(x, y) = v_n + v_{n-1} + \dots + v_0$ , where  $v_n$  is a homogeneous function of  $x$  and  $y$  of the  $n^{\text{th}}$  degree, the constant part forming the right member of equation (5120) takes the value

$$-v_{n-1} - 2v_{n-2} - \dots - (n-1)v_1 - nv_0.$$

By Euler's theorem (1621) and  $\phi(x, y) = 0$ .

The equation of the normal at  $xy$  is

**5122**  $\eta - y = -\frac{dx}{dy} (\xi - x), \quad (4122)$

**5123** or  $\xi x_y + \eta = xx_y + y,$

**5124** or  $\xi u_y - \eta u_x = xu_y - yu_x. \quad (1708)$

## POLAR EQUATIONS OF THE TANGENT AND NORMAL.

Let  $r, \theta$  be the coordinates of  $P$  (Fig. 91), and  $R, \Theta$  those of  $S$ , any point on the tangent at  $P$ ; and let  $u = r^{-1}$ ,  $U = R^{-1}$ ,  $\tau = \Theta - \theta$ ; the polar equation of the tangent at  $P$  will be

**5125**  $R = \frac{r^2}{d_\theta(r \sin \tau)}, \quad \text{or} \quad U = u \cos \tau + u_\theta \sin \tau.$

The polar equation of the normal is

**5127**  $R = \frac{rr_\theta}{d_\theta(r \cos \tau)}, \quad \text{or} \quad U = u \cos \tau - u^2 \theta_\theta \sin \tau.$

PROOF.—From  $\frac{r}{R} = \frac{OP}{OS} = \frac{\sin OSP}{\sin OPS} = \frac{\sin(\phi - \tau)}{\sin \phi}$ , and from  $\tan \phi = r\theta_r$  (5112). Similarly for the normal.



Let  $OY \equiv p$  be the perpendicular from the pole upon the tangent, then

$$5129 \quad p = r \sin \phi = (u^2 + u_\theta^2)^{-1}. \quad (5112)$$

$$5131 \quad \frac{xy_x - y}{\sqrt{(1 + y_x^2)}} = \frac{x\phi_x + y\phi_y}{\sqrt{(\phi_x^2 + \phi_y^2)}}. \quad (4064 \text{ \& } 5119, '20)$$

$OS$ , drawn at right angles to  $r$  to meet the tangent, is called the *polar sub-tangent*.

$$5133 \quad \text{Polar sub-tangent} = r^2 \theta_r. \quad (5112)$$

### RADIUS OF CURVATURE AND EVOLUTE.

Let  $\xi, \eta$  be the centre of curvature for a point  $xy$  on the curve, and  $\rho$  the radius of curvature; then

$$5134 \quad (x - \xi)^2 + (y - \eta)^2 = \rho^2 \dots\dots\dots (1).$$

$$5135 \quad (x - \xi) + (y - \eta) y_x = 0 \dots\dots\dots (2),$$

$$1 + y_x^2 + (y - \eta) y_{2x} = 0 \dots\dots\dots (3).$$

PROOF.—(2) and (3) are obtained from (1) by differentiating for  $x$ , considering  $\xi, \eta$  constants.

The following are different values of  $\rho$ :

$$5137 \quad \rho = \frac{(1 + y_x^2)^{\frac{3}{2}}}{y_{2x}} = \frac{(\phi_x^2 + \phi_y^2)^{\frac{3}{2}}}{2\phi_{xy}\phi_x\phi_y - \phi_{2x}\phi_y^2 - \phi_{2y}\phi_x^2}$$

$$5139 \quad = \frac{(x_t^2 + y_t^2)^{\frac{3}{2}}}{y_{2t}x_t - x_{2t}y_t} = \frac{1}{y_{2s}x_s - x_{2s}y_s}$$

$$5141 \quad = \frac{1}{\sqrt{(x_{2s}^2 + y_{2s}^2)}} = \frac{x_s}{y_{2s}} = -\frac{y_s}{x_{2s}}$$

$$5144 \quad = \frac{(r^2 + r_\theta^2)^{\frac{3}{2}}}{r^2 + 2r_\theta^2 + rr_{2\theta}} = \frac{(u + u_\theta^2)^{\frac{3}{2}}}{u^3(u + u_{2\theta})}$$

$$5146 \quad = s_\phi = p + p_{2\phi} = rr_p.$$

PROOFS.—For (5137), eliminate  $x - \xi$  and  $y - \eta$  between equations (1), (2), and (3).

(5138) is obtained from the preceding value by substituting for  $y_x$  and

$y_{2x}$  the values (1708, '9). The equation of the curve is here supposed to be of the form  $\phi(x, y) = 0$ .

For (5139); change the variable to  $t$ . For (5140); make  $t = s$ .

For (5141-3); let  $PQ = QR = ds$  (Fig. 92) be equal consecutive elements of the curve. Draw the normals at  $P, Q, R$ , and the tangents at  $P$  and  $Q$  to meet the normals at  $Q$  and  $R$  in  $T$  and  $S$ . Then, if  $PN$  be drawn parallel and equal to  $QS$ , the point  $N$  will ultimately fall on the normal  $QO$ . Now the difference of the projections of  $PT$  and  $PN$  upon  $OX$  is equal to the projection of  $TN$ . Projection of  $PT = dsx_1$ ; that of  $PN$  or  $QS = ds(x_1 + x_2, ds)$  (1500); therefore the difference  $= dsx_2, ds = TN \cos a$ . But  $TN : ds = ds : \rho$ , therefore  $\rho x_2 = \cos a$ . Similarly  $\rho y_2 = \sin a$ .

For (5144); change (5137) to  $r$  and  $\theta$ , by (1768, '9).

(5145) is obtained from  $\rho = rr_p = -\frac{u_p}{u^3}$  and (5129); or change (5144) from  $r$  to  $u$  by  $r = u^{-1}$ .

(5146.) In Fig. (93),  $PQ = \rho$ ,  $PP' = ds$ , and  $PQP' = d\psi$ .

(5147.) In Fig. (93), let  $PQ, P'Q$  be consecutive normals;  $PT, P'T'$  consecutive tangents;  $OT, OT', ON, ON'$  perpendiculars from the origin upon the tangents and normals. Then, putting  $p$  for  $OT = PN$ ,  $q$  for  $PT = ON$ , and  $d\psi$  for  $\angle TPT' = PQP'$ , &c., we have

$$q = \frac{dp}{d\psi}, \quad QN = \frac{dq}{d\psi}, \quad \text{and} \quad \rho = PQ = p + QN = p + p_{2\psi}.$$

(5148.)  $dp = r \cos \phi d\psi$  and  $\cos \phi = r_1$ . Eliminate  $\cos \phi$ .

**5149** DEF.—The *evolute* of a curve is the locus of its centre of curvature. Regarding the evolute as the principal curve, the original curve is called its *involute*.

**5150** The normal of any curve is a tangent to its evolute.

PROOF.—By differentiating equation (5135) on the hypothesis that  $\xi$  and  $\eta$  are variables dependent upon  $x$ , and combining the result with (3), we obtain  $y_x \eta_1 = -1$ .

In (Fig. 94), the normal at  $P$  of the curve  $AP$  touches the evolute at  $Q$ . Otherwise the evolute is the envelope of the normals of the given curve.

If  $xy$  and  $\xi\eta$  are the points  $P, Q$ , we have the relations

$$5151 \quad \frac{\xi - x}{\rho} d\xi + \frac{\eta - y}{\rho} d\eta = d\rho, \quad \frac{d\xi}{\xi - x} = \frac{d\eta}{\eta - y} = \frac{d\rho}{\rho}.$$

PROOF.—Take  $Qn = d\xi$  and  $ns = d\eta$ , then  $Qs = d\rho$ . The projection of  $Qn, ns$  gives  $d\rho$  in (5151) and proportion gives (5152).

**5153** The evolute and involute are connected by the formulæ below, in which  $r', p', s'$  in the evolute correspond to  $r, p, s$  in the involute.

$$5154 \quad \rho \pm s' = \text{constant}; \quad p'^2 = r^2 - p^2; \quad r'^2 = r^2 + \rho^2 - 2p\rho.$$

PROOF.—From Fig. (94),  $dp = \pm ds'$ , &c.,  $s$  being the arc  $RQ$  measured from a fixed point  $R$ . Hence, if a string is wrapped upon a given curve, the free end describes an involute of the curve. (3155, '6) from Fig. (93).

**5157** To obtain the equation of the evolute; eliminate  $x$  and  $y$  from equations (5135, '6) and the equation of the curve.

**5158** To obtain the polar equation of the evolute; eliminate  $r$  and  $p$  from (5156) and (5157) and the given equation of the curve  $r = F(p)$ .

**5159** Ex.—To find the evolute of the catenary  $y = \frac{c}{2}(e^{\frac{x}{c}} + e^{-\frac{x}{c}})$ . Here  $y_x = \frac{1}{2}(e^{\frac{x}{c}} - e^{-\frac{x}{c}}) = \frac{\sqrt{(y^2 - c^2)}}{c}$ ;  $y_{xx} = \frac{1}{2c}(e^{\frac{x}{c}} + e^{-\frac{x}{c}}) = \frac{y}{c^2}$ ; so that equations (5135, '6) become

$$(x - \xi) + (y - \eta) \frac{\sqrt{(y^2 - c^2)}}{c} = 0 \quad \text{and} \quad 1 + \frac{y^2 - c^2}{c^2} + (y - \eta) \frac{y}{c^2} = 0.$$

From these we find  $y = \frac{\eta}{2}$ ,  $x = \xi - \frac{\eta}{4c} \sqrt{(\eta^2 - 4c^2)}$ . Substituting in the equation of the curve, we obtain the required equation in  $\xi$  and  $\eta$ .

## INVERSE PROBLEM AND INTRINSIC EQUATION.

An inverse question occurs when the arc is a given function of the abscissa, say  $s = \phi(x)$ ; the equation of the curve in rectangular coordinates will then be

**5160** 
$$y = \int \sqrt{(s_x^2 - 1)} dx. \quad [\text{From (5113).}]$$

**5161** The *intrinsic equation* of a curve is an equation independent of coordinate axes. Let  $y = \phi(x)$  be the ordinary equation, taking for origin a point  $O$  on the curve (Fig. 95), and the tangent at  $O$  for  $x$  axis. Let  $s = \text{arc } OP$ , and  $\psi$  the inclination of the tangent at  $P$ ; then the intrinsic equation of the curve is

**5162** 
$$s = \int \sec \psi x_\psi d\psi;$$

where  $x_\psi$  is found from  $\tan \psi = \phi'(x)$ .

To obtain the Cartesian equation from the intrinsic equation :

**5163** Let  $s = F(\psi)$  be the intrinsic equation. Eliminate  $\psi$  between this and the equations

$$x = \int \cos \psi ds, \quad y = \int \sin \psi ds.$$

**5165** The intrinsic equation of the evolute obtained from the intrinsic equation of the curve,  $s = F(\psi)$ , is

$$\frac{ds}{d\psi} + s' = l, \quad \text{a constant.} \quad (5154)$$

**5166** The intrinsic equation of the involute obtained from  $s' = F(\psi)$ , the equation of the curve, is

$$s = \int \{l - F(\psi)\} d\psi.$$

For  $d\psi$  is the same for both curves (Fig. 94),  $\psi$  only differing by  $\frac{1}{2}\pi$ , and  $s = \int \rho d\psi$ .

## ASYMPTOTES.

**5167** DEF.—An asymptote of a curve is a straight line or curve which the former continually approaches but never reaches. (*Vide* 1185).

### GENERAL RULES FOR RECTILINEAR ASYMPTOTES.

**5168** RULE I.—Ascertain if  $y_x$  has a limiting value when  $x = \infty$ . If it has, find the intercept on the  $x$  or  $y$  axis, that is,  $x - yx_y$  or  $y - xy_x$  (5104).

There will be an asymptote parallel to the  $y$  axis when  $y_x$  is infinite, and the  $x$  intercept finite, or one parallel to the  $x$  axis when  $y_x$  is zero and the  $y$  intercept finite.

**5169** RULE II.—When the equation of the curve consists of homogeneous functions of  $x$  and  $y$ , of the  $m^{\text{th}}$ ,  $n^{\text{th}}$ , &c. degrees, so that it may be written

$$x^m \phi\left(\frac{y}{x}\right) + x^n \chi\left(\frac{y}{x}\right) + \&c. = 0 \dots\dots\dots (1);$$

put  $\mu x + \beta$  for  $y$  and expand  $\phi\left(\mu + \frac{\beta}{x}\right)$ , &c., by (1500). Divide (1) by  $x^m$ , and make  $x$  infinite; then  $\phi(\mu) = 0$  determines  $\mu$ .

Next, put this value of  $\mu$  in (1), divide by  $x^{m-1}$ , and make  $x$  infinite; thus  $\beta\phi'(\mu) + \chi(\mu) = 0$  determines  $\beta$ . Should the last equation be indeterminate, then

$$\frac{1}{2}\beta^2\phi''(\mu) + \beta\chi'(\mu) + \psi(\mu) = 0$$

gives two values for  $\beta$ , and so on.

When  $n$  is  $< m-1$ ,  $\beta = 0$ , and when  $n$  is  $> m-1$ ,  $\beta = \infty$ .

**5170** RULE III.—If  $\phi(x, y) = 0$  be a rational integral equation, to discover asymptotes parallel to the axes, equate to zero the coefficients of the highest powers of  $x$  and  $y$ , if those coefficients contain  $y$  or  $x$  respectively.

To find other asymptotes—Substitute  $\mu x + \beta$  for  $y$  in the original equation, and arrange according to powers of  $x$ . To find  $\mu$ , equate to zero the coefficient of the highest power of  $x$ . To find  $\beta$ , equate to zero the coefficient of the next power of  $x$ , or, if that equation be indeterminate, take the next coefficient in order, and so on.

**5171** RULE IV.—If the polar equation of the curve be  $r = f(\theta)$  and if  $r = \infty$  makes the polar subtangent  $r^2\theta_r = c$ , a finite quantity, there is an asymptote whose equation is  $r \cos(\theta - a) = c$ ; where  $a \pm \frac{1}{2}\pi = f^{-1}(\infty) =$  the value of  $\theta$  of the curve when  $r$  is infinite.

**5172** Asymptotic curves.—In these the difference of corresponding ordinates continually diminishes as  $x$  increases.

As an example, the curves  $y = \phi(x)$  and  $y = \phi(x) + \frac{a}{x}$  are asymptotic.

**5173** Ex. 1.—To find the asymptotes of the curve

$$(a + 3x)(x^2 + y^2) = 4x^3 \dots\dots\dots (1).$$

The coefficient of  $y^2$ ,  $a + 3x = 0$ , gives an asymptote parallel to the  $y$  axis.

Putting  $y = \mu x + \beta$ , (1) becomes

$$(a + 3x)(x^2 + \mu^2 x^2 + 2\mu\beta x + \beta^2) - 4x^3 = 0 \dots\dots\dots (2).$$

The coefficient of  $x^2$ ,  $3(1 + \mu^2) - 4 = 0$  gives  $\mu = \pm \frac{1}{\sqrt{3}}$ . Substituting this value of  $\mu$  in (2), the coefficient of  $x^3$  becomes  $\frac{4a}{\sqrt{3}} \pm \frac{6\beta}{\sqrt{3}}$ ; and this, equated to zero, gives  $\beta = \mp \frac{2a}{3\sqrt{3}}$ . Hence the equations of two more asymptotes are  $3y\sqrt{3} = \pm (3x - 2a)$ .

Ex. 2.—To find an asymptote of the curve  $r \cos \theta = a \cos 2\theta$ . Here

$$r^2 \frac{d\theta}{dr} = \frac{a^2 \cos 2\theta}{a \cos 2\theta \sin \theta - 2a \sin 2\theta \cos \theta}.$$

When  $r = \infty$ ,  $\theta = \frac{1}{2}\pi$ , and  $r^2 \theta_r = -a$ . Hence the equation of the asymptote is  $r \cos \theta = -a$ .

## SINGULARITIES OF CURVES.

**5174** *Concavity and Convexity.*—A curve is reckoned convex or concave towards the axis of  $x$  according as  $yy_{2x}$  is positive or negative.

### POINTS OF INFLEXION.

**5175** A point of inflexion (Fig. 96) exists where the tangent has a limiting position, and therefore where  $y_x$  takes a maximum or minimum value.

**5176** Hence  $y_{2x}$  must vanish and change sign, as in (1832).

**5177** Or, more generally, an even number of consecutive derivatives of  $y = \phi(x)$  must vanish, and the curve will pass from positive to negative, or from negative to positive, with respect to the axis of  $x$ , according as the next derivative is negative or positive. [See (1833).]

### MULTIPLE POINTS.

**5178** A multiple point, known also as a node or crunode, exists when  $y_x$  has more than one value, as at  $B$  (Fig. 98). If  $\phi(x, y) = 0$  be the curve,  $\phi_x$  and  $\phi_y$  must both vanish, by (1713). Then, by (1704), two values of  $y_x$  determining a double point, will be given by the quadratic

$$\phi_{2y}y_x^2 + 2\phi_{xy}y_x + \phi_{2x} = 0 \dots\dots\dots (1).$$

**5179** If  $\phi_{2x}$ ,  $\phi_{2y}$ ,  $\phi_{xy}$  also vanish; then, by (1705), three values of  $y_x$ , determining a triple point, will be obtained from the cubic

$$\phi_{3y}y_x^3 + 3\phi_{2yx}y_x^2 + 3\phi_{y2x}y_x + \phi_{3x} = 0 \dots\dots\dots (2).$$

**5180** Generally, when all the derivatives of  $\phi$  of an order



less than  $n$  vanish, the equation for determining  $y_x$  (put  $= z$ ) may be written

$$(zd_y + d_x)^n \phi(x, y) = 0.$$

PROOF.—Let  $ab$  be the multiple point. Then, by (1512),

$$\phi(a+h, b+k) = \frac{1}{n!} (hd_x + kd_y)^n \phi(x, y)$$

+ terms of higher order which vanish when  $h$  and  $k$  are small.

And  $\frac{k}{h} = -\frac{\phi_x}{\phi_y} = \frac{dy}{dx}$  in the limit.

### CUSPS.

**5181** When two branches of a curve have a common tangent at a point, but do not pass through the point, they form a *cusp*, termed also a *spinode* or *stationary point*.

**5182** In the *first species*, or *ceratoid cusp* (Fig. 100), the two values of  $y_{2x}$  have opposite signs.

**5183** In the *second species*, or *ramphoid cusp* (Fig. 101), they have the same sign.

### CONJUGATE POINTS.

**5184** A *conjugate point*, or *acnode*, is an isolated point whose coordinates satisfy the equation of the curve. A necessary condition for the existence of a conjugate point is that  $\phi_x$  and  $\phi_y$  must both vanish.

PROOF.—For the tangent at such a point may have any direction, therefore  $\frac{\phi_x}{\phi_y}$  is indeterminate (1713).

**5185** There are four species of the triple point according as it is formed by the union of

- (i.) three crunodes, as in (Fig. 102);
- (ii.) two crunodes and a cusp, as in (Fig. 103);
- (iii.) a crunode and two cusps, as in (Fig. 104);
- (iv.) when only one real tangent exists at the point.

**5186** Ex.—The equation  $y^2 = (x-a)(x-b)(x-c)^*$  when  $a < b < c$  represents a curve, such as that drawn in (Fig. 97).

\* Salmon's *Higher Plane Curves*, Arts. 39, 40.

When  $b = c$  the curve takes the form in (Fig. 98). But if, instead,  $b = a$ , the oval shrinks into a point  $A$  (Fig. 99).

If  $a = b = c$  the point  $A$  becomes a cusp, as in (Fig. 100).

*A geometrical method of investigating singular points.*

**5187** Describe an elementary circle of radius  $r$  round the point  $x, y$  on the curve  $\phi(x, y) = 0$ , intersecting the curve in the point  $x+h, y+k$ . Let  $h = r \cos \theta$ ,  $k = r \sin \theta$ . Expand  $\phi(x+h, y+k) = 0$  by (1512), and put  $\phi_x = K \sin \gamma$ ,  $\phi_y = K \cos \gamma$ . We thus obtain

$$K \sin(\gamma + \theta) + \frac{r}{2}(\phi_{2x} \cos^2 \theta + 2\phi_{xy} \sin \theta \cos \theta + \phi_{2y} \sin^2 \theta) + \frac{R}{r} = 0$$

$R$  being put for the rest of the expansion .....(1),

According as the quadratic in  $\tan \theta$ ,

$$\phi_{2x} + 2\phi_{xy} \tan \theta + \phi_{2y} \tan^2 \theta = 0,$$

has real, equal, or imaginary roots; i.e., according as  $\phi_{xy}^2 - \phi_{2x}\phi_{2y}$  is positive, zero, or negative,  $xy$  will be a crunode, a cusp, or an acnode. By examining the sign of  $R$ , the species of cusp and character of the curvature may be determined.

Figures (105) and (106), according as  $R$  and  $\phi_{2y}$  have opposite or like signs, show the nature of a crunode; and figures (107) and (108) show a cusp.

PROOF.—At an ordinary point the circle cuts the curve at the two points given by  $\theta = -\gamma$ ,  $\theta = \pi - \gamma$ . But, if  $\phi_x$  and  $\phi_y$  both vanish, there is a singular point. Writing  $A, B, C$  for  $\phi_{2x}, \phi_{xy}, \phi_{2y}$ , equation (1) now becomes

$$C \cos^2 \theta \left\{ \tan^2 \theta + \frac{2B}{C} \tan \theta + \frac{A}{C} \right\} + \frac{2R}{r^2} = 0 \dots\dots\dots(2).$$

(i.) If  $B^2 > AC$ , this may be put in the form

$$C \cos^2 \theta (\tan \theta - \tan \alpha)(\tan \theta - \tan \beta) + \frac{2R}{r^2} = 0,$$

and the points of intersection with the circle are given by  $\theta = \alpha, \beta, \pi + \alpha$ , and  $\pi + \beta$ . (Figs. 105 and 106.)

(ii.) When  $B^2 = AC$ , we may write equation (1)

$$C \cos^2 \theta (\tan \theta - \tan \alpha)^2 + \frac{2R}{r^2} = 0.$$

If  $R$  and  $C$  have opposite signs, there is a cusp with  $\alpha$  for the inclination of the tangent (Fig. 107). So also, if  $R$  and  $C$  have the same sign, the inclination and direction being  $\pi + \alpha$  (Fig. 108). The cusps exist in this case because  $R$  changes its sign when  $\pi$  is added to  $\theta$ ,  $R$  being a homogeneous function of the third degree in  $\sin \theta$  and  $\cos \theta$ .

(iii.) If  $B^2 < AC$ , there are no real points of intersection, and therefore  $xy$  is an acnode.

## CONTACT OF CURVES.

**5188** A contact of the  $n^{\text{th}}$  order exists between two curves when  $n$  successive derivatives,  $y_x, \dots y_{nx}$  or  $r_\theta, \dots r_{n\theta}$ , correspond. The curves cross at the point if  $n$  be even. No curve can pass between them which has a contact of a lower order with either.

Ex.—The curve  $y = \phi(x)$  has a contact of the  $n^{\text{th}}$  order, at the point where  $x = a$ , with the curve  $y = \phi(a) + (x-a)\phi'(a) + \dots + \frac{(x-a)^n}{n!}\phi^n(a)$ .

**5189** COR.—If the curve  $y = f(x)$  has  $n$  parameters, they may be determined so that the curve shall have a contact of the  $(n-1)^{\text{th}}$  order with  $y = \phi(x)$ .

A contact of the first order between two curves implies a common tangent, and a contact of the second order a common radius of curvature.

*Conic of closest contact with a given curve.*

**5190** Lemma.—In a central conic (Fig. of 1195),

$$\tan CPG = \frac{1}{3} \frac{d\rho}{ds}.$$

PROOF.—Putting  $PCT \equiv \theta$ ,  $CPT \equiv \phi$ ,  $CP \equiv r$ ,  $CD \equiv R$ , we have, by (1211),  $r^2 + R^2 = a^2 + b^2$ ,  $\therefore rr_\theta = -RR_\theta$ , ..... (i).

Also  $Rr \sin \phi = ab$ , by (1194),  $\therefore Rr^2\theta_\theta = ab$ , by (5110) ..... (ii).

Now  $\tan CPG = -\cot \phi = -\frac{r_\theta}{r}$  (5112)  $= \frac{-r_\theta}{r\theta_\theta} = \frac{R^2R_\theta}{ab}$ , by (i) and (ii).

But  $\rho = \frac{R^3}{ab}$  (4538),  $\therefore \frac{1}{3} \frac{d\rho}{ds} = \frac{R^2R_\theta}{ab} = \tan CPG$ .

**5191** To find the conic having a contact of the fourth order with a given curve at a given point  $P$ .

If  $O$  be the conic's centre, the radius  $r \equiv OP$ , and the angle  $\nu$  between  $r$  and the normal are found from the equations

$$\tan \nu = \frac{1}{3} \frac{d\rho}{ds}, \quad \frac{\cos \nu}{r} = \frac{1}{\rho} - \frac{d\nu}{ds},$$

and these determine the conic.

PROOF.—In Fig. 93, let  $O$  be the centre of the conic and  $P$  the point of contact. The five disposable constants of the general equation of a conic will be determined by the following five data: two coordinates of  $O$ , a common point  $P$ , a common tangent at  $P$ , and the same radius of curvature  $PQ$ .

Since  $\nu = POT$ ,  $d\theta = POP'$ ,  $d\psi = TOT'$ , and  $ds = PP'$ , we have, in passing from  $P$  to  $P'$ ,  $d\nu = P'OT' - POT = d\psi - d\theta$ . Now  $rd\theta = ds \cos \nu$ , therefore  $\frac{\cos \nu}{r} = \frac{d\psi - d\nu}{ds \cos \theta} = \frac{1}{\rho} - \frac{d\nu}{ds}$ ; and  $\tan \nu$  has been found in the lemma.

The squares of the semi-axes of the same conic are the roots of the equation

$$(9 + b^2 - 3ac)^3 x^2 - 9a^2 (18 + 2b^2 - 3ac)(9 + b^2 - 3ac) x + 729a^4 = 0,$$

$a, b, c$  being written for  $\rho, \rho_s, \rho_{2s}$ . The eccentricity is found

from 
$$\frac{9(e^2 - 2)^2}{1 - e^2} = \frac{(18 + 2b^2 - 3ac)^2}{9 + b^2 - 3ac}.$$

Also the equation of the conic referred to the tangent and normal at the point is

$$Ax^2 + 2Bxy + Cy^2 = 2y,$$

where  $A = \frac{1}{\rho}$ ,  $B = -\frac{\rho_s}{3\rho}$ ,  $C = \frac{1}{\rho} + \frac{2\rho_s^2}{9\rho} - \frac{\rho_{2s}}{3}$ .

*Ed. Times, Math. Reprint, Vol. xxi., p. 87, where the demonstrations by Prof. Wolstenholme will be found.*

## ENVELOPES.

**5192** An envelope of a curve is the locus of the ultimate intersections of the different curves of the same species, got by varying continuously a parameter of the curve; and the envelope touches all the intersecting curves so obtained.

**5193** RULE.—If  $F(x, y, a) = 0$  be a curve having the parameter  $a$ , the envelope is the curve obtained by eliminating  $a$  between the equations

$$F(x, y, a) = 0 \quad \text{and} \quad d_a F(x, y, a) = 0.$$

PROOF.—Let  $a$  change to  $a+h$ . The coordinates of the point of intersection of  $F(x, y, a) = 0$  and  $F(x, y, a+h) = 0$  satisfy the equation

$$\frac{F(x, y, a+h) - F(x, y, a)}{h} = 0, \quad \text{i.e.,} \quad \frac{dF(x, y, a)}{da} = 0. \quad (1404)$$

**5194** If  $F(x, y, a, b, c, \dots) = 0$  be the equation of a curve having  $n$  parameters  $a, b, c, \dots$  connected by  $n-1$  equations, then, by varying the parameters, a series of intersecting curves may be obtained. The envelope of these curves will be found by differentiating all the equations with respect to  $a, b, c$ , &c., and eliminating  $da, db, \dots$  and  $a, b, \dots$

**5195** EX.—In (2) of (5135), we have the equation of the normal of a curve at a given point  $xy$ ;  $\xi, \eta$  being the variable coordinates, and  $x, y$  the parameters connected by the equation of the curve  $F(x, y) = 0$ . By differentiating for  $x$  and  $y$ , (5136) is found, and the elimination as directed in (5157) produces the equation of the evolute which, by (5194), is the envelope of the curve.

## INTEGRALS OF CURVES AND AREAS.

### FORMULÆ FOR THE LENGTH OF AN ARC $s$ .

$$\mathbf{5196} \quad s = \int ds = \int \sqrt{1+y_x^2} dx = \int \sqrt{1+x_y^2} dy \quad (5113)$$

$$\mathbf{5200} \quad = \int \sqrt{x_t^2 + y_t^2} dt = \int \sqrt{r^2 + r_\theta^2} d\theta \quad (5116)$$

$$\mathbf{5201} \quad = \int \sqrt{r^2 \theta_r^2 + 1} dr = \int \frac{r dr}{\sqrt{r^2 - p^2}}. \quad (5111)$$

$$\mathbf{5203} \quad \text{Legendre's formula, } s = p_\psi + \int p d\psi.$$

$$\mathbf{5204} \quad \text{The whole contour of a closed curve} = \int_0^{2\pi} p d\psi.$$

PROOF.—In figure (93), let  $P, P'$  be an element  $ds$  of the curve;  $PT, P'T'$  tangents, and  $OT, OT'$  the perpendiculars upon them from the origin;  $OT \equiv p$ ,  $PT \equiv q$ . Then  $ds + P'T' - PT = TL$ , i.e.,  $ds + dq = p d\psi$ ; therefore  $s + q = \int p d\psi$ . But  $q d\psi = -dp$ ; therefore  $s = p_\psi + \int p d\psi$ . Also, in integrating all round the curve,  $P'T' - PT$  taken for every point vanishes in the summation, or  $\int dq = 0$ . Therefore  $\int ds = \int_0^{2\pi} p d\psi$ .

FORMULÆ FOR PLANE AREAS.

**5205** If  $y = \phi(x)$  be the equation of a curve, the area bounded by the curve, two ordinates ( $x=a$ ,  $x=b$ ), and the  $x$  axis, is, as in (1902),

$$A = \int_a^b \phi(x) dx.$$

**5206** With polar coordinates the area included between two radii ( $\theta=a$ ,  $\theta=\beta$ ) and the curve is

$$A = \frac{1}{2} \int_a^\beta r^2 d\theta = \frac{1}{2} \int p ds = \frac{1}{2} \int \frac{pr dr}{\sqrt{(r^2 - p^2)}}.$$

PROOF.—From figure (91) and the elemental area  $OPP'$ .

**5209** The area bounded by two circles of radii  $a$ ,  $b$ , and the two curves  $\theta = \phi(r)$ ,  $\theta = \psi(r)$  (Fig. 109).

$$A = \int_a^b \int_{\phi(r)}^{\psi(r)} r dr d\theta = \int_a^b r \{ \psi(r) - \phi(r) \} dr.$$

Here  $r \{ \psi(r) - \phi(r) \} dr$  is the elemental area between the dotted circumferences.

**5211** The area bounded by two radii of curvature, the curve, and its evolute (Fig. 110).

$$A = \frac{1}{2} \int \rho^2 d\psi = \frac{1}{2} \int \rho ds.$$

PROOF.—From figure (93) and the elemental area  $QPP'$ .

INVERSE CURVES.

The following results may be added to those given in Arts. (1000–15).

**5212** Let  $r$ ,  $r'$  be corresponding radii of a curve and its inverse, so that  $rr' = k^2$ ;  $s$ ,  $s'$  corresponding arcs, and  $\phi$ ,  $\phi'$  the angles between the radius and tangents, then

$$\frac{ds}{ds'} = \frac{r}{r'} \quad \text{and} \quad \phi = \phi'.$$

PROOF.—Let  $PQ$  be the element of arc  $ds$ ,  $P'Q'$  the element  $ds'$ , and  $O$  the origin.

Then  $OP \cdot OP' = OQ \cdot OQ'$ , therefore  $OPQ$ ,  $OQ'P'$  are similar triangles; therefore  $PQ : P'Q' :: OP : OQ' = r : r'$ ; also  $\angle OPQ = \angle OQ'P'$ .



**5214** If  $\rho, \rho'$  be the radii of curvature,

$$\frac{r}{\rho} + \frac{r'}{\rho'} = 2 \sin \phi.$$

PROOF.—From  $p = r \sin \phi$ ,  $p' = r' \sin \phi$ , we have

$$p' = k^2 \frac{p}{r^2}, \text{ therefore } \frac{dp'}{dr} = k^2 \frac{r^2 p_r - 2rp}{r^4} \dots\dots\dots (i.).$$

Also  $r = \frac{k^2}{r'}$ , therefore  $\frac{dr}{dr'} = -\frac{k^2}{r'^2} \dots\dots\dots (ii.).$

Now  $\rho' = r' \frac{dr'}{dp'}$  (5148), therefore  $\frac{1}{\rho'} = \frac{1}{r'} \frac{dp'}{dr} \frac{dr}{dr'} = \frac{2p}{rr'} - \frac{r}{\rho r'}$ , by (i.) and (ii.).

Therefore  $\frac{r'}{\rho'} + \frac{r}{\rho} = \frac{2p}{r} = 2 \sin \phi.$

**5215** To find the equation of the inverse of a curve in rectangular coordinates, substitute

$$\frac{k^2 x}{x^2 + y^2} \quad \text{and} \quad \frac{k^2 y}{x^2 + y^2}$$

for  $x$  and  $y$  in the equation of the given curve.

**5216** The inverse of the algebraic curve

$$u_n + u_{n-1} + u_{n-2} + \dots + u_1 + u_0 = 0,$$

where  $u_n$  is a homogeneous function of the  $n^{\text{th}}$  degree, will be

$$k^{2n} u_n + k^{2(n-1)} u_{n-1} (x^2 + y^2) + k^{2(n-2)} u_{n-2} (x^2 + y^2)^2 + \dots \\ \dots + u_0 (x^2 + y^2)^n = 0.$$

**5217** The inverse of the conic  $u_2 + u_1 + u_0 = 0$  is

$$k^4 u_2 + k^2 u_1 (x^2 + y^2) + u_0 (x^2 + y^2)^2 = 0.$$

**5218** If the origin be on the curve, this equation becomes

$$k^2 u_2 + u_1 (x^2 + y^2) = 0.$$

**5219** The angle  $\phi$  will also be unaltered in any curve,  $r = f(\theta)$ , if the inversion be effected by putting

$$r = kr'^n \quad \text{and} \quad \theta = n\theta'.$$

PROOF.—

$$\tan \phi' = r' \theta'_r \quad (5112) = r' \theta'_r r_r = r' \frac{\theta_r}{n} k n r'^{n-1} = k r'^n \theta_r = r \theta_r = \tan \phi.$$

PEDAL CURVES.

**5220** The locus of the foot of the perpendicular from the origin upon the tangent is called a *pedal curve*. The pedal of the pedal curve is called the second pedal, and so on. Reversing the order, the envelope of the right lines drawn from each point of a curve at right angles to the radius vector is called the *first negative pedal*, and so on.

**5221** The pedal and the reciprocal polar are inverse curves  
(1000, 4844.)

AREA OF A PEDAL CURVE.

**5222** Let  $C$ ,  $P$ ,  $Q$  be the respective areas of a closed curve, the pedal of the curve, and the pedal of the evolute; then

$$P - Q = C, \quad P + Q = \frac{1}{2} \int r^2 d\psi, \quad 2P = C + \frac{1}{2} \int r^2 d\psi.$$

PROOF.—With figure (93) and the notation of (5204), we have, by (5206),  $P = \frac{1}{2} \int p^2 d\psi$ ,  $Q = \frac{1}{2} \int q^2 d\psi$ ; therefore  $P + Q = \frac{1}{2} \int (p^2 + q^2) d\psi = \frac{1}{2} \int r^2 d\psi$ . Also, taking two consecutive positions of the triangle  $OPT \equiv A$ , we get  $OPT - OP'T' = \delta A = \delta C + \delta Q - \delta P$ . Therefore, integrating all round,

$$\int_A^A dA = 0 = C + Q - P.$$

**5225** *Steiner's Theorem*.—If  $P$  be the area of the pedal of a closed curve when the pole is the origin, and  $P'$  the area of the pedal when the pole is the point  $xy$ ,

$$P' - P = \frac{1}{2}\pi (x^2 + y^2) - ax - by,$$

where  $a = \int_0^{2\pi} p \cos \theta d\theta$  and  $b = \int_0^{2\pi} p \sin \theta d\theta$ ;

$\theta$  being the inclination of  $p$ .

PROOF.—(Fig. 111.) Let  $LM$  be a tangent,  $S$  the point  $xy$ , perpendiculars  $OM \equiv p$  and  $SR \equiv p'$ . Draw  $SN$  perpendicular to  $OM$ , and let  $ON = p_1$ ; then  $P' = \frac{1}{2} \int p'^2 d\psi = \frac{1}{2} \int (p - p_1)^2 d\psi = \frac{1}{2} \int p^2 d\psi + \frac{1}{2} \int p_1^2 d\psi - \int p p_1 d\psi$   
 $= P + \frac{\pi}{2} OS^2 - \int_0^{2\pi} p (x \cos \theta + y \sin \theta) d\theta$ , by (4094), and  $d\theta = d\psi$ .

And  $\frac{1}{2} \int_0^{2\pi} p_1^2 d\psi =$  twice the area of the circle whose diameter is  $OS$ .

**5226** COR. 1.—If  $P'$  be given, the locus of  $xy$  is a circle whose equation is (5225), and the centre of this circle is the same for all values of  $P'$ , the coordinates of the centre being  $\frac{a}{\pi}$  and  $\frac{b}{\pi}$ .

**5227** COR. 2.—Let  $Q$  be the fixed centre referred to, and let  $QS = c$ . Let  $P''$  be the area of the pedal whose origin is  $Q$ ; then

$$P' - P'' = \frac{1}{2}\pi c^2.$$

For  $a$  and  $b$  must vanish in (5225) when the origin is at the centre  $Q$ , and  $x^2 + y^2$  then  $= c^2$ .

**5228** COR. 3.—Hence  $P''$  is the minimum value of  $P'$ .\*

## ROULETTES.

**5229** DEF.—A *Roulette* is the locus of a point rigidly connected with a curve which rolls upon a fixed right line or curve.

### AREA OF A ROULETTE.

**5230** When a closed curve rolls upon a right line, the area generated in one revolution by the normal to the roulette at the generating point is twice the area of the pedal of the rolling curve with respect to the generating point.

PROOF.—(Fig. 112.) Let  $P$  be the point of contact of the rolling curve and fixed straight line,  $Q$  the point which generates the roulette. Let  $R$  be a consecutive point, and when  $R$  comes into contact with the straight line, let  $P'Q'$  be the position of  $RQ$ . Then  $PQ$  is a normal to the roulette at  $Q$ , and  $P$  is the instantaneous centre of rotation. Draw  $QN, QS$  perpendiculars on the tangents at  $P$  and  $R$ . The elemental area  $PQQ'P'$ , included between the two normals  $QP, Q'P'$ , is ultimately equal to  $PQR + QRQ'$ . But  $PQR = dC$ , an element of the area of the curve swept over by the radius vector  $QP$  or  $r$  round the pole  $Q$ ; and  $QRQ' = \frac{1}{2}r^2 d\psi$ ; therefore, whole area of roulette  $= C + \frac{1}{2} \int_0^{2\pi} r^2 d\psi = 2P$ , by (5224).

**5231** Hence, by (5228), there is one point in any closed curve for which the area of the corresponding roulette is a

\* For a discussion of the pedal curves of an ellipse by the Editor of the *Educ. Times* and others, see *Reprint*, Vol. i., p. 23; Vol. xvi., p. 77; Vol. xvii., p. 92; and Vol. xx., p. 106.

minimum. Also the area of the roulette described by any other point, distant  $c$  from the origin of the minimum roulette, exceeds the area of the latter by  $\pi c^2$ .

**5232** When the line rolled upon is a curve, the whole area generated in one revolution of the rolling curve becomes

$$C + \frac{1}{2} \int_0^{2\pi} r^2 \left(1 + \frac{\rho}{\rho'}\right) d\psi,$$

where  $\rho, \rho'$  are the radii of curvature of the rolling and fixed curves, and  $C$  is the area of the former.

PROOF.—(Fig. 113.) Instead of the angle  $d\psi$ , we now have the sum of the angles of contingence at  $P$  of the rolling curve and fixed curve, viz.,

$$d\psi + d\psi' = d\psi \left(1 + \frac{d\psi'}{d\psi}\right) = d\psi \left(1 + \frac{\rho}{\rho'}\right),$$

since  $\rho d\psi = ds = \rho' d\psi'$ , by (5146).

#### LENGTH OF THE ARC OF A ROULETTE.

**5233** If  $\sigma$  and  $\zeta$  be corresponding arcs of the roulette and the pedal whose origin is the generating point; then, when the fixed line is straight,  $\sigma = \zeta$ ; and when it is a curve,

$$\int d\sigma = \int \left(1 + \frac{\rho}{\rho'}\right) d\zeta.$$

PROOF.—(Fig. 112.) Let  $R$  be the point which has just left the straight line,  $Q$  the generating point,  $N, S$  consecutive points on the pedal curve. Draw the circle circumscribing  $RQNS$ , of which  $RQ = r$  is a diameter, and let the diameter which bisects  $NS$  meet the circle in  $K$ . Then, when the points  $P, R, P'$  coincide,  $KN$  and  $RQ$  are diameters, and  $SKN = SPN = d\psi = QRQ'$ ; therefore  $SN$  or  $d\zeta = rd\psi = QQ'$  or  $d\sigma$ . When the fixed line is a curve,  $d\sigma = rd\psi \left(1 + \frac{\rho}{\rho'}\right)$ , as in (5232).

#### RADIUS OF CURVATURE OF A ROULETTE.

**5235** Let  $\alpha$  (Fig. 113) be the angle between the generating line  $r$  and the normal at the point of contact;  $\rho, \rho'$  the radii of curvature of the fixed and rolling curves, and  $R$  the radius of curvature of the roulette; then,

$$R = \frac{r^2}{\frac{\rho\rho'}{\rho+\rho'} \cos \alpha - r}.$$

4 Y

PROOF.—Let consecutive normals of the roulette meet in  $O$ ; then  $OQ = R$ ,  $PQ = r$ ,  $MPT = \alpha$ .

$$\frac{R-r}{R} = \frac{PM}{QQ'} = \frac{ds \cos \alpha}{d\sigma}, \text{ and } d\sigma = r(d\psi + d\psi') = r\left(\frac{ds}{\rho} + \frac{ds}{\rho'}\right),$$

from which  $R$  is obtained. If the curvature of the roulette is convex towards  $P$  (Fig. 114), we must write  $R+r$  instead of  $R-r$  above.

**5236** The curvature is convex towards  $P$  when  $R$  is positive, that is, when the carried point  $Q$  falls within the circle whose diameter measured on the normal of the rolling curve  $= \frac{\rho\rho'}{\rho + \rho'}$ . When  $Q$  falls without this circle, the curvature is concave; and when  $Q$  falls upon the circumference, the point is one of inflexion. The circle has for this reason been called the *circle of inflexions*.

**5237** In figure (163) let  $PA = \rho$ ,  $PB = \rho'$ ,  $PQ = r$ ,  $OQ = R$ , as in (5235). Draw  $PCD$ , the circle of inflexions, with its diameter  $PC = \frac{\rho\rho'}{\rho + \rho'}$ , and therefore  $PD = \frac{\rho\rho'}{\rho + \rho'} \cos \alpha$ . From these values and proportion it follows that  $BC : BP : BA$  and  $QD : QP : QO$ . Also, if the circle on diameter  $PE = PC$  be drawn,  $AE : AP : AB$  and  $OF : OP : OQ$ .

**5238** A simple construction for the centre of curvature of the roulette is the following. (Fig. 164, with letters as in 5237.) At  $P$  draw a perpendicular to  $PQ$  to meet  $QB$  in  $N$ . Join  $NA$ , which will meet  $QP$  produced in  $O$ , the required point.

PROOF.—From equation (5235), assuming  $O$  to be the centre of curvature, we can deduce the relation  $(BA : AP)(PO : OQ)(QN : NB) = 1$ , therefore, by (968),  $A$ ,  $O$ ,  $N$  are collinear points.

#### THE ENVELOPE OF A CARRIED CURVE.

**5239** When a curve is rigidly connected with a rolling curve, it will have an envelope. The path of its point of contact with the envelope is a tangent to both curves, and therefore the normal, common to the carried curve and its envelope, passes through the point of contact  $P$  of the rolling and fixed curve.

**5240** The centre of curvature of the envelope is obtained as follows.



In Fig. (163), from  $P$  draw a normal to the carried curve meeting it in  $Q$ , and let  $S$  on  $PQ$  be the centre of curvature of the envelope for the point  $Q$ ; and  $O$  that of the carried curve. Then  $PS$  is found from

$$\frac{1}{\rho} + \frac{1}{\rho'} = \cos \alpha \left( \frac{1}{PS} + \frac{1}{PO} \right).$$

**5241** When the envelope is a right line, the centre of curvature lies on the circle of inflexions (5236). When the carried curve is a right line, the same point lies on the circle  $PEF$  (Fig. 163), and if the right line always passes through a fixed point, that point lies on the circle  $PEF$ .

**5242** If  $p$  be the perpendicular from a fixed point upon a carried right line whose inclination to a fixed line is  $\psi$ ; the radius of curvature of the envelope is  $\rho = p + p_{2\psi}$ , by (5147).

#### INSTANTANEOUS CENTRE.

**5243** When a plane figure moves in any manner in its own plane, the *instantaneous centre of rotation* is the intersection of the perpendiculars at two points to the directions in which the points are moving; and a line from the instantaneous centre to any point of the figure is the normal to the path of that point.

Ex.—Let a triangle  $ABC$  slide with its vertices  $A, B$  always upon the right lines  $OA, OB$ . The perpendiculars at  $A, B$  to  $OA, OB$  meet in  $Q$ , the instantaneous centre, and  $QC$  is the normal at  $C$  to the locus of  $C$ .

Since  $AB$  and the angle  $AOB$  are of constant magnitude,  $OQ$ , the diameter of the circle circumscribing  $OAQB$ , is of constant magnitude. Hence the locus of the instantaneous centre  $Q$  is a circle of centre  $O$  and radius  $OQ$ .

**5244** *Holditch's Theorem*.—If a chord of a given length  $LM$  moves completely round a closed curve, the area enclosed between the curve and the locus of a point  $P$  on the chord is equal to  $\pi cc'$  where  $c = LK, c' = MK$ .

**5245** If the ends of  $LM$  move on different closed curves whose areas are  $\lambda, \mu$ , while the area described by  $K$  is  $\kappa$ , then

$$\kappa = \frac{\lambda c' + \mu c}{c + c'} - \pi cc'.$$

PROOF.—(5244). Let the innermost oval in figure (134) be the envelope of  $LM$ ,  $e$  its area, and  $E$  the point of contact. Let  $EL \equiv l, EM \equiv m$ ,



$EK \equiv k$ ,  $l + m = a = c + c'$ ;  $\theta$ , the inclination of  $LM$ . Then, integrating in every case from 0 to  $2\pi$ ,

$$\left. \begin{aligned} \frac{1}{2} \int l^2 d\theta &= \lambda - \epsilon \\ \frac{1}{2} \int m^2 d\theta &= \mu - \epsilon \end{aligned} \right\} \therefore \frac{1}{2} \int (l^2 - m^2) d\theta = \frac{a}{2} \int (l - m) d\theta = \lambda - \mu.$$

$$\text{Also } \frac{a}{2} \int (l + m) d\theta = \pi a^2,$$

$\therefore a \int l d\theta = \pi a^2 + \lambda - \mu \dots\dots (i.)$ . Similarly  $c \int l d\theta = \pi c^2 + \lambda - \kappa \dots\dots (ii.)$ , the last being obtained from  $\frac{1}{2} \int (l^2 - k^2) d\theta = \lambda - \kappa$ .  $\kappa$  is then found by eliminating the integral between (i.) and (ii.).

(5245.) If the curves  $\lambda$ ,  $\mu$  coincide,  $\lambda = \mu$  and therefore  $\lambda - \kappa = \pi c c'$ .

### TRAJECTORIES.

**5246** DEF.—A *trajectory* is a curve which cuts according to a given law a system of curves obtained by varying a single parameter.

The differential equation of the trajectory which cuts at a constant angle  $\beta$  the system of curves represented by  $\phi(x, y, c) = 0$  is obtained by eliminating  $c$  between the equations

$$\phi(x, y, c) = 0 \quad \text{and} \quad \tan \beta = \frac{\phi_x + \phi_y y_x}{\phi_y - \phi_x y_x},$$

the derivatives of  $\phi$  being partial, and  $y_x$  referring to the trajectory.\*

PROOF.—At a point of intersection we have for the given curve  $m = -\phi_x \div \phi_y$ , and for the trajectory  $m' = y_x$ . Employ (4070).

If the trajectory is to be orthogonal,  $\tan \beta = \infty$ , and the second equation becomes

$$\phi_y - \phi_x y_x = 0.$$

EX.—To find the curve which cuts at a constant angle all right lines passing through the origin.

Let  $y = cx$  represent these lines by varying  $c$ ; then, writing  $n$  for  $\tan \beta$ , the two equations become  $y - cx = 0$  and  $n(1 + cy_x) = y_x - c$ . Eliminating  $c$ ,  $xy_x - y = n(yy_x + x)$ . Divide by  $x^2 + y^2$  and integrate; thus

$$\tan^{-1} \frac{y}{x} = n \log \sqrt{(x^2 + y^2)} + C,$$

which is equivalent to  $r = ae^{\frac{\theta}{n}}$ , the equation of the logarithmic spiral (5289).

\* For a very full investigation of this problem, see Euler, *Novi Com. Petrop.*, Vol. xiv., p. 46, xvii., p. 205; and *Nova Acta Petrop.*, Vol. i., p. 3.

## CURVES OF PURSUIT.

**5247** DEF.—A *curve of pursuit* is the locus of a point which moves with uniform velocity towards another point while the latter describes a known curve also with uniform velocity.

Let  $f(x, y) = 0$  be the known curve,  $xy$  the moving point upon it,  $\xi\eta$  the pursuing point, and  $n:1$  the ratio of their velocities. The differential equation of the path of  $\xi\eta$  is obtained by eliminating  $x$  and  $y$  between the equations

$$f(x, y) = 0 \dots\dots\dots (i.), \quad y - \eta = \eta_\xi (x - \xi) \dots\dots\dots (ii.),$$

$$\sqrt{(x_\xi^2 + y_\xi^2)} = n \sqrt{(1 + \eta_\xi^2)} \dots\dots\dots (iii.).$$

PROOF.—(ii.) expresses the fact that  $xy$  is always in the tangent of the path of  $\xi\eta$ .

(iii.) follows from  $1:n = \sqrt{(d\xi^2 + d\eta^2)} : \sqrt{(dx^2 + dy^2)}$ ; the elements of arc described being proportional to the velocities.

Ex.—The simplest case, being the problem usually presented, is that in which the point  $xy$  moves in a right line. Let  $x = a$  be this line, and let the point  $\xi\eta$  start from the origin when the point  $xy$  is on the  $x$  axis. The equations (i.), (ii.), (iii.) now become, since  $x_\xi = 0$ ,

$$x = a, \quad y = \eta + \eta_\xi (a - \xi), \quad y_\xi = n \sqrt{(1 + \eta_\xi^2)}.$$

From the second  $y_\xi = \eta_\xi (a - \xi)$ , therefore  $(a - \xi) \eta_\xi = n \sqrt{(1 + \eta_\xi^2)}$ .

Putting  $\eta_\xi = p$ ,

$$\frac{dp}{\sqrt{(1 + p^2)}} = \frac{n d\xi}{(a - \xi)}.$$

Integrating by (1928), we find

$$\log(p + \sqrt{1 + p^2}) = -n \log(a - \xi) + n \log a,$$

so that  $p$  and  $\xi$  vanish together at the origin;

therefore  $\sqrt{1 + p^2} + p = \left(\frac{a}{a - \xi}\right)^n$ , and therefore  $\sqrt{1 + p^2} - p = \left(\frac{a - \xi}{a}\right)^n$ ;

therefore

$$\frac{d\eta}{d\xi} = \frac{1}{2} \left\{ \left(\frac{a}{a - \xi}\right)^n - \left(\frac{a - \xi}{a}\right)^n \right\},$$

therefore

$$\eta = \frac{1}{2} \left\{ \frac{a^n (a - \xi)^{1-n}}{n-1} + \frac{a^{-n} (a - \xi)^{1+n}}{n+1} \right\} + \frac{an}{1-n^2}.$$

the equation of the required locus, the constant being taken so that  $\xi = \eta = 0$  together. If, however,  $n = 1$ , the integral is

$$\eta = \frac{\xi^2 - 2a\xi}{4a} - \frac{a}{2} \log \frac{a - \xi}{a}.$$

## CAUSTICS.

**5248** DEF.—If right lines radiating from a point be reflected from a given plane curve, the envelope of the reflected rays is called the *caustic by reflexion* of the curve.

Let  $\phi(x, y) = 0$ ,  $\psi(x, y) = 0$  be the equations of the tangent and normal of the curve, and let  $hk$  be the radiant point; then the equation of the reflected ray will be

$$\phi(h, k) \psi(x, y) + \psi(h, k) \phi(x, y) = 0,$$

and the envelope obtained by varying the coordinates of the point of incidence, as explained in (5194), will be the caustic of the curve.

Ex.—To find the caustic by reflexion of the circle  $x^2 + y^2 = r^2$ , the radiant point being  $hk$ .

Taking for the tangent and normal, as in (4140),  $x \cos \alpha + y \sin \alpha = r$ , and  $x \sin \alpha - y \cos \alpha = 0$ , the reflected ray is

$$(h \cos \alpha + k \sin \alpha - r)(x \sin \alpha - y \cos \alpha) + (h \sin \alpha - k \cos \alpha)(x \cos \alpha + y \sin \alpha - r) = 0.$$

Reducing this to the form

$$A \cos 2\alpha + B \sin 2\alpha + C \sin \alpha - D \cos \alpha = 0,$$

and differentiating for  $\alpha$ ,

$$-2A \sin 2\alpha + 2B \cos 2\alpha + C \cos \alpha + D \sin \alpha = 0.$$

The result of eliminating  $\alpha$  is

$$\{4(h^2 + k^2)(x^2 + y^2) - r^2(x + h)^2 - r^2(y + k)^2\}^2 = 27(kx - hy)^2(x^2 + y^2 - h^2 - k^2)^2,$$

the envelope and caustic required.

**5249** *Quetelet's Theorem*.—The caustic of a curve is the evolute of the locus of the image of the radiant point with respect to the tangent of the curve.

Thus, in the Fig. of (1178), if  $S$  be the radiant point,  $W$  is the image in the tangent at  $P$ . The locus of  $W$  is, in this case, a circle, and the evolute and caustic reduce to the single point  $S'$ .

Since the distance of the image from the radiant point is twice the perpendicular on the tangent, it follows that the locus of the image will always be got by substituting  $2r$  for  $r$  in the polar equation of the pedal, or  $\frac{2k^2}{r}$  for  $r$  in the polar equation of the reciprocal of the given curve with respect to the radiant point and a circle of radius  $k$ .

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TRANSCENDENTAL AND OTHER CURVES.

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THE CYCLOID.\*

(Fig. 115)

**5250** DEF.—A *cycloid* is the roulette generated by a circle rolling upon a right line, the carried point being on the circumference. When the carried point is without the circumference, the roulette is called a *prolate cycloid*; and, when it is within, a *curtate cycloid*.

**5251** The equations of the cycloid are

$$x = a(\theta + \sin \theta), \quad y = a(1 - \cos \theta),$$

where  $\theta$  is the angle rolled through, and  $a$  the radius of the generating circle.

PROOF.—(Fig. 115.) Let the circle  $KPT$  roll upon the line  $DE$ , the point  $P$  meeting the line at  $D$  and again at  $E$ . Arc  $KP = KD$ ; therefore arc  $PT = AK = OT$ . Also  $\theta = PCT$ , the angle rolled through from  $A$ , the centre of the base  $ED$ . Then

$$x = OT + TN = a\theta + a \sin \theta; \quad y = PN = a - a \cos \theta.$$

**5253** If  $s$  be the arc  $OP$  and  $\rho$  the radius of curvature at  $P$ ,

$$s = 2PT = \sqrt{(8ay)}, \quad \rho = 2PK.$$

PROOF.—(i.) The element  $Pp = Bh = 2(OB - Ob)$  ultimately; therefore, by summation,  $s = 2OB$ . Also  $OB = PT = \sqrt{(TK \cdot TR)} = \sqrt{(2ay)}$ .

(ii.) Let two consecutive normals at  $P$  and  $p$  intersect in  $L$ . Then  $PL$ ,  $pl$  are parallel to  $BA$ ,  $bA$ ; therefore  $PLp$  is similar to  $BAi$ . But  $Pp = 2Bi$ ; therefore  $\rho$  or  $PL = 2BA = 2PK$ .

**5255** COR.—The locus of  $L$ , that is the evolute of the cycloid, consists of two half-cycloids as shown in the diagram.

**5256** The area of a cycloid is equal to three times the area of the generating circle, and the curve length is four times the diameter of the same circle.

PROOF.—(i.) Area  $PpnN = PprR = BbqQ$  ultimately. Therefore, by summation,  $DE \cdot AO - \text{cycloid} = \pi a^2$ . But  $DE \cdot AO = 2\pi a \cdot 2a = 4\pi a^2$ ; therefore cycloid =  $3\pi a^2$ .

(ii.) Total curve length =  $8a$ , by (5253).

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\* The earliest notice of this curve is to be found in a MSS. by Cardinal de Cusa, 1454 See *Leibnitz, Opera*, Vol. III., p. 95.

**5257** The intrinsic equation of the cycloid is

$$s = 4a \sin \psi.$$

PROOF:  $s = 2PT = 4a \sin PKT$ , and  $PKT = PTN = \psi$ .\*

#### THE COMPANION TO THE CYCLOID.

**5258** This curve is the locus of the point  $R$  in Fig. (115). Its equation is

$$y = a \left( 1 - \cos \frac{x}{a} \right).$$

PROOF.—From  $x = a\theta$  and  $y = a(1 - \cos \theta)$ .

**5259** The locus of  $S$ , the intersection of the tangents at  $P$  and  $B$ , is the involute of the circle  $ABO$ .

PROOF:  $BS = BP = \text{arc } OB$ .

#### PROLATE AND CURTATE CYCLOIDS. (5250)

**5260** The equations in every case are

$$x = a(\theta + m \sin \theta), \quad y = a(1 - m \cos \theta).$$

The cycloid is prolate when  $m$  is  $> 1$  (Fig. 116), and curtate when  $m$  is  $< 1$  (Fig. 117),  $m$  being the ratio of  $CP$  to the radius  $a$ .

#### EPITROCHOIDS AND HYPOTROCHOIDS. (Fig. 118)

**5262** These curves are the roulettes formed by a circle rolling upon the convex or concave circumference respectively of a fixed circle, and carrying a generating point either within or without the rolling circle.

The equations of the epitrochoid are

$$\mathbf{5263} \quad x = (a+b) \cos \theta - mb \cos \frac{a+b}{b} \theta,$$

$$\mathbf{5264} \quad y = (a+b) \sin \theta - mb \sin \frac{a+b}{b} \theta,$$

\* For other properties, see Pascal, *Histoire de la Roulette*; Carlo Dati, *History of the Cycloid*; Wallis, *Traité de Cycloïdes*; Groningius, *Historia Cycloidis, Bibliotheca Univ.*; and Lalouère, *Geometria promota in septem de Cycloïde libris*; Bernoulli, *Op.*, Vol. iv., p. 98; Euler, *Comm. Pet.*, 1766; and Legendre, *Exercice du Calcul. Int.*, Tom. II, p. 491.

where  $a, b$  are the radii of the fixed and rolling circle (Fig. 118),  $\theta$  is the angle  $OCX$ ,  $Q$  is the generating point initially in contact with the  $x$  axis, and  $m$  is the ratio  $OQ : b$ . The dotted line shows the curve described. For the hypotrochoid change the sign of  $b$ .

PROOF:  $x = CN + MQ$ ;  $CN = (a + b) \cos \theta$ ;

$MQ = OQ \cos OQM = -OQ \cos (\phi + \theta)$ , where  $\phi = POR$  and  $b\phi = a\theta$ .

**5265** The length of the arc of an epitrochoid is

$$s = (a + b) \int \left\{ 1 + m^2 - 2m \cos \frac{a\theta}{b} \right\}^{\frac{1}{2}} d\theta,$$

which is expressed as an elliptic integral  $E(k, \phi)$  by substituting  $a\theta = 2b\phi$ .

For the arc of a hypotrochoid, change the sign of  $b$ .

PROOF:  $s = \int s, d\theta = \int \sqrt{(x'^2 + y'^2)} d\theta$  (5113). Find  $x,$  and  $y,$  from (5263-4).

#### EPICYCLOIDS AND HYPOCYCLOIDS. (Fig. 118)

**5266** For the equations of these curves make  $m = 1$ , in (5263, '4).  $P$  is then the generating point, and the curve is shown by a solid line in Figure (118).\*

**5267** If  $\psi$  be the inclination of the tangent at a point  $P$  on any of these curves,

$$\tan \psi = - \frac{\cos \theta - m \cos \frac{a+b}{b} \theta}{\sin \theta - m \sin \frac{a+b}{b} \theta} = \tan \frac{a+2b}{2b} \theta, \text{ if } m = 1.$$

**5268** Hence, in the epicycloid,  $\psi = \frac{a+2b}{2b} \theta$ ,

and the equation of the tangent is

$$x \sin \frac{a+2b}{2b} \theta - y \cos \frac{a+2b}{2b} \theta = (a+2b) \sin \frac{a}{2b} \theta.$$

**5269** The equation of the normal will be

$$x \cos \frac{a+2b}{2b} \theta + y \sin \frac{a+2b}{2b} \theta = a \cos \frac{a}{2b} \theta.$$

\* Prof. Wolstenholme has investigated these curves considered as the envelopes of a chord whose extremities move on a fixed circle with uniform velocities in the ratio  $m : n$  or  $m : (-n)$ .—*Proc. Lond. Math. Soc.*, Vol. IV., p. 321.



**5270** The length of the arc of an epicycloid or hypocycloid included between two successive cusps is

$$\frac{8b}{a}(a \pm b), \text{ and the included area is } \frac{\pi b^3}{a}(3a \pm 2b).$$

PROOF.—Putting  $m = 1$  into (5265) and  $a\theta = b\phi$ , the length becomes

$$\frac{2b}{a}(a \pm b) \int_0^{2\pi} \sin \frac{\phi}{2} d\phi = \frac{8b}{a}(a \pm b).$$

Otherwise by (5234); the pedal being the cardioid whose perimeter  $= 8a$  (5333).

(ii.) The area, by (5232), is  $\pi b^2 + \frac{1}{2} \int_0^{2\pi} 4b^2 \sin^2 \frac{\phi}{2} \left(1 + \frac{b}{a}\right) d\phi$ ; since, in Fig. (118),  $d\psi$  of (5232)  $= dPOR = d\phi$  and  $r = PR = 2b \sin \frac{\phi}{2}$ .

**5271** The evolute of an epicycloid is a similar epicycloid.

PROOF.—The equation of the tangent referred to an  $x$  axis drawn through the *summit* of the curve will be (by turning axes through an angle  $b\pi \div a$ ),

$$x \cos \frac{a+2b}{2b} \theta + y \sin \frac{a+2b}{2b} \theta = (a+2b) \cos \frac{a}{2b} \theta.$$

Comparing this with (5270), which is the equation of the tangent of the evolute, we see that the epicycloid and its evolute are similar curves having their parameters in the ratio  $a+2b : a$ ; and that the radius drawn through a cusp of either of the curves passes through a summit of the other.

**5272** When  $b = -\frac{1}{2}a$ , the hypocycloid becomes a straight line, namely, a diameter of the fixed circle.

#### THE CATENARY.

(Fig. 119)

**5273** *Characteristic.*—The perpendicular  $TP$  from the foot of the ordinate upon the tangent is of a constant length  $c$ , and therefore equal to  $OA$ , the perpendicular from the origin on the tangent at the vertex.  $c$  is the parameter of the curve. The equation is

**5274** 
$$y = \frac{c}{2}(e^{\frac{x}{c}} + e^{-\frac{x}{c}}).$$

PROOF:  $\tan PCT = \frac{dx}{dy} = \frac{c}{\sqrt{(y^2 - c^2)}}$ ,  $\therefore x = c \{ \log(y + \sqrt{y^2 - c^2}) - \log c \}$  (1928), since  $x = 0$  when  $y = c$ . Therefore

$$e^{\frac{x}{c}} = \frac{1}{c} \{y + \sqrt{(y^2 - c^2)}\} \quad \text{therefore} \quad e^{-\frac{x}{c}} = \frac{1}{c} \{y - \sqrt{(y^2 - c^2)}\}.$$

**5275** If  $s \equiv \text{arc } AC$ ,  $s = \frac{c}{2}(e^{\frac{x}{c}} - e^{-\frac{x}{c}}) = CP$ .

PROOF:  $s = \int \sqrt{1+y_x^2} dx$  (5197)  
 $= \int \sqrt{1 + \frac{y^2 - c^2}{c^2}} dx = \int \frac{y}{c} dx = \frac{c}{2}(e^{\frac{x}{c}} - e^{-\frac{x}{c}}) = \sqrt{y^2 - c^2} = CP$ .

**5276** The area  $OACT = cs$ . (5205)

**5277** The radius of curvature at  $C = \frac{y^2}{c}$ , and is therefore equal to the tangent intercepted by the axis of  $x$ .

PROOF:  $\cos \psi = \frac{c}{y}$ ,  $\therefore -\sin \psi \psi_s = -\frac{c}{y^2} y_n$ ,  $\therefore \rho = s_s = \frac{y^2}{c}$  (5146).

**5278** The catenary derives its name from a chain, which, when suspended from its extremities, takes the form of this curve.

For the equation of the evolute of the catenary, see (5159).

# THE TRACTRIX.

(Fig. 119)

**5279** *Characteristic.*—The length of the tangent intercepted by the  $x$  axis is constant. This curve is the involute of the catenary, being the locus of  $P$  in Figure (119).

The equation of the tractrix is

**5280**  $x = c \log \{c + \sqrt{c^2 - y^2}\} - c \log y - \sqrt{c^2 - y^2}$ .

PROOF.—Let the tangent  $PT = c$ , then the differential equation of the curve is therefore  $yx_y = -\sqrt{c^2 - y^2}$ . Substitute  $z = \sqrt{c^2 - y^2}$ , and integrate by (1937).

**5281** The area included by the four branches  $= \pi c^2$ .

PROOF.—Area  $= 4 \int y dx = -4 \int_0^c \sqrt{c^2 - y^2} dy = \pi c^2$ , by (1933).

# THE SYNTRACTRIX.

**5282** This curve is the locus of a point  $Q$  on the tangent of the tractrix in Fig. (119). Let  $QT$  be equal to a given constant length  $d$ ; then the equation of the syntractrix will be

**5283**  $x = c \log \{d + \sqrt{d^2 - y^2}\} - c \log y - \sqrt{d^2 - y^2}$ .

## THE LOGARITHMIC CURVE.\*

(Fig. 120)

**5284** *Characteristic.*—The subtangent is constant.

The equation of the curve is either

$$\mathbf{5285} \quad y = ae^{\frac{x}{n}}, \quad \text{or} \quad x = n \log \frac{y}{a},$$

where  $n = NT$ , the constant subtangent, and  $a$  is the intercept on the  $y$  axis.**5287** If  $n$  be an even integer,  $y$  may take negative values. The most general form of the equation may perhaps be assumed to be

$$y = e^{\frac{x}{n}} \left( \cos \frac{2r\pi}{n} + i \sin \frac{2r\pi}{n} \right). \dagger$$

## THE EQUIANGULAR SPIRAL.

(Fig. 121)

**5288** *Characteristic.*—The angle  $OPS$  between the tangent and radius is constant. The equation of the curve is either

$$\mathbf{5289} \quad r = ae^{\frac{\theta}{n}} \quad \text{or} \quad \theta = n \log \frac{r}{a}.$$

$$\mathbf{5291} \quad \tan \phi = n, \quad s = r \sec \phi,$$

measuring  $s$  from the pole.

PROOF.—By (5112) and (5200).

**5293** Hence the length of the spiral measured from the pole  $O$  to a point  $P$  (Fig. 121) is equal to  $PS$ , the intercept on the tangent made by the polar subtangent  $OS$ .**5294** The locus of  $S$  is a similar spiral, and is also an involute of the original curve.**5295** The pedal curve, which is the locus of  $Y$ , is also a similar equiangular spiral.PROOF.—The constancy of the angle  $\phi$  makes the figure  $OPYS$  always similar to itself. Therefore  $P$ ,  $Y$ , and  $S$  describe similar curves. Hence, if  $ST$  is the tangent to the locus of  $S$ ,  $OST = \phi = OPS$ ; therefore  $PST$  is a right angle; therefore the locus of  $S$  is an involute of the original spiral. ‡\* Originated by James Gregory, *Geometriae Pars Universalis*, 1668.† See Euler, *Anal. Infin.*, Vol. II., p. 290; Vincent, *Ann. de Gergonne*, Vol. xv., p. 1; Gregory, *Camb. Math. Journal*, Vol. I., pp. 231, 264; Salmon, *Higher Plane Curves*, p. 274.‡ For additional properties, see Bernoulli, *Opera*, p. 497.

THE SPIRAL OF ARCHIMEDES.\* (Fig. 122)

**5296** *Characteristic.*—The distance from the pole is proportional to the angle described. Hence the equation is

**5297**  $r = a\theta$ . Also  $\tan \phi = \theta$ . By (5112).

**5299** The intercept,  $PQ$ , on any radius between two successive convolutions of the spiral, is constant and  $= 2a\pi$ .

**5300** The area swept over by any radius is one third of the corresponding circular sector of that radius.

**5301** This curve is one of the class the general equation of which is

$$r = a\theta^n, \quad \text{with} \quad \tan \phi = \frac{\theta}{n}.$$

THE HYPERBOLIC OR RECIPROCAL SPIRAL. (Fig. 123)

**5302** The equation is  $r = \frac{a}{\theta}$ .

**5303** An asymptote is the line  $y = a$ . (5171)

**5304** The spiral is also an asymptote to itself.

For when the radius is of the first order of smallness, the distance between two successive convolutions is of the second order. Hence the distance to the pole measured along the curve is infinite.

The area between the radiants  $r_1, r_2$  is  $= \frac{1}{2}a(r_1 - r_2)$ .

**5305** The equation of the *Lituus* is  $r = \frac{a}{\sqrt{\theta}}$ .

THE INVOLUTE OF THE CIRCLE. (Fig. 124)

**5306** The equation is

$$\sqrt{(r^2 - a^2)} = a \left( \theta + \cos^{-1} \frac{a}{r} \right).$$

PROOF:  $\phi = OPY = \cos^{-1} \frac{a}{r}$  and  $\sqrt{(r^2 - a^2)} = BP = \text{arc } AB = a(\theta + \phi)$ .

**5307** The pedal of the involute is the spiral of Archimedes.

\* Invented by Conon, B.C. 250.

PROOF.—Let  $r', \theta'$  be the coordinates of  $Y$  on the pedal curve. Then  $r' = BP = \text{arc } AB = a(\theta' + \frac{1}{2}\pi)$ . (See 5297).

**5308** The reciprocal of the involute is the hyperbolic spiral.

PROOF.—(Fig. 124.) Let  $P'$  on  $OY$  correspond to  $P$ , and let  $r', \theta'$  be the polar coordinates of  $P'$ . Then  $r' = OP' = \frac{a}{OY}$ .

But  $OY = BP = \text{arc } AB = a(\theta' + \frac{1}{2}\pi)$ ,  $\therefore r' = \frac{a}{\theta' + \frac{1}{2}\pi}$ . See (5302).

#### THE CISSOID.\*

(Fig. 125)

**5309** *Characteristic.*—A line drawn from the end,  $O$ , of a fixed diameter of a circle to the end,  $Q$ , of any perpendicular ordinate intersects the parallel ordinate equidistant from the centre in a point,  $P$ , whose locus is the cissoid. The equation of the curve is

$$\mathbf{5310} \quad y^2(2a-x) = x^3 \quad \text{and} \quad \frac{dy}{dx} = \frac{(6a-2x)\sqrt{x}}{2\sqrt{(2a-x)^3}}.$$

PROOF.—By similar triangles,  $y : x = \sqrt{(2ax-x^2)} : 2a-x$ . Two mean proportionals between the radius  $a$  and  $CS$  are given by the curve, for it appears that  $a^2 : CT^2 :: CT : CS$ , and therefore  $a : CT : \sqrt{CS \cdot CT} : CS$ .

**5311** The tangent of the circle at  $B$ , the other end of the diameter, is an asymptote to both branches of the cissoid.

**5312** The area between the curve and its asymptote is equal to three times the area of the circle.

PROOF: In  $\int_0^{2a} y dx$  substitute  $x = 2a \sin^2 \theta$ .

#### THE CASSINIAN OR OVAL OF CASSINI. (Fig. 126)

**5313** *Characteristic.*—The product  $PA \cdot PB$  of the distances of any point on the curve from two fixed points  $A, B$  is constant; the equation is consequently

$$\{y^2 + (a+x)^2\} \{y^2 + (a-x)^2\} = m^4$$

or

$$(x^2 + y^2 + a^2)^2 - 4a^2x^2 = m^4,$$

where  $2a = AB$ . The equation in polar coordinates is

$$r^4 - 2a^2r^2 \cos 2\theta + a^4 - m^4 = 0.$$

\* Diocles, A.D. 500.

**5314** If  $a$  be  $> m$ , there are two ovals, as shown in the figure. In that case, the last equation shows that if  $OPP'$  meets the curve in  $P$  and  $P'$ , we have  $OP \cdot OP' = \sqrt{(a^4 - m^4)}$ ; and therefore the curve is its own inverse with respect to a circle of radius  $= \sqrt[4]{(a^4 - m^4)}$ .

**5315**  $O$  being the centre, the normal  $PG$  makes the same angle with  $PB$  that  $OP$  does with  $PA$ .

PROOF.—From  $(r + dr)(r' - dr') = m^2$  and  $rr' = m^2$ ; therefore  $rdr' = r'dr$  or  $r : r' = dr : dr' = \sin \theta : \sin \theta'$ , if  $\theta, \theta'$  be the angles between the normal and  $r, r'$ . But  $OP$  divides  $APB$  in a similar way in reverse order.

**5316** Let  $OP = R$ , then the normal  $PG$ , and the radius of curvature at  $P$ , are respectively equal to

$$\frac{m^2 R}{R^2 + a^2} \quad \text{and} \quad \frac{2m^2 R^3}{3R^4 + a^4 + m^4}.*$$

#### THE LEMNISCATE. †

(Fig. 126)

**5317** *Characteristic.*—This curve is what a Cassinian becomes when  $m = a$ . The above equations then reduce to

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2) \quad \text{and} \quad r^2 = 2a^2 \cos 2\theta.$$

**5318** The lemniscate is the pedal of the rectangular hyperbola, the centre being the pole.

**5319** The area of each loop  $= a^2$ . (5206)

#### THE CONCHOID. ‡

(Fig. 127)

**5320** *Characteristic.*—If a radiant from a fixed point  $O$  intersects a fixed right line, the directrix, in  $R$ , and a constant length,  $RP = b$ , be measured in either direction along the radiant, the locus of  $P$  is a *conchoid*. If  $OB \equiv a$ , be the perpendicular from  $O$  upon the directrix, the equation of the curve with  $B$  for the origin or  $O$  for the pole is

**5321**  $x^2 y^2 = (a + y)^2 (b^2 - y^2)$  or  $r = a \sec \theta \pm b$ .

\* B. Williamson, M.A., *Educ. Times Math.*, Vol. xxv., p. 81.

† Bernoulli, *Opera*, p. 609.

‡ Nicomedes, about A.D. 100.



**5323** When  $a < b$ , there is a loop; when  $a = b$ , a cusp; and when  $a > b$ , there are two points of inflexion.

**5324** To draw the normal at any point of the curve, erect perpendiculars, at  $R$  to the directrix, and at  $O$  to  $OP$ . They will meet in  $S$  the instantaneous centre, and  $SP$  will be the normal at  $P$  (5242).

**5325** To trisect a given angle  $BON$  by means of this curve, make  $AB = 2ON$ , and draw the conchoid, thus determining  $Q$ ; then  $\angle ON = 3\angle OQ$ .

PROOF.—Bisect  $QT$  in  $S$ ;  $QT = AB = 2ON$ , therefore  $SN = SQ = ON$ ; therefore  $\angle NOS = \angle NSO = 2\angle NQO = 2\angle AOQ$ .

**5326** The total area of the conchoid between two radiants each making an angle  $\theta$  with  $OA$  is

$$a^3 \tan \theta + 2b^3 \theta + 3a \sqrt{(b^2 - a^2)} \quad \text{or} \quad a^3 \tan \theta + 2b^3 \theta,$$

according as  $b$  is or is not  $> a$ .

$$\left. \begin{array}{l} \text{The area above the directrix} \\ \text{between the same radiants} \end{array} \right\} = 2ab \log \tan \left( \frac{\pi}{4} + \frac{\theta}{2} \right) + b^3 \theta.$$

The area of the loop which exists when  $b$  is  $> a$  is

$$b^3 \cos^{-1} \frac{a}{b} - 2ab \log \frac{a + \sqrt{(b^2 - a^2)}}{a - \sqrt{(b^2 - a^2)}} + a \sqrt{(b^2 - a^2)}.$$

### THE LIMAÇON.\*

(Fig. 128)

**5327** *Characteristic.*—As in the conchoid, if, instead of the fixed line for directrix, we take a fixed circle upon  $OB$  as diameter. This curve is also the inverse of a conic with respect to the focus. The equation, with  $OB$  for the initial line and axis of  $x$  is

$$\mathbf{5328} \quad r = a \cos \theta \pm b \quad \text{or} \quad (x^2 + y^2 - ax)^2 = b^2 (x^2 + y^2),$$

where  $a = OB$ ,  $b = PQ$ .

**5330** With  $b > a$ ,  $O$  is a conjugate point.

With  $b < a$ ,  $O$  is a node. [For  $m = a$ , see (5332).

**5331** The area  $= \pi (\frac{1}{2}a^2 + b^2)$ .

When  $a = 2b$ , the limaçon has been called the *trisectrix*.

\* Blaise Pascal, 1643.

THE VERSIERA.\*  
(Or Witch of Agnesi.)

(Fig. 130)

**5335** *Characteristic.*—If upon a diameter  $OA$  of a circle as base a rectangle of variable altitude be drawn whose diagonal cuts the circle in  $B$ , the locus of  $P$ , the point in which the perpendicular from  $B$  meets the side parallel to  $OA$ , is the curve in question. Its equation is

$$\mathbf{5336} \quad xy = 2a \sqrt{(2ax - x^2)},$$

where  $a = OC$  the radius.

**5337** There are points of inflexion where  $x = \frac{3}{2}a$ .  
The total area is four times the area of the circle.

THE QUADRATRIX.†

(Fig. 131)

**5338** *Characteristic.*—The curve is the locus of the intersection,  $P$ , of the radius  $OD$  and the ordinate  $QN$ , when these move uniformly, so that  $x : a :: \theta : \frac{1}{2}\pi$ , where  $x = ON$ ,  $a = OA$ , and  $\theta = BOD$ . The equation is

$$y = x \tan \left( \frac{a-x}{a} \cdot \frac{\pi}{2} \right).$$

**5339** The curve effects the quadrature of the circle, for  $OC : OB :: OB : \text{arc } ADB$ .

PROOF:  $OC : OB :: CP : BD$ . But  $CP = x$  in the limit when it is small, therefore  $CP : BD :: a : ADB$ .

**5340** The area enclosed above the  $x$  axis  $= 4a^2\pi^{-1} \log 2$ .

PROOF.—In the integral  $\int x \tan \left( \frac{a-x}{a} \frac{\pi}{2} \right) dx$  substitute  $\pi(a-x) = 2ay$ , and integrate  $\int y \tan y dy$  by parts, using (1940). The integrated terms produce  $\log \cos \frac{1}{2}\pi - \log \cos \frac{1}{2}\pi$  at the limit  $\frac{1}{2}\pi$ , which vanishes though of the form  $\infty - \infty$ . The remaining integral is  $\int \log \cos y dy$ , and will be found at (2635).

THE CARTESIAN OVAL.

(Fig. 134)

**5341** *Characteristic.*—The sum or difference of certain fixed multiples of the distances of a point  $P$  on the curve from two

\* Donna Maria Agnesi, *Instituzioni Analitiche*, 1748, Art. 238. † Dinostratus, 370 B.C.

fixed points  $A, B$ , called the foci, is constant. The equations of the inner and outer ovals are respectively

$$5342 \quad mr_1 + lr_2 = nc_3, \quad mr_1 - lr_2 = nc_3,$$

where  $r_1 = AP$ ,  $r_2 = BP$ ,  $c_3 = AB$ , and  $n > m > l$ .

5343 To draw the curve, put  $\frac{l}{m} = \mu$  and  $\frac{nc_3}{m} = a$ ; therefore  $r_1 \pm \mu r_2 = a$ , where  $a$  is  $> AB$  and  $\mu < 1$  (1). Describe the circle centre  $A$ , and radius  $AR = a$ . Draw any radiant  $AQ$ , and let  $P, Q$  be the points in which it cuts the ovals, then, by (1),

$$5344 \quad PR = \mu PB \quad \text{and} \quad QR = \mu QB \dots \dots \dots (2).$$

Hence, by (932), we can draw the circle which will cut  $AR$  in the required points  $P, Q$ . Thus any number of points on the oval may be found.

5345 By (2) and Euc. vi. 3, it follows that the chord  $RBr$  bisects the angle  $PBQ$ .

Draw  $Ap$  through  $r$ , and let  $PB, QB$  produced meet  $Ar$  in  $p$  and  $q$ . The triangles  $PBR, qBr$  are similar, therefore  $qr = \mu qB$ ; therefore  $q$  is on the inner oval. Similarly  $p$  is on the outer oval. By Euc. vi. B.,  $PB \cdot QB = PR \cdot QR + BR^2$ ; therefore, by (2),  $(1 - \mu^2) PB \cdot QB = BR^2$ . Combining this with  $PB : Bq = BR : Br$ , from similar triangles, we get

$$5346 \quad BQ \cdot Bq = \frac{BR \cdot Br}{1 - \mu^2} = \frac{a^2 - c_3^2}{1 - \mu^2} \dots \dots \dots (3).$$

5347 Draw  $QC$  to make  $\angle BQC = BAq$ ; therefore,  $A, Q, C, q$  being concyclic, we have, by (3),

$$BQ \cdot Bq = AB \cdot BC = \frac{a^2 - c_3^2}{1 - \mu^2} \dots \dots \dots (4).$$

Hence  $C$  can be found if  $a, \mu$ , and the points  $A, B$  are given.  $C$  is the third focus of the ovals, and the equation of either oval may be referred to any two of the three foci.

Putting  $BC = c_1$ ,  $AC = c_2$ ,  $AB = c_3$ , the equation between  $l, m, n$  is obtained from (4) thus:  $c_3 c_1 (1 - \mu^2) = a^2 - c_3^2$ ; therefore  $c_3 (c_3 + c_1) = a^2 + \mu^2 c_1 c_3$ . But  $c_3 + c_1 = c_2$ ,  $a = \frac{nc_3}{m}$ ,  $\mu = \frac{l}{m}$ , and the result is

$$5348 \quad l^2 c_1 + n^2 c_3 = m^2 c_2 \quad \text{or} \quad l^2 BC + m^2 CA + n^2 AB = 0 \dots (5),$$

where  $CA = -AC$ .

Putting  $r_1, r_2, r_3$  for  $PA, PB, PC$ , the equations of the curves are as follows—

Inner Oval.

Outer Oval.

$$5349 \quad mr_1 + lr_2 = nc_3 \dots (6), \quad mr_1 - lr_2 = nc_3 \dots (7),$$

$$5351 \quad nr_1 + lr_3 = mc_2 \dots (8), \quad nr_1 - lr_3 = mc_2 \dots (9),$$

$$5353 \quad mr_3 - nr_2 = lc_1 \dots (10), \quad nr_2 - mr_3 = lc_1 \dots (11).$$

That (6) and (7) are equations of the curve has been shown. To deduce the other four, we have  $\angle APB = \angle qB = \angle ACQ$  (5347); therefore  $ACQ$ ,  $APB$  are similar triangles. But, by (6),  $mAP + lBP = nAB$ , therefore  $mAC + lCQ = nAQ$  or  $nAQ - lCQ = mAC$ , which is equation (9). Again,  $ABQ$ ,  $APC$  are similar. But, by (7),  $mAQ - lBQ = nAB$ ; therefore  $mAC - lCP = nAP$  or  $nAP + lCP = mAC$ , which is equation (8).

Equations (10) and (11) are obtained by taking (6) from (8) and (7) from (9), and employing (5).

$$5355 \quad AP.AQ = AB.AC = \text{constant}.$$

PROOF.—Since  $A, Q, C, q$  are concyclic,  $\angle QCA = \angle qCA = \angle ABB$ ; therefore  $P, Q, C, B$  are concyclic; therefore  $AP.AQ = AB.AC = \text{constant}$  (12).

$$5356 \quad CP.CP' = CA.CB = \text{constant}.$$

PROOF:  $\angle PCB = \angle PQB = \angle Bpq = \angle BCq$ . Hence, if  $CP$  meets the inner oval again in  $P'$ ,  $CBq$ ,  $CBP'$  are similar triangles. Again, because  $\angle BPC = \angle BQC = \angle BAq = \angle BAP'$ , the points  $A, B, P', P$  are concyclic; therefore  $CP.CP' = CA.CB = \text{constant}$ . Q. E. D.

Hence, by making  $P, P'$  coincide, we have the theorem:—

$$5357 \quad \text{The tangent from the external focus to a series of tri-} \\ \text{confocal Cartesians is of constant length, and} = \sqrt{CB.CA}.$$

5358 To draw the tangents to the ovals at  $P$  and  $Q$ . Describe the circle round  $PQCB$ , and produce  $BR$  to meet the circumference in  $T$ ; then  $TP, TQ$  are the normals at  $P$  and  $Q$ .

The proof is obtained from the similar triangles  $TQR, TBQ$ , which show that  $\sin TQA : \sin TQB = l : m$ , by (2), and from differentiating equation (7), which produces  $\frac{dr_1}{ds} : \frac{dr_2}{ds} = l : m$ .\*

5359 The *Semi-cubical parabola*  $y^2 = ax^3$  is the evolute of a parabola (4549). The length of its arc measured from the

$$\text{origin is} \quad s = \frac{8}{27a} \left\{ \left( 1 + \frac{9}{4} ax \right)^{\frac{2}{3}} - 1 \right\}.$$

\* For the length of an arc of a Cartesian oval expressed by Elliptic Functions, see a paper by S. Roberts, M.A., in *Proc. Lond. Math. Soc.*, Vol. v., p. 6.

**5360** The *Folium of Descartes*,  $x^3 - 3axy + y^3 = 0$ , has two infinite branches, and the asymptote  $x + y + a = 0$ .

For the lengths of arcs and for areas of conics, see (6015), *et seq.*

## LINKAGES AND LINKWORK.

**5400** A *plane linkage*, in its extended sense, consists of a series of triangles in the same plane connected by hinges, so as to have but one degree of freedom of motion; that is, if any two points of the figure be fixed, and a third point be made to move in some path, every other point of the figure will, in general, also describe a definite path. With two points actually fixed, the linkage is commonly called a *piece-work*, and if straight bars take the place of the triangles, it is called a *link-work*.

### THE FIVE-BAR LINKAGE.

**5401** Mr. Kempe's fundamental five-bar linkage is shown in Figure (135).  $A, B, D'$  are fixed pivots indicated by small circles.  $C, D, B', C'$ , in the same plane, are moveable pivots indicated by dots. The lengths of the bars  $AB, BC, CD, DA$  are denoted by  $a, b, c, d$ . The lengths of  $AB', B'C', C'D', D'A$  are proportional to the former, and are equal to  $ka, kb, kc, kd$ , respectively. Hence  $ABCD, AB'C'D'$  are similar quadrilaterals, and  $\angle AD'C' = ADC$ .  $P$  being any assigned point on  $BC$  and  $BP = \lambda$ ,  $P'$  must be taken on  $D'C'$  so that  $D'P' = \lambda \frac{cd}{ab}$ . Draw  $PN, P'N'$  perpendiculars to  $AB$ . Then, throughout the motion of the linkage in one plane,  $NN'$  is a constant length.

PROOF:  $NN' = BD' - (BN + N'D')$ . But  $BD' = a - kd$ , and

$$\begin{aligned} BN + N'D' &= \lambda \cos B - \lambda \frac{cd}{ab} \cos D = \frac{\lambda}{2ab} (2ab \cos B - 2cd \cos D) \\ &= \frac{\lambda}{2ab} (a^2 + b^2 - c^2 - d^2) \text{ (702). Hence} \end{aligned}$$



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**5402**  $NN' = a - kd - \frac{\lambda}{2ab}(a^2 + b^2 - c^2 - d^2).$

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**5403** CASE I.—(Fig. 136.) If  $\lambda = \frac{(a - kd)ab}{a^2 + b^2 - c^2 - d^2}$ , then  $NN' = \frac{BD'}{2}$ ; consequently, if the bars  $PO = BB$  and  $P'O = P'D'$  be added, the point  $O$  will move in the line  $AB$ .

If, in this case,  $d = ka$  and  $b = c$ , then  $\lambda = b$  and  $P$  coincides with  $C$ ,  $P'$  with  $C'$ , and  $B'$  with  $D$ ,  $O$  as before moving in the line  $AB$ .

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**5404** CASE II.—(Fig. 137.) If, in Case I.,  $kd = a$  and  $a^2 + b^2 = c^2 + d^2$ ,  $\lambda$  is indeterminate; that is,  $P$  may then be taken anywhere on  $BC$ .  $D'$  coincides with  $B$ , and  $NN' = 0$ .

$PP'$  is now always perpendicular to  $AB$ . If the bars  $PO$ ,  $P'O$  be added, of lengths such that  $PO^2 - P'O^2 = PB^2 - P'B^2$ ,  $O$  will move in the line  $AB$ . If, on the other side of  $PP'$ , bars  $PO' = P'B$  and  $P'O' = PB$  be attached, then  $O'$  will move in a perpendicular to  $AB$  through  $B$ .

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**5405** CASE III.—(Fig. 138.) If, in Case I.,  $kd = a$ ,  $b = d$ , and  $c = -a$ , the figure  $ABCD$  is termed a contra-parallelogram.  $BP = \lambda$  is indeterminate,  $BC' = kc = -\frac{a^2}{d}$  and  $BP' = -\lambda$ .

Hence  $BC'$  and  $BP'$  are measured in a reversed direction;  $PP'$  is always perpendicular to  $AB$ , and if any two equal bars  $PO$ ,  $P'O$  are added,  $O$  will move in the line  $AB$ .

**5406** If three or more similar contra-parallelograms be added to the linkage in this way, as in Figure (139), having the common pivot  $B$  and the bars  $BA$ ,  $BC$ ,  $BE$ ,  $BG$  in geometrical progression; then, if the bars  $BA$ ,  $BG$  are set to any angle, the other bars will divide that angle into three or more equal parts.

**5407** If, in Figure (138),  $AD$  be fixed and  $DC$  describe an angle  $ADC$ , then  $B'C'$  describes an equal angle in the opposite direction. Mr. Kempe terms such an arrangement a *reversor*, and the linkage in Figure (139) a *multiplicator*. With the aid





of these, and with a *translator* (Fig. 140), for moving a bar  $AB$  anywhere parallel to itself, he shows that any plane curve of the  $n^{\text{th}}$  degree may, theoretically, be constructed by link-work.\*

**5408** CASE IV.—(Fig. 141.) If, in the original linkage (Fig. 135)  $kd = a$ ,  $D'$  coincides with  $B$ . Then, if the bars  $RPO$ ,  $RP'O'$  be added by pivots at  $P$ ,  $P'$ , and  $R$ ; and if  $OP = PR = BP'$  and  $O'P' = P'R = BP$ ; the points  $O$ ,  $O'$  will move in perpendiculars to  $AB$ . For by projecting the equal lines upon  $AB$ , we get  $NL = BN'$  and  $BN = N'L'$ , therefore  $BL = BL' = NN' = \text{a constant}$ , by (5402).

**5409** CASE V.—(Fig. 142.) Make  $ka = d$  and  $\lambda = b$ . Then  $B'$  coincides with  $D$ ,  $P$  with  $C$ , and  $P'$  with  $C'$ . Replace  $D'C'$ ,  $C'D$  by the bars  $DK$ ,  $KD'$  equal and parallel to the former. Also add the bars  $CO = DK$  and  $OK = CD$ . Draw the perpendiculars from  $O$ ,  $C$  and  $C'$  to  $AB$ . Then by projection,  $NL = N'D'$ ; therefore  $BL = BN + NL = BN + N'D' = BD' - NN' = \text{constant}$ . Hence the point  $O$  will move perpendicularly to  $AB$ .

**5410** CASE VI.—(Fig. 143.) In the last case take  $k = 1$ . Therefore  $d = a$ ,  $D'$  coincides with  $B$ ,  $BK = BC$ , and  $CDKO$  is a rhombus. This is Peaucellier's linkage.

**5411** CASE VII.—(Fig. 144.) In the fundamental linkage (Fig. 135), transfer the fixed pivots from  $A$ ,  $B$  to  $P$ ,  $S$ , adding the bar  $SA$ , so that  $PBSA$  shall be a parallelogram. Then, since  $NN'$  is constant (5402), the point  $P'$  will move perpendicularly to the fixed line  $PS$ .

**5412** Join  $AC$  cutting  $PS$  in  $U$ , and draw  $UV$  parallel to  $AD$ . Then  $UV : AD = PU : AB = CP : CB = \text{constant}$ ; therefore  $PU$  and  $UV$  are constant lengths. Hence it follows that the parallelism of  $AB$  to itself may be secured by a fixed pivot at  $U$  and a bar  $UV$  instead of the pivot  $S$  and bar  $SA$ .

**5413** In Case VII. (Fig. 144), with fixed pivots  $P$  and  $S$

\* *Proc. of the Lond. Math. Soc.*, Vol. VII., p. 213.

and bar  $SA$ , make  $b = a$ ,  $d = c$ ,  $ka = d$ ,  $\lambda = b$ . Then  $B'$  coincides with  $D$ ,  $N'$  with  $N$ ,  $P$  with  $C$  and  $L$ , and  $P'$  with  $C'$ ; and we have Figure 145.  $DC$ ,  $DC'$  are equal, and they are equally inclined to  $AB$  or  $CS$ ; because, in similar quadrilaterals, it is obvious that  $AB$  and  $CD$  and the homologous sides  $DC'$  and  $AD'$  include equal angles. Therefore  $CC'$  is perpendicular to  $CS$ , and  $C'$  moves in that perpendicular only.

**5414** If two equal linkages like that in (5413), Figure (145), but with the bars  $AS$ ,  $CS$  removed, be joined at  $D$  (Fig. 146) and constructed so that  $CD\gamma'$ ,  $\gamma DC'$  form two rigid bars, then  $AB$ ,  $\alpha\beta$  will always be in one straight line. Let  $A$ ,  $B$  be made fixed pivots, then, while  $C$  describes a circle, the motion of the bar  $\alpha\beta$  will be that of a carpenter's plane.

**5415** On the other hand, if the linkage of Figure (145), with  $AS$  and  $CS$  removed as before, be united to a similar inverted linkage (Fig. 147), with  $DC$ ,  $DC'$  common, then, with fixed pivots  $A$ ,  $B$ ,  $D'$ , the motion of the bar  $\alpha\beta$  will be that of a lift, directly to and from  $AB$ .

**5416** The crossing of the links may be obviated by the arrangement in Figure (148). Here the bars  $C'\beta$ ,  $C'D$ ,  $C'D'$  are removed, and the bars  $FD$ ,  $FE$ ,  $FG$  added in parallel ruler fashion.

**5417** CASE VIII.—(Fig. 149.) In Case VII., substitute the pivot  $U$  and the bar  $UV$  for  $S$  and  $SA$ . Make  $d = a$ , and therefore  $k = 1$ . Then  $b' = b$  and  $c' = c$ , making  $BCDC'$  a contra-parallelogram;  $D'$  coincides with  $B$ , and  $B'$  with  $D$ . The bars  $AB$ ,  $AD$  are now superfluous. Take  $BP = \lambda$ ; then  $BP' = \lambda \frac{c}{b}$ ; therefore  $PP'$  is parallel to  $CC'$ , therefore to  $BD$ , therefore to  $PV$  (5412); therefore  $V$ ,  $P$ ,  $P'$  are always in one right line.  $P'$ , as in Case VII., moves perpendicularly to  $PU$  and  $AB$ . This arrangement is Hart's *five-bar linkage*.

**5418** When a point  $P$  (Fig. 152) moves in a right line  $PS$ , it is easy to connect to  $P$  a linkage which will make another point move in any other given line we please in the same

plane. Let  $QR$  be such a line cutting  $PS$  in  $Q$ . Make  $Q$  a fixed pivot, and let  $OQ$ ,  $OP$ ,  $OR$  be equal bars on a free pivot  $O$ . Then, if the angle  $POR$  be kept constant by the tie-bar  $PR$ ,  $PQR$ , being one half of  $POR$  (Euc. III. 21), will also be constant, and therefore, while  $P$  describes one line,  $R$  describes the other.

If the bar  $PO$  carries a plane along with it, every point in that plane on the circumference of the circle  $PQR$  will move in a right line passing through  $Q$ .

#### THE SIX-BAR INVERTOR.\*

**5419** If in the linkwork (5410, Fig. 143) the bar  $AD$  be removed, and  $D$  be made to describe any curve,  $O$  will describe the inverse curve, just as, when  $D$  described a circle,  $O$  moved in a right line which is the inverse of a circle.

PROOF.—Let  $BOD$  and  $CK$  intersect in  $E$ . Then  $BO \cdot OD = BE^2 - OE^2 = BC^2 - OC^2 =$  a constant called the *modulus* of the cell.

#### THE EIGHT-BAR DOUBLE INVERTOR.

**5420** Two jointed rhombi (Fig. 150) having a common diameter  $AB$  form a double *Peaucellier cell* termed positive or negative according as  $P$  or  $Q$  is made the fulcrum. We have  $PQ \cdot PR = PQ \cdot QS = AP^2 - AQ^2$ , the constant modulus of the cell.

#### THE FOUR-BAR DOUBLE INVERTOR.

**5421** If, on the bars of a contra-parallelogram  $ABCD$  (Fig. 151) four points  $p, q, r, s$  be taken in a line parallel to  $AC$  or  $BD$ , then in every deformation of the linkage, the points  $p, q, r, s$  will lie in a right line parallel to  $AC$ ; and  $pq \cdot pr = pq \cdot qs =$  a constant modulus. Thus, if  $p$  be a fulcrum and  $r$  describes a curve,  $q$  will describe the inverse curve. If  $q$  be the fulcrum,  $p$  will describe the inverse curve.

PROOF.—Let  $Ap = mAB$ , therefore  $pq = mBD$ , and  $pr = (1-m)AC$ , therefore  $pq \cdot pr = m(1-m)AC \cdot BD = m(1-m)(AD^2 - AB^2) =$  constant.

\* Since the curve described is the inverse and not the polar reciprocal of the guiding curve, it seems better to call this linkage an *invertor* rather than a *reciprocator*.

## THE QUADRUPLANE, OR VERSOR INVERTOR.

**5422** Let the bars of the contra-parallelogram inverter (5421, Fig. 151) carry planes, and let  $P, Q, R, S$  be points in the planes similarly situated with respect to the bars which contain  $p, q, r, s$  respectively, so that  $\angle PAp = QAq$  and  $AP : Ap = AQ : Aq$ ; and similarly at  $C$ . Then, if  $P$  be the fulcrum and  $R$  traces a curve,  $Q$  will trace the inverse curve and the angle  $QPR$  will be constant.

PROOF.—Let  $PA = nAB$  and  $PB = n'AB$ , therefore, by similar triangles,  $PAQ, BAD, PQ = nBD$ . Also, by the triangles  $PBR, ABC, PR = n'AC$ ; therefore  $PQ.PR = mn'AC.BD = \frac{PA.PB}{AB^2}(AD^2 - AB^2)$ , a constant.

Again, the inclination of  $PQ$  to  $BD =$  that of  $AP$  to  $AB$ , which is constant. Similarly, by the triangles  $PBR, ABC$ , the inclination of  $PR$  to  $AC =$  that of  $BR$  to  $BC$ , which is also constant; therefore  $QPR$ , the sum of these two inclinations, is a constant angle.

## THE PENTOGRAPH, OR PROPORTIONATOR.

**5423** Let  $ABCD$  (Fig. 153) be a jointed parallelogram,  $A, B$  fixed pivots,  $q$  a tracer placed at any assigned point in  $BC$  produced; then a pencil at  $p$  will evidently reproduce any figure traced by  $q$  diminished in linear proportions in the ratio of  $Bq$  to  $BC$ .

## THE PLAGIOGRAPH, OR VERSOR PROPORTIONATOR.

**5424** In the same figure, make an angle  $qBQ = pDP$ ,  $BQ = Bq$ , and  $DP = Dp$ , and let a tracer  $Q$  and pencil  $P$  be rigidly connected to the arms  $BC$  and  $DC$ . Then  $P$  will produce a similar reduced figure as before, but no longer similarly situated. It will be turned round through an angle  $QBq$ . This is Prof. Sylvester's *Plagiograph*.

PROOF.—Let  $BC = k.Bq$ ; therefore  $AD = kBQ, DP = kAB$ , and  $\angle ABQ = PDA$ ; therefore (Enc. vi. 6)  $AP = kAQ$ . Also  $PAQ$  is a constant angle, for  $PAQ = BAD - BAQ - PAD = BAD - BAQ - BQA = BAD - (\pi - ABQ) = BAD - \pi + ABC + QBq = QBq$ .

## THE ISOKLINOSTAT,\* OR ANGLE-DIVIDER.

**5425** This linkage (Fig. 154) accomplishes the division of an angle into any desired number of equal parts. The dia-

\* Invented and so named by Prof. Sylvester.

gram shows the trisection of an angle by it. A number of equal bars are hinged together end to end, and also pivoted on their centres to the same number of equal bars which radiate, fan-like, from a common pivot. The alternate radial bars make equal angles with each other.

The same thing is accomplished in a different way by Kempe's Multiplier (5406, Fig. 139).

#### A LINKAGE FOR DRAWING AN ELLIPSE.

**5426** In the arrangement of (5413, Fig. 145) the locus of any point  $P$ , on  $DC'$ , excepting  $D$  and  $C'$ , is an ellipse.

PROOF.—Take  $CS, CC'$  for  $x$  and  $y$  axes;  $P$  the point  $xy$ ;  $SCD = \theta$ , and therefore  $CDC' = 2\theta$ ;  $PD = h$ . Then we have  $x = (c-h)\cos\theta$ ,  $y = (c+h)\sin\theta$ , therefore  $\frac{x^2}{(c-h)^2} + \frac{y^2}{(c+h)^2} = 1$  is the equation of the locus.

Any point on a plane carried by  $DC'$  also describes an ellipse round  $C$ ; but if the point lies on a circle whose centre is  $D$  and radius  $DC$ , the ellipse becomes a right line passing through  $C$ , as appears from (5418).

#### A LINKAGE FOR DRAWING A LIMAÇON, AND ALSO A BICIRCULAR QUARTIC.\*

**5427** (Fig. 155.) Let four bars  $AP', AQ', BC, CD$  be pivoted at  $A, B, C, D$ , and let  $AB = BC = BQ' = a$ ;  $AD = DC = DP' = b$ . Take a fulcrum  $F$  on  $BC$ , a tracer at  $P$ , and a follower at  $Q$ , so that  $PQ$  is parallel to  $BD$ . Let  $FP = \rho$ ,  $FQ = r$ ; then, if  $P$  traces out a circle passing through  $F$ ,  $Q$  will describe a limaçon.

PROOF.—Let  $BQ = ma$ , therefore  $PD = mb$ ;  $r = 2m \cdot BN$ ,  $\rho = (m+1) \cdot DN + (1-m) \cdot BN$ . Also  $BN^2 - DN^2 = a^2 - b^2$ . Eliminate  $BN$  and  $DN$ , and the equation between  $r$  and  $\rho$  is

$$r^2 + (1-m) \rho^2 - m\rho^2 = m(m+1)^2(a^2 - b^2) = k^2.$$

If  $P$  describes the circle  $\rho = c \cos \theta$ ,  $Q$  describes the locus

$$r^2 + (1-m) c^2 \cos^2 \theta - mc^2 \cos^2 \theta = k^2,$$

which is the inverse of a conic, that is, a limaçon (5327).

If  $C$  be made the fulcrum, the equation reduces to  $r^2 - \rho^2 = 4(a^2 - b^2)$ .

**5428** With the same fulcrum  $F$ , drawing  $FH$  parallel to  $AC$ , if a tracer at  $H$  describes the circle, then a follower at  $K$  on  $CD$  will trace out a bicircular quartic.

\* W. Woolsey Johnson, *Mess. of Math.*, Vol. v., p. 159.



PROOF.—Draw  $FL, LK$  parallel to  $BA, AD$ . Let  $FH = \rho, FK = r, CK = \beta, CF = a = nFB$ , and therefore  $CL = n\rho$ . Now

$$2(a^2 + \beta^2) = r^2 + n^2\rho^2 + \frac{(a^2 - \beta^2)^2}{r^2}.$$

Therefore, if  $H$  moves on the circle  $\rho = c \cos \theta$ ,  $K$  will describe the curve

$$r^4 + n^2 c^2 r^2 \cos^2 \theta - 2(a^2 + \beta^2) r^2 + (a^2 - \beta^2)^2 = 0,$$

or 
$$(x^2 + y^2)^2 + (n^2 c^2 - 2a^2 - 2\beta^2) x^2 - 2(a^2 + \beta^2) y^2 + (a^2 - \beta^2)^2 = 0.$$

#### A LINKAGE FOR SOLVING A CUBIC EQUATION.\*

**5429** Let the three-bar linkwork (Fig. 156) have the bars  $AB, DC$  produced to cross each other. Let  $AB = AD = a, BC = b, CD = c$ ; and let  $b$  and  $c$  be adjustable lengths.

Suppose  $x^3 - qx + r = 0$  a given cubic equation.

Make  $c = \frac{1}{2} \sqrt{\left(q + \frac{r}{a}\right)}, b = \frac{1}{2} \sqrt{\left(q - \frac{r}{a}\right)}$ ; then deform the quadrilateral until  $EC = CD$ ;  $DE$  will then be equal to a real root of the cubic.

PROOF: 
$$\cos E = \frac{x^2 + c^2 - b^2}{2cx} = \frac{(x+a)^2 + 4c^2 - a^2}{4c(x+a)},$$

from which 
$$x^3 - 2(c^2 + b^2)x + 2a(c^2 - b^2) = 0.$$

Equate coefficients with the given cubic.†

#### ON THREE-BAR MOTION IN A PLANE.

**5430** If a triangle  $ABC$  (Fig. 157) be connected by the bars  $AO, BO'$  to the fulcra  $O, O'$ , the locus of  $C$  is called a three-bar curve.

$OA, O'B$  meet in  $Q$ , the instantaneous centre of rotation of the triangle, since  $QA, QB$  are perpendicular to the movements of  $A$  and  $B$  respectively. Therefore  $CQ$  is the normal to the locus of  $C$ .

**5431** If a triangle similar to  $ABC$  be placed upon  $OO'$  (homologous to  $AB$ ), the circum-circle of the triangle will pass through the node, and the vertices of the triangle are called the foci of the curve.

\* M. Saint Loup, *Comptes Rendus*, 1874.

† The foregoing account of linkages is taken chiefly from a paper by A. B. Kempe, F.R.S., in the *Proc. of the Royal Soc.* for 1875, Vol. xxiii. Other results by the same author will be found in the *Proc. of the Lond. Math. Soc.*, Vol. ix., p. 133; and by H. Hart, M.A., *ibid.*, Vol. vi., p. 137, and Vol. viii., p. 286. See also *The Messenger of Mathematics*, Vol. v.



Figures (158) and (159) exhibit different varieties of the curve according to the relative proportions between the lengths of the bars.\*

## MECHANICAL CALCULATORS.

### *The Mechanical Integrator.†*

**5450** This instrument computes not only the area of any closed plane curve, but the moment and also the moment of inertia of the area about a fixed line. The principle of its action is shown in Figure (160).  $OP$  is a bar carrying a tracer at  $P$ , and a roller  $A$  at some point of its length. The end  $O$  is constrained to move in the fixed line  $ON$ . When the tracer  $P$  moves round a closed curve, the length  $OP$  multiplied by the entire advance recorded by the roller is equal to the area of the curve.

PROOF.—Let the motion of the tracer from  $P$  to a consecutive point  $Q$  be decomposed into  $PP'$  and  $P'Q$  parallel and perpendicular to  $ON$ . Let  $OP = a$  and  $PON = \theta$ . When the pointer moves from  $P$  to  $P'$ , the roll accomplished is  $PP' \sin \theta$ . The roll due to the motion from  $P'$  to  $Q$  will be neutralized by the exactly equal and opposite roll in the motion of the pointer from  $q$  to  $p'$ , since the bar will there have again the same inclination. Consequently the product of the entire roll and the length  $a$  is equal to the sum of such terms as  $aPP' \sin \theta$ . But this is the area  $OPP'O' = NPP'N'$ . The algebraic addition of such rectangles gives the entire area, and the instrument effects this, for the area  $SN$  is subtracted, by the motion of the roller, from the area  $QN$  which is added.

**5451** The instrument itself is shown in Figure (161). A frame moving parallel to  $OX$  by means of the guide  $BB$  carries two equal horizontal wheels geared to a central wheel which has two circumferences, such that its rate of angular motion is half that of the lower wheel and one third of that of the upper. The latter wheels carry two rollers,  $M$  and  $I$ , on horizontal axles; and the middle wheel carries an arm  $OP$ , a pointer at  $P$ , and a roller  $A$ . In the initial position, the

\* The curve is a tricircular trinodal sextic, and is completely discussed by S. Roberts, F.R.S., and Prof. Cayley, in the *Proc. of the Lond. Math. Soc.*, Vol. vii., pp. 14, 136.

† Invented and manufactured by Mr. J. Amsler-Laffon, of Schaffhausen. The demonstrations (which in clearness and elegance cannot be surpassed) of the action of this instrument, and of the Planimeter which follows, were communicated to the author by Mr. J. Macfarlane Gray, of the Board of Trade.

rollers  $A$  and  $I$  are parallel, while  $M$  is at right angles to  $A$ . The frame is thus supported above the paper on the three rollers; and if the arm  $OP$  be moved through an angle  $AOA'$ , the axles of the rollers  $M$  and  $I$  will describe twice and three times that angle respectively. Putting  $OP = a$  as above, and  $A$ ,  $M$ , and  $I$  for the linear circumferential advances recorded by the three equal rollers respectively, we have the following results—

I.—The area traced out by the pointer  $P = Aa$ .

II.—The moment of the area about  $OX = M \frac{a^2}{4}$ .

III.—The moment of inertia about  $OX = (3A + I) \frac{a^3}{12}$ .

PROOF.—I. Since  $O$  moves in the line  $OX$ , while the pointer  $P$  moves round a curve, the roller  $A$  will, as shown above, make the rolling  $\Sigma h \sin \theta$ , where  $h = PP'$  in Figure (160), and the area of the curve  $= a \Sigma h \sin \theta$  or  $a \times \text{roll}$ .

II. The moment of the area about  $OX$

$$= \Sigma \left( ah \sin \theta \times \frac{a \sin \theta}{2} \right) = \frac{a^2}{4} (\Sigma h - \Sigma h \cos 2\theta).$$

Now  $\Sigma h$  vanishes when  $P$  returns to the starting point, and  $-\Sigma h \cos 2\theta$  is the roll recorded by  $M$ . For, when  $OP$  makes an angle  $\theta$  with  $OX$ , the axis of  $M$  will make an angle  $-(90^\circ + 2\theta)$  with  $OX$ . In this position, while  $P$  makes a parallel movement  $h$ , the roll produced thereby in  $M$  will be  $-h \sin (90^\circ + 2\theta) = -h \cos 2\theta$ . Therefore  $\frac{a^2}{4} \times \text{roll of } M = \text{moment of area}$ .

III. Lastly, the moment of inertia of the area about  $OX$

$$= \Sigma \left( ah \sin \theta \times \frac{a^2 \sin^2 \theta}{3} \right) = \frac{a^3}{12} \Sigma (3h \sin \theta - h \sin 3\theta).$$

Now, when  $OP$  makes an angle  $\theta$  with  $OX$ , the axis of  $I$  makes  $-3\theta$ ; therefore  $-\Sigma h \sin 3\theta$  is the entire roll of  $I$ . Hence the moment of inertia

$$= \frac{a^3}{4} \times \text{roll of } A + \frac{a^3}{12} \times \text{roll of } I.$$

### The Planimeter.

(Fig. 162)

**5452** This instrument \* is a simpler form of area computer.  $O$  is a fixed pivot;  $OA$ ,  $AP$  are two rods having a free pivot at  $A$ ;  $C$  is the roller, and  $P$  the pointer. The area of a closed curve traced by the pointer is equal to the total roll multiplied by the length  $AP$ .

\* Like *The Integrator*, the invention of Mr. Amaler.

PROOF.—Decompose the elementary motion  $PQ$  of the pointer into  $PP'$ , effected with a constant radius  $OP$ , and  $P'Q$  along the radius  $OP'$ , and so all round the curve. The roll of  $C$  accomplished while  $P$  moves from  $P'$  to  $Q$  will be neutralized by the equal contrary roll when  $P$  moves from  $q$  to  $p'$  on the radius  $Op' = OP$ . Thus the total roll recorded will be the sum of the rolls due to the movements  $PP'$ ,  $QQ'$ , &c.

Draw  $OB$  perpendicular to  $AP$ , and, when  $P$  comes to  $R$ , let  $B'$  be the altered position of  $B$ . The area  $PQSR = \frac{1}{2}(OP^2 - OR^2)\omega$ , where  $\omega = POQ$ . But  $OP^2 = OA^2 + PA^2 - 2PA \cdot AC - 2PA \cdot BC$  (Euc. II. 13); therefore, since  $BC$  is the only varying length on the right, we have  $PQSR = PA(BC - B'C)\omega$ . But  $BC\omega$  is the roll of  $C$  due to the angular motion  $\omega$  of the rigid frame  $OAP$ , and the subtraction of the area  $OSR$  from  $OPQ$  is effected by the instrument, since when the pointer moves from  $S$  to  $R$  the direction of the roll must be reversed. Hence the total area  $= PA \times$  the total recorded roll.

## APPENDIX ON BIANGULAR COORDINATES.\*

**5453** In the figure of (1178), the biangular coordinates of a point  $P$  are defined to be  $\theta = PSS'$  and  $\phi = PS'S$ , or  $\alpha = \cot \theta$  and  $\beta = \cot \phi$ .

**5454** The equation of a right line  $YY'$  is

$$a\alpha + b\beta = 1,$$

where  $a = \cot SYS'$  and  $b = \cot SY'S'$ .

PROOF.—Supplying the ordinate  $PN$  in the figure and denoting the angle  $S'SY$  by  $\psi$ , the equation is obtained from  $CN \cos \psi + PN \sin \psi = p$  the perpendicular on the tangent,  $SS' \sin \psi = YY'$  and  $SS' \cos \psi = SY - S'Y'$ .

**5455**  $\cot \psi = a - b.$

**5456** Equation of a line through  $C$ :  $\alpha - \beta = \text{const.}$

**5457** Equation of the line at infinity:  $\alpha + \beta = 0.$

**5458** Let  $SS' = c$ , then the distance between two points  $\alpha_1\beta_1, \alpha_2\beta_2$  is

$$= c^2 \left\{ \left( \frac{1}{\alpha_1 + \beta_1} - \frac{1}{\alpha_2 + \beta_2} \right)^2 + \left( \frac{\alpha_1}{\alpha_1 + \beta_1} - \frac{\alpha_2}{\alpha_2 + \beta_2} \right)^2 \right\}.$$

\* *Quarterly Journal of Mathematics*, Vols. 9 and 13; W. Walton, M.A.

**5459** The equation of a line through the two points is

$$\frac{a-a_1}{\beta-\beta_1} = \frac{a_1-a_2}{\beta_1-\beta_2}.$$

**5460** The length of the perpendicular from  $a'\beta'$  upon the line  $aa+b\beta=1$  is

$$P = \frac{c}{a'+\beta'} \cdot \frac{aa'+b\beta'-1}{\sqrt{\{(a-b)^2+1\}}}.$$

**5461** Cor.—The perpendiculars from the poles  $S, S'$  are therefore

$$SY = \frac{bc}{\sqrt{\{(a-b)^2+1\}}}, \quad S'Y' = \frac{ac}{\sqrt{\{(a-b)^2+1\}}}.$$

**5463** When the point  $a'\beta'$  is on  $SS'$  at a distance  $h$  from  $S$ ,

$$p = \frac{(a-b)h+bc}{\sqrt{\{(a-b)^2+1\}}}.$$

With two lines  $aa+b\beta=1, a'a+b'\beta=1$ , the condition

**5464** of parallelism is  $a-b=a'-b'$ ,

**5465** of perpendicularity  $(a-b)(a'-b')+1=0$ .

**5466** The equation of the line bisecting the angle between the same lines is

$$\frac{aa+b\beta-1}{\sqrt{\{(a-b)^2+1\}}} = \frac{a'a+b'\beta-1}{\sqrt{\{(a'-b')^2+1\}}}.$$

**5467** The equation of the tangent at a point  $a'\beta'$  on the curve  $F(a, \beta)=0$  is

$$(a-a')F_a + (\beta-\beta')F_{\beta} = 0.$$

**5468** And the equation of the normal is

$$\frac{a-a'}{(a'\beta'-1)F_{\beta'} + (1+a'^2)F_a} = \frac{\beta-\beta'}{(a'\beta'-1)F_a + (1+\beta'^2)F_{\beta'}}.$$

**5469** The equation of a circle through  $S, S'$  is

$$a\beta-1 = m(a+\beta),$$

where  $m = \cot SPS'$  the angle of the segment.

**5470** If  $C$  be the centre, the equation becomes

$$a\beta = 1.$$

**5471** And, in this case, the equations of the tangent and normal at  $a'\beta'$  are respectively

$$\frac{a}{a'} + \frac{\beta}{\beta'} = 2 \quad \text{and} \quad a - \beta = a' - \beta'.$$

**5472** The equation of the radical axis of two circles whose centres are  $S, S'$ , and radii  $a, b$ , is

$$(c^2 - a^2 + b^2) a = (c^2 + a^2 - b^2) \beta.$$

PROOF.—By equating the tangents from  $a\beta$  to the two circles, their lengths being respectively

$$\frac{c^2(1+a^2)}{(a+\beta)^2} - a^2 \quad \text{and} \quad \frac{c^2(1+\beta^2)}{(a+\beta)^2} - b^2, \text{ by (5458).}$$

**5473** To find the equation of the asymptotes of a curve when they exist,—

Eliminate  $a$  and  $\beta$  between the equations of the line at infinity

$$a + \beta = 0,$$

the curve

$$F(a, \beta) = 0,$$

and the tangent  $(a - a') F_a + (\beta - \beta') F_\beta = 0$ .

Ex.—The hyperbola  $a^2 + \beta^2 = m^2$  has, for the equation of its asymptotes,  $a - \beta = \pm m\sqrt{2}$ .

# SOLID COORDINATE GEOMETRY.

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## SYSTEMS OF COORDINATES.

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### CARTESIAN OR THREE-PLANE COORDINATES.

**5501** The position of a point  $P$  in this system (Fig. 168) is determined by its distances,  $x = PA$ ,  $y = PB$ ,  $z = PC$ , from three fixed planes  $YOZ$ ,  $ZOX$ ,  $XOY$ , the distances being measured parallel to the mutual intersections  $OX$ ,  $OY$ ,  $OZ$  of the planes, which intersections constitute the axes of coordinates. The point  $P$  is referred to as the point  $xyz$ , and in the drawing  $x$ ,  $y$ ,  $z$  are all reckoned positive,  $ZOX$  being the plane of the paper and  $P$  being situated in front of it, to the right of  $YOZ$  and above  $XOY$ . If  $P$  be taken on the other side of any of these planes, its coordinate distance from that plane is reckoned negative.

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### FOUR-PLANE COORDINATES.

**5502** In this system the position of a point is determined by four coordinates  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$ , which are its perpendicular distances from four fixed planes constituting a tetrahedron of reference. The system is in Solid Geometry precisely what trilinear coordinates are in Plane. The relation between the coordinates of a point corresponding to (4007) in trilinears is

**5503** 
$$A\alpha + B\beta + C\gamma + D\delta = 3V,$$

where  $A, B, C, D$  are the areas of the faces of the tetrahedron of reference, and  $V$  is its volume.

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### TETRAHEDRAL COORDINATES.

**5504** In this system the coordinates of a point are the volumes of the pyramids of which the point is the vertex and



the faces of the tetrahedron of reference the bases : viz.,  $\frac{1}{3}A\alpha$ ,  $\frac{1}{3}B\beta$ ,  $\frac{1}{3}C\gamma$ ,  $\frac{1}{3}D\delta$ . The relation between them is

$$5505 \quad \alpha' + \beta' + \gamma' + \delta' = V.$$

#### POLAR COORDINATES.

**5506** Let  $O$  be the origin (Fig. 168),  $XOZ$  the plane of reference in rectangular coordinates, then the polar coordinates of a point  $P$  are  $r$ ,  $\theta$ ,  $\phi$ , such that  $r = OP$ ,  $\theta = \angle POZ$ , and  $\phi = \angle XOC$  between the planes of  $XOZ$  and  $POZ$ .

#### THE RIGHT LINE.

**5507** The coordinates of the point dividing in a given ratio the distance between two given points are as in (4032), with a similar value for the third coordinate  $z$ .

**5508** The distance  $P, Q$  between the two points  $xyz$ ,  $x'y'z'$  is

$$PQ = \sqrt{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}}. \text{ (Euc. I. 47).}$$

**5509** The same with oblique axes, the angles between the axes being  $\lambda$ ,  $\mu$ ,  $\nu$ .

$$PQ = \sqrt{\{(x-x')^2 + (y-y')^2 + (z-z')^2 + 2(y-y')(z-z') \cos \lambda + 2(z-z')(x-x') \cos \mu + 2(x-x')(y-y') \cos \nu\}}. \text{ (By702).}$$

**5510** The same in polar coordinates, the given points being  $r\theta\phi$ ,  $r'\theta'\phi'$ ,

$$PQ = \sqrt{[r^2 + r'^2 - 2rr' \{\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi')\}]}$$

PROOF.—Let  $P, Q$  be the points,  $O$  the origin. Describe a sphere cutting  $OP, OQ$  in  $B, C$  and the  $z$  axis in  $A$ ; then, by (702),  $PQ^2 = OP^2 + OQ^2 - 2OP \cdot OQ \cos POQ$  and  $\cos POQ$ , or  $\cos a$  in the spherical triangle  $ABC$ , is given by formula (882), since  $b = \theta$ ,  $c = \theta'$ , and  $A = \phi - \phi'$ .

#### DIRECTION RATIOS.

**5511** Through any point  $Q$  on a right line  $QP$  (Fig. 169), draw lines  $QL, QM, QN$  parallel to the axes, and through any other point  $P$  on the line draw planes parallel to the coordi-

nate planes cutting the lines just drawn in  $L, M, N$ ; then the direction ratios of the line  $OP$  are

$$5512 \quad l = \frac{QL}{QP}, \quad m = \frac{QM}{QP}, \quad n = \frac{QN}{QP}.$$

The angles  $PQL, PQM, PQN$  are denoted by  $\alpha, \beta, \gamma$ ; and the angles  $YOZ, ZOY, XOY$  between the axes by  $\lambda, \mu, \nu$ .

**5513** When  $\lambda, \mu, \nu$  are right angles, the axes are called rectangular, and the direction-ratios are called direction-cosines, being in that case severally equal to  $\cos \alpha, \cos \beta, \cos \gamma$ .

**5514** When  $L, M, N$  are the direction-ratios (or numbers proportional to them) of a line which passes through a point  $abc$ , the line may be referred to as the line  $(LMN, abc)$ , or, if direction only is concerned, merely the line  $LMN$ .

#### EQUATIONS BETWEEN THE CONSTANTS OF A LINE.

**5515** The relation between the constants of a line with rectangular axes is

$$l^2 + m^2 + n^2 = 1;$$

and with oblique axes, it is

$$5516 \quad l \cos \alpha + m \cos \beta + n \cos \gamma = 1.$$

PROOF.—The first by (Euc. I. 47). The second by projecting the bent line  $QLCP$  (Fig. 169) upon  $PQ$ , thus  $PQ = QL \cos \alpha + LC \cos \beta + CP \cos \gamma$ , and  $QL = PQ \cdot l$ , &c., by (5512).

**5517** Also, when the axes are oblique,

$$\begin{aligned} \cos \alpha &= l + m \cos \nu + n \cos \mu, \\ \cos \beta &= m + n \cos \lambda + l \cos \nu, \\ \cos \gamma &= n + l \cos \mu + m \cos \lambda. \end{aligned}$$

PROOF.—By projecting  $QP$  in figure (169) and the bent line  $QLCP$  upon each axis in turn, and equating results; thus  $PQ \cos \alpha = QL + LC \cos \beta + CP \cos \gamma$ , applying (5512).

**5518** The relation between  $l, m, n$  and  $\lambda, \mu, \nu$  is

$$l^2 + m^2 + n^2 + 2mn \cos \lambda + 2nl \cos \mu + 2lm \cos \nu = 1.$$

PROOF.—By eliminating  $\cos \alpha, \cos \beta, \cos \gamma$  between (5516) and (5517).

**5519** The relation between  $\cos \alpha$ ,  $\cos \beta$ ,  $\cos \gamma$  and  $\lambda$ ,  $\mu$ ,  $\nu$  is

$$\begin{aligned} & \cos^2 \alpha \sin^2 \lambda + \cos^2 \beta \sin^2 \mu + \cos^2 \gamma \sin^2 \nu \\ & + 2 \cos \beta \cos \gamma (\cos \mu \cos \nu - \cos \lambda) \\ & + 2 \cos \gamma \cos \alpha (\cos \nu \cos \lambda - \cos \mu) \\ & + 2 \cos \alpha \cos \beta (\cos \lambda \cos \mu - \cos \nu) \\ & = 1 - \cos^2 \lambda - \cos^2 \mu - \cos^2 \nu + 2 \cos \lambda \cos \mu \cos \nu. \end{aligned}$$

PROOF.—By eliminating  $l$ ,  $m$ ,  $n$  between the four equations in (5516) and (5517).

**5520** The angle  $\theta$  between two right lines  $lmn$ ,  $l'm'n'$ , the axes being rectangular :

$$\cos \theta = ll' + mm' + nn'.$$

PROOF.—In Figure (169), let  $QP$  be a segment of the line  $lmn$ . The projection of  $QP$  upon the line  $l'm'n'$  will be  $QP \cos \theta$ . And this will also be equal to the projection of the bent line  $QLCP$ , upon  $l'm'n'$ , for, if planes be drawn through  $Q$ ,  $L$ ,  $C$ , and  $P$ , at right angles to the second line  $l'm'n'$ , the segment on that line intercepted between the first and last plane will be  $= QP \cos \theta$ , and the three segments which compose this will be severally equal to  $QL.l'$ ,  $LC.m'$ ,  $CP.n'$ , the projections of  $QL$ ,  $LC$ ,  $CP$ . Then, by (5512),  $QL = QP.l$ , &c.

$$\mathbf{5521} \quad \sin^2 \theta = (mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2.$$

PROOF.—From

$$1 - \cos^2 \theta = (l^2 + m^2 + n^2)(l'^2 + m'^2 + n'^2) - (ll' + mm' + nn')^2 \quad (5515, '20).$$

**5522** With oblique axes,

$$\begin{aligned} \cos \theta = ll' + mm' + nn' + (mn' + m'n) \cos \lambda \\ + (nl' + n'l) \cos \mu + (lm' + l'm) \cos \nu. \end{aligned}$$

PROOF.—As in (5520), substituting from (5517) the values of  $\cos \alpha$ , &c.

#### EQUATIONS OF THE RIGHT LINE.

$$\mathbf{5523} \quad \frac{x-a}{L} = \frac{y-b}{M} = \frac{z-c}{N} \quad \text{or} \quad \frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}.$$

Here  $abc$  is a datum point on the line, and if  $r$  be put for the value of each of the fractions,  $r$  is the distance to a variable point  $xyz$ .  $L$ ,  $M$ ,  $N$  are proportional to the direction ratios of

the line, which ratios must therefore have the values

$$5524 \quad l = \frac{L}{\sqrt{L^2 + M^2 + N^2}}, \quad m = \frac{M}{\sqrt{L^2 + M^2 + N^2}}, \\ n = \frac{N}{\sqrt{L^2 + M^2 + N^2}}.$$

5525 NOTE.—Instead of  $a, b, c$  in the equation we may use  $kL + a, kM + b, kl + c$ , where  $k$  is an arbitrary constant.

5526 The equations of a line may also be written in the forms  $x = \lambda z + \alpha, \quad y = \mu z + \beta.$

5527 These are the equations of the traces on the planes of  $xz$  and  $yz$ , and are equivalent to

$$\frac{x - \alpha}{\lambda} = \frac{y - \beta}{\mu} = \frac{z - 0}{1}.$$

5528 If the line is determined as the intersection of the two planes  $Ax + By + Cz = D$  and  $A'x + B'y + C'z = D'$ , we may write equations (5523) by taking

$$L = BC' - B'C, \quad M = CA' - C'A, \quad N = AB' - A'B,$$

$$a = \frac{DB' - D'B}{N}, \quad b = \frac{DA' - D'A}{N}, \quad c = 0.$$

PROOF.—Eliminate  $z$  between the equations of the planes, then the reciprocals of the coefficients of  $x$  and  $y$  will be  $L$  and  $M$ .

5529 The projection of the line joining the points  $xyz$  and  $abc$  upon the line  $lmn$  is

$$l(x - a) + m(y - b) + n(z - c).$$

5530 Hence, when the line passes through  $abc$ , the square of the perpendicular from  $xyz$  upon it is equal to

$$(x - a)^2 + (y - b)^2 + (z - c)^2 - \{l(x - a) + m(y - b) + n(z - c)\}^2.$$

5531 Condition of parallelism of two lines  $LMN, L'M'N'$ :

$$L : L' = M : M' = N : N'.$$

**5532** Condition of perpendicularity :

$$LL' + MM' + NN' = 0. \quad (5520)$$

**5533** Condition of the intersection of the lines  $(LMN, abc)$  and  $(L'M'N', a'b'c')$  (5514) :

$$(a-a')(MN'-M'N) + (b-b')(NL'-N'L) + (c-c')(LM'-L'M) = 0.$$

PROOF.—Eliminate  $x, y, z$  between the equations

$$\frac{x-a}{L} = \frac{y-b}{M} = \frac{z-c}{N} = r \quad \text{and} \quad \frac{x-a'}{L'} = \frac{y-b'}{M'} = \frac{z-c'}{N'} = r',$$

by subtracting in pairs, and then eliminate  $r$  and  $r'$ .

**5534** The shortest distance between the same lines is

$$\frac{(a-a')(MN'-M'N) + (b-b')(NL'-N'L) + (c-c')(LM'-L'M)}{\sqrt{\{(MN'-M'N)^2 + (NL'-N'L)^2 + (LM'-L'M)^2\}}}$$

PROOF.—Assume  $\lambda, \mu, \nu$  for the dir-cos. of the shortest distance. Then, by projecting the line joining  $abc, a'b'c'$  upon the shortest distance, we get  $p = (a-a')\lambda + (b-b')\mu + (c-c')\nu$ . Also, by (5520),  $L\lambda + M\mu + N\nu = 0$  and  $L'\lambda + M'\mu + N'\nu = 0$ , giving the ratios  $\lambda : \mu : \nu = MN' - M'N : NL' - N'L : LM' - L'M$ ; and (5524) then gives the values of  $\lambda, \mu, \nu$ .

**5535** The equation of the line of shortest distance between the lines  $(lmn, abc)$  and  $(l'm'n', a'b'c')$  is given by the intersection of the two planes

$$l(x-a) + m(y-b) + n(z-\gamma) = \frac{u+u'\cos\theta}{\sin^2\theta} \dots\dots(i.),$$

$$l'(x-a') + m'(y-b') + n'(z-\gamma') = \frac{u'+u\cos\theta}{\sin^2\theta} \dots(ii.),$$

where  $u = l(a'-a) + m(b'-b) + n(c'-c),$

$$u' = l'(a-a') + m'(b-b') + n'(c-c'),$$

and  $\cos\theta = ll' + mm' + nn'.$

PROOF.—(Fig. 170.) Let  $O$  be the point  $xyz$  on the line of shortest distance  $AB$ ;  $P, Q$  the points  $abc, a'b'c'$  on the given lines  $AP, BQ$ . Draw  $BR$  and  $PR$  parallel to  $AP$  and  $AB$ ;  $RT$  perpendicular to  $BQ$ ; and  $QN, TM$  perpendicular to  $BR$ . Then  $\angle RBQ = \theta$ ,  $RN = u$ ,  $QT = u'$ , therefore  $NM = u'\cos\theta$  and  $RM = RN + NM = u + u'\cos\theta$ , and in the right-angled triangle  $RTB$ ,  $RM \operatorname{cosec}\theta = RB$ , the projection of  $OP$  upon  $AP$ , that is, the

left member of equation (i.). Similarly for equation (ii.). It should be observed that (i.) and (ii.) represent planes through  $AB$  respectively perpendicular to the given lines  $AP$  and  $BQ$ .

**5536** Otherwise, the line of shortest distance is the intersection of the two planes whose equations are

$$\frac{l'(x-a) + m'(y-b) + n'(z-c)}{l(x-a) + m(y-b) + n(z-c)} = \cos \theta$$

$$= \frac{l(x-a') + m(y-b') + n(z-c')}{l'(x-a') + m'(y-b') + n'(z-c')}.$$

For these equations state that  $\cos \theta$  is the ratio of the projections of  $OP$  or of  $OQ$  upon the given lines, and this fact is apparent from the figure.

**5537** Equations of the line passing through the two points  $abc, a'b'c'$ :

$$\frac{x-a}{a-a'} = \frac{y-b}{b-b'} = \frac{z-c}{c-c'}.$$

**5538** A line passing through the point  $abc$  and intersecting at right angles the line  $lmn$ :

$$\frac{x-a}{L} = \frac{y-b}{M} = \frac{z-c}{N},$$

where  $L = lm(b-b') + nl(c-c') - (m^2 + n^2)(a-a')$ ,

and symmetrical values exist for  $M$  and  $N$ .

PROOF.—The condition of perpendicularity to  $lmn$  is

$$Ll + Mm + Nn = 0; \quad (5520)$$

and the condition of intersecting the line is

$$(a-a')(Mn-mN) + (b-b')(Nl-nL) + (c-c')(Lm-lM) = 0.$$

These equations determine the ratios  $L : M : N$ .

**5539** Equations of the line passing through the point  $abc$ , parallel to the plane  $Lx + My + Nz = D$ , and intersecting the line  $(l'm'n', a'b'c')$ :

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n},$$

where  $l, m, n$  are found, as in the last, from

$$Ll + Mm + Nn = 0, \quad \text{and}$$

$$(a-a')(mn'-m'n) + (b-b')(nl'-n'l) + (c-c')(lm'-l'm) = 0.$$



**5540** Equations of the bisector of the angle between the two lines  $l_1m_1n_1, l_2m_2n_2$ :

$$\frac{x}{l_1+l_2} = \frac{y}{m_1+m_2} = \frac{z}{n_1+n_2}.$$

PROOF.—Taking the intersection of the lines for origin, let  $x_1y_1z_1, x_2y_2z_2$  be points on the given lines equidistant from the origin; then, if  $xyz$  be a point on the bisector midway between the former points,  $x = \frac{1}{2}(x_1+x_2)$ , &c. (4033); and the direction-cosines of a line through the origin are proportional to the coordinates.

**5541** The equations of a right line in four plane coordinates

are 
$$\frac{\alpha-\alpha'}{L} = \frac{\beta-\beta'}{M} = \frac{\gamma-\gamma'}{N} = \frac{\delta-\delta'}{R} \dots\dots\dots (i.),$$

where  $\alpha\beta\gamma\delta$  is a variable point, and  $\alpha'\beta'\gamma'\delta'$  a fixed point on the line. The relation between  $L, M, N, R$  is

**5542** 
$$AL+BM+CN+DR=0 \dots\dots\dots (ii.).$$

PROOF.—For, since equation (5503) holds for  $\alpha\beta\gamma\delta$  and also for  $\alpha'\beta'\gamma'\delta'$ , we have  $A(\alpha-\alpha') + B(\beta-\beta') + C(\gamma-\gamma') + D(\delta-\delta') = 0$ .

Substitute from (i.)  $\alpha-\alpha' = rL, \beta-\beta' = rM, \&c.$

**5543** In tetrahedral coordinates the same equation (i.) subsists, but the relation between  $L, M, N, R$  becomes, by changing  $A\alpha$  into  $\alpha$ , &c.,

**5544** 
$$L+M+N+R=0.$$

## THE PLANE.

**5545** General equation of a plane:

$$Ax+By+Cz+D=0.$$

**5546** Equation in terms of the intercepts on the axes:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

**5547** Equation in terms of  $p$ , the perpendicular from the origin upon the plane, and  $l, m, n$ , the direction-cosines of  $p$ :

$$lx+my+nz=p.$$

PROOF.—If  $P$  be any point  $xyz$  upon the plane, and  $O$  the origin, the projection of  $OP$  upon the normal through  $O$  is  $p$  itself; but this projection is  $lx + my + nz$ , as in (5520).

**5548** The values of  $l, m, n, p$  for the general equation (5545) are

$$l = \frac{A}{\sqrt{A^2 + B^2 + C^2}}, \text{ \&c., } p = \frac{-D}{\sqrt{A^2 + B^2 + C^2}}.$$

PROOF.—Similar to that for (4060-2): by equating coefficients in (5545) and (5547) and employing  $l^2 + m^2 + n^2 = 1$ .

**5550** The equation of a plane in four-plane coordinates is

$$la + m\beta + n\gamma + r\delta = 0,$$

with 
$$l : m : n : r = \frac{a_1}{p_1} : \frac{\beta_1}{p_2} : \frac{\gamma_1}{p_3} : \frac{\delta_1}{p_4},$$

where  $a_1, \beta_1, \gamma_1, \delta_1$  are the perpendiculars upon the plane from  $A, B, C, D$ , the vertices of the tetrahedron of reference, and  $p_1, p_2, p_3, p_4$  are the perpendiculars from the same points upon the opposite faces of the tetrahedron.

PROOF.—Put  $\gamma = \delta = 0$  for the point where the plane cuts an edge of the tetrahedron, and then determine the ratio  $l : m$  by proportion.

See *Frost and Wolstenholme*, Art. 81.

**5551** The equation of a plane in tetrahedral coordinates is also of the form in (5550), but the ratios are, in that case,

$$l : m : n : r = a_1 : \beta_1 : \gamma_1 : \delta_1.$$

The relation between the three-plane and four-plane coordinates is

$$a = p - lx - my - nz.$$

**5552** The equation of a plane in polar coordinates is

$$r \{ \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi') \} = p.$$

PROOF.—Here  $p$  is the perpendicular from the origin on the plane, and  $p, \theta', \phi'$  the polar coordinates of the foot of the perpendicular. Then, if  $\psi$  is the angle between  $p$  and  $r$ , we have  $p = r \cos \psi$  and  $\cos \psi$  from (882).

**5553** The angle  $\theta$  between two planes

$$lx + my + nz = p \quad \text{and} \quad l'x + m'y + n'z = p'$$

5 D

is given by formula (5520), and the conditions of parallelism and perpendicularity by (5531) and (5532), since the mutual inclination of the planes is the same as that of their normals.

**5554** The length of the perpendicular from the point  $x'y'z'$  upon the plane  $Ax + By + Cz + D = 0$  is

$$\pm \frac{Ax + By + Cz + D}{\sqrt{A^2 + B^2 + C^2}} = p - lx' - my' - nz'.$$

PROOF.—As in (4094).

**5556** The same in oblique coordinates

$$= \frac{(Ax + By + Cz + D)}{\rho} = p - x \cos \alpha - y \cos \beta - z \cos \gamma,$$

where  $\rho$  is found from (5519) by putting  $A, B, C$  for  $\rho \cos \alpha, \rho \cos \beta, \rho \cos \gamma$ . This gives

$$\mathbf{5558} \quad \rho^2 = \frac{\left\{ \begin{array}{l} A^2 \sin^2 \lambda + B^2 \sin^2 \mu + C^2 \sin^2 \nu + 2BC (\cos \mu \cos \nu - \cos \lambda) \\ + 2CA (\cos \nu \cos \lambda - \cos \mu) + 2AB (\cos \lambda \cos \mu - \cos \nu) \end{array} \right\}}{1 - \cos^2 \lambda - \cos^2 \mu - \cos^2 \nu + 2 \cos \lambda \cos \mu \cos \nu}.$$

**5559** The distance  $r$  of the point  $x'y'z'$  from the plane  $Ax + By + Cz + D = 0$ , measured in the direction  $lmn$ , the axes being oblique:

$$r = - \frac{Ax' + By' + Cz' + D}{Al + Bm + Cn}.$$

PROOF.—By determining  $r$  from the simultaneous equations of the line and the plane, viz.,

$$\frac{x-x'}{l} = \frac{y-y'}{m} = \frac{z-z'}{n} = r \quad \text{and} \quad Ax + By + Cz + D = 0.$$

Otherwise, by dividing the perpendicular from  $x'y'z'$  (5554) by the cosine of its inclination to  $lmn$ , viz.,  $\frac{Al + Bm + Cn}{\sqrt{A^2 + B^2 + C^2}}$ .

#### EQUATIONS OF PLANES UNDER GIVEN CONDITIONS.

**5560** A plane passing through the point  $abc$  and perpendicular to the direction  $lmn$ :

$$l(x-a) + m(y-b) + n(z-c) = 0.$$

**5561** A plane passing through two points  $abc, a'b'c'$  :

$$\lambda \frac{x-a}{a-a'} + \mu \frac{y-b}{b-b'} + \nu \frac{z-c}{c-c'} = 0,$$

**5562** with  $\lambda + \mu + \nu = 0$ .

PROOF.—By eliminating  $n$  between the equations

$l(x-a) + m(y-b) + n(z-c) = 0, \quad l(a-a') + m(b-b') + n(c-c') = 0,$   
and altering the arbitrary constant.

**5563** A plane passing through the point of intersection of the three planes  $u = 0, v = 0, w = 0$  :

$$lu + mv + nw = 0.$$

**5564** A plane passing through the line of intersection of the two planes  $u = 0, v = 0$  :

$$lu + mv = 0.$$

**5565** A plane passing through the two points given by  $u = 0, v = 0, w = 0$  and  $u = a, v = b, w = c$  :

$$lu + mv + nw = 0 \quad \text{with} \quad la + mb + nc = 0.$$

**5566** The equation of a plane passing through the three points  $x_1y_1z_1, x_2y_2z_2, x_3y_3z_3$  or  $A, B, C$ , is given by the determinant annexed, in which the coefficients of  $x, y, z$  represent twice the projections of the area  $ABC$  upon the coordinate planes.

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

PROOF.—The determinant is the eliminant of  $Ax + By + Cz = 1$ , and three similar equations. Expanded it becomes

$$x(y_2z_3 - y_3z_2 + y_3z_1 - y_1z_3 + y_1z_2 - y_2z_1) + y(\&c.) + z(\&c.) + x_1y_2z_3 - \&c. = 0.$$

Hence, by (4036), we see that the coefficients are twice the projections of  $ABC$ , as stated.

**5567** The sum of squares of the coefficients is equal to four times the square of the area  $ABC$ .

PROOF.—For, if  $l, m, n$  are the dir-cos. of the plane, and  $ABC = S$ , the coefficients are  $= 2Sl, 2Sm, 2Sn$ , by projection.

**5568** The determinant  $(x_1, y_2, z_3)$ , that is, the absolute term in equation (5566), represents six times the volume of the tetrahedron  $OABC$ , where  $O$  is the origin.

PROOF.—Writing the equation of the plane  $ABC$ ,  $Ax + By + Cz + D = 0$ , we have for the perpendicular from the origin, disregarding sign,

$$p = \frac{D}{\sqrt{A^2 + B^2 + C^2}} = \frac{D}{2S} \quad (5567),$$

therefore  $D = 2pS = 6 \times$  the tetrahedron  $OABC$ .

**5569** If  $xyz$  be a fourth point,  $P$ , not in the plane of  $ABC$ , the determinant in (5566) represents six times the volume of the tetrahedron  $PABC$ .

PROOF.—By the last theorem the four component determinants represent six times  $(OBCP + OCAP + OABP + OABC)$  for an origin  $O$  within the tetrahedron.

**5570** A plane passing through the points  $abc$ ,  $a'b'c'$ , and parallel to the direction  $lmn$ :

$$\left\| \begin{array}{ccc} \frac{x-a}{a-a'} & \frac{y-b}{b-b'} & \frac{z-c}{c-c'} \\ l & m & n \\ \frac{l}{a-a'} & \frac{m}{b-b'} & \frac{n}{c-c'} \end{array} \right\| = 0.$$

PROOF.—Eliminate  $\lambda, \mu, \nu$  between the equations (5561-2) and  $\frac{l\lambda}{a-a'} = \frac{m\mu}{b-b'} + \frac{n\nu}{c-c'} = 0$ , the condition of perpendicularity between  $lmn$  and the normal of the plane (5561).

**5571** A plane passing through the point  $abc$  and parallel to the lines  $lmn, l'm'n'$ :

$$\left| \begin{array}{ccc} x-a & l & l' \\ y-b & m & m' \\ z-c & n & n' \end{array} \right| = 0.$$

PROOF.—The equation is of the form  $\lambda(x-a) + \mu(y-b) + \nu(z-c) = 0$ , and the conditions of perpendicularity between the normal of the plane and the given lines are  $l\lambda + m\mu + n\nu = 0$ ,  $l'\lambda + m'\mu + n'\nu = 0$ . Form the eliminant of the three equations.

**5572** A plane equidistant from the two right lines  $(abc, lmn)$  and  $(a'b'c', l'm'n')$ :

$$\left| \begin{array}{ccc} x - \frac{1}{2}(a+a') & l & l' \\ y - \frac{1}{2}(b+b') & m & m' \\ z - \frac{1}{2}(c+c') & n & n' \end{array} \right| = 0.$$

By (5571).

**5573** A plane passing through the line  $(abc, lmn)$  and perpendicular to the plane  $l'x + m'y + n'z = p$ :

The equation is that in (5571).

For proof, assume  $\lambda, \mu, \nu$  for dir-cos. of the normal of the required plane, and write the conditions that the plane may pass through  $abc$  and that the normal may be perpendicular to the given line and to the normal of the given plane.

## TRANSFORMATION OF COORDINATES.

**5574** To change any axes of reference to new axes parallel to the old ones:

Let the coordinates of the new origin referred to the old axes be  $a, b, c$ ;  $xyz$  and  $x'y'z'$ , the same point referred to the old and new axes respectively; then

$$x = x' + a, \quad y = y' + b, \quad z = z' + c.$$

**5575** To change rectangular axes of reference to new rectangular axes with the same origin:

Let  $OX, OY, OZ$  be the original axes, and  $OX', OY', OZ'$  the new ones,

$l_1 m_1 n_1$  the dir-cos. of  $OX'$  referred to  $OX, OY, OZ$ ,

$l_2 m_2 n_2$  do.  $OY'$  do. do.

$l_3 m_3 n_3$  do.  $OZ'$  do. do.

$xyz, \xi\eta\zeta$  the same point referred to the old and new axes respectively. Then the equations of transformation are

$$\begin{aligned} \mathbf{5576} \quad x &= l_1 \xi + l_2 \eta + l_3 \zeta \dots\dots\dots \text{(i.)}, \\ y &= m_1 \xi + m_2 \eta + m_3 \zeta \dots\dots\dots \text{(ii.)}, \\ z &= n_1 \xi + n_2 \eta + n_3 \zeta \dots\dots\dots \text{(iii.)}. \end{aligned}$$

And the nine constants are connected by the six equations

$$\begin{aligned} \mathbf{5577} \quad l_1^2 + m_1^2 + n_1^2 &= 1 \dots \text{(iv.)}, \quad l_2 l_3 + m_2 m_3 + n_2 n_3 = 0 \dots \text{(vii.)}, \\ l_2^2 + m_2^2 + n_2^2 &= 1 \dots \text{(v.)}, \quad l_3 l_1 + m_3 m_1 + n_3 n_1 = 0 \dots \text{(viii.)}, \\ l_3^2 + m_3^2 + n_3^2 &= 1 \dots \text{(vi.)}, \quad l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \dots \text{(ix.)}, \end{aligned}$$

so that three constants are independent.



PROOF.—By (5515) and (5532), since  $OX', OY', OZ'$  are mutually at right angles.

**5578** The relations (iv. to ix.) may also be expressed thus—

$$\frac{l_1}{m_2 n_3 - m_3 n_2} = \frac{m_1}{n_2 l_3 - n_3 l_2} = \frac{n_1}{l_2 m_3 - l_3 m_2} = \pm 1 \dots\dots (x.),$$

$$\frac{l_2}{m_3 n_1 - m_1 n_3} = \frac{m_2}{n_3 l_1 - n_1 l_3} = \frac{n_2}{l_3 m_1 - l_1 m_3} = \pm 1 \dots\dots (xi.),$$

$$\frac{l_3}{m_1 n_2 - m_2 n_1} = \frac{m_3}{n_1 l_2 - n_2 l_1} = \frac{n_3}{l_1 m_2 - l_2 m_1} = \pm 1 \dots\dots (xii.).$$

Obtained by eliminating the third term from any two of equations (vii.—ix.). Also, by squaring each fraction in (x.) and adding numerators and denominators, we get

$$\frac{l_1^2 + m_1^2 + n_1^2}{(l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1 l_2 + m_1 m_2 + n_1 n_2)} = 1, \text{ by (5577).}$$

**5579** If the transformation above is *rotational*, that is, if it can be effected by a rotation about a fixed axis, the position of that axis and the angle of rotation  $\theta$  are found from the equations

$$2 \cos \theta = l_1 + m_2 + n_3 - 1,$$

$$\mathbf{5580} \quad \frac{\cos^2 \alpha}{m_2 + n_3 - l_1 - 1} = \frac{\cos^2 \beta}{n_3 + l_1 - m_2 - 1} = \frac{\cos^2 \gamma}{l_1 + m_2 - n_3 - 1},$$

where  $\alpha, \beta, \gamma$  are the angles which the axis makes with the original coordinate axes.

PROOF.—(Fig. 171.) Let the original rectangular axes and the axis of rotation cut the surface of a sphere, whose centre is the origin  $O$ , in the points  $x, y, z$ , and  $I$  respectively. Then, if the altered axes cut the sphere in  $\xi, \eta, \zeta$ , we shall have  $\theta = \angle xI\xi$  in the spherical triangle;  $Ix = I\xi = \alpha$ ;  $Iy = I\eta = \beta$ ;  $Iz = I\zeta = \gamma$ , and by (882) applied to the isosceles spherical triangles  $xI\xi$ , &c.,  $l_1 = \cos x\xi = \cos^2 \alpha + \sin^2 \alpha \cos \theta$ ,  $m_2 = \cos y\eta = \cos^2 \beta + \sin^2 \beta \cos \theta$ ,  $n_3 = \cos z\zeta = \cos^2 \gamma + \sin^2 \gamma \cos \theta$ . From these  $\cos \theta$ ,  $\cos \alpha$ ,  $\cos \beta$ , and  $\cos \gamma$  are found.

**5581** Transformation of rectangular coordinates to oblique:

Equations (i. to vi.) apply as before, but (vii. to ix.) no longer hold, so that there are now six independent constants.

## THE SPHERE.

**5582** The equation of a sphere when the point  $abc$  is the centre and  $r$  is the radius,

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

**5583** The general equation is

$$x^2 + y^2 + z^2 + Ax + By + Cz + D = 0.$$

The coordinates of the centre are then  $-\frac{A}{2}$ ,  $-\frac{B}{2}$ ,  $-\frac{C}{2}$ ; and the radius  $= \frac{1}{2}\sqrt{(A^2 + B^2 + C^2 - 4D)}$ .

PROOF.—By equating coefficients with (5582).

**5584** If  $xyz$  be a point not on the sphere, the value of  $(x-a)^2 + (y-b)^2 + (z-c)^2 - r^2$  is the product of the segments of any right line drawn through  $xyz$  to cut the sphere.

PROOF.—From Euc. III., 35, 36.

## THE RADICAL PLANE.

**5585** The radical planes of the two spheres whose equations are  $u = 0$ ,  $u' = 0$ , is

$$u - u' = 0.$$

**5586** The radical planes of three spheres have a common section, and the radical planes of four spheres intersect in the same point.

PROOF.—By adding their equations, and by the principle of (4608) extended to the equations of planes.

## POLES OF SIMILITUDE.

**5587** DEF.—A *pole of similitude* is a point such that the tangents from it to two spheres are proportional to the radii.

**5588** The *external* and *internal poles of similitude* are the vertices of the common enveloping cones.

**5589** The locus of the pole of similitude of two spheres is a sphere whose diameter contains the centres and is divided harmonically by them.

## CYLINDRICAL AND CONICAL SURFACES.

**5590** DEF.—A *conical surface* is generated by a right line which passes through a fixed point called the *vertex* and moves in any manner.

**5591** If the point be at infinity, the line moves always parallel to itself and generates a *cylindrical surface*.

**5592** Any section of the surface by a plane may be taken for the *guiding curve*.

**5593** To find the equation of a cylindrical or conical surface.

RULE.—Eliminate  $xyz$  from the equations of the guiding curve and the equations  $\frac{x-a}{l} = \frac{y-\beta}{m} = \frac{z}{n}$  of any generating line; and in the result put for the variable parameters of the line their values in terms of  $x$ ,  $y$ , and  $z$ .

**5594** Ex. 1.—To find the equation of the cylindrical surface whose generating lines have the direction  $lmn$ , and whose guiding curve is given by  $b^2x^2 + a^2y^2 = a^2b^2$  and  $z = 0$ .

At the point where the line  $\frac{x-a}{l} = \frac{y-\beta}{m} = \frac{z}{n}$  meets the ellipse,  $z = 0$ ,  $x = a$ ,  $y = \beta$ . Therefore  $b^2a^2 + a^2\beta^2 = a^2b^2$ . Substitute in this, for the variable parameters,  $a$ ,  $\beta$ ,  $a = x - \frac{lz}{n}$ ,  $\beta = y - \frac{mz}{n}$ ; and we get, for the cylindrical surface  $b^2(na - lz)^2 + a^2(ny - mz)^2 = a^2b^2n^2$ .

**5595** A conical surface whose vertex is the origin and guiding curve the ellipse  $b^2x^2 + a^2y^2 = a^2b^2$ ,  $z = c$ , is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

PROOF.—Here the generating line is  $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ . At the point of inter-

section of the line and curve  $z = c$ ,  $x = \frac{lc}{n}$ ,  $y = \frac{mc}{n}$ ;  $\therefore \frac{b^2 l^2 c^2}{n^2} + \frac{a^2 m^2 c^2}{n^2} = a^2 b^2$ .

Substitute for the variable parameters  $l : m : n$  the values  $x : y : z$ , and the result is obtained.

### CIRCULAR SECTIONS.

**5596** RULE.—To find the circular sections of a quadric curve, express the equation in the form  $A(x^2 + y^2 + z^2 + c^2) + \&c. = 0$ . If the remaining terms can be resolved into two factors, the circular sections are defined by the intersection of a sphere and two planes.

**5597** Generally the two quadrics

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$$

$$\text{and } (a + \lambda)x^2 + (b + \lambda)y^2 + (c + \lambda)z^2 + 2fyz + 2gzx + 2hxy = 1$$

have the same circular sections.

PROOF.—Let  $r, \rho$  be coincident radii of the two surfaces having  $lmn$  for a common direction. Then  $\frac{1}{r^2} = al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm$  and  $\frac{1}{\rho^2} = \text{the same} + \lambda$ . Therefore, if  $r$  has a constant value throughout any section,  $\rho$  is also constant throughout that section.

**5598** Ex.—An oblique circular cone whose vertex is the point  $a, 0, b$ , and guiding curve the circle  $x^2 + y^2 = c^2$ ;  $z = 0$ ; is

$$(az - bx)^2 + b^2 y^2 = c^2 (z - b)^2.$$

The equation may be written

$$b^2 (x^2 + y^2 + z^2 - c^2) = z \{ 2abx + (b^2 + c^2 - a^2)z - 2bc^2 \},$$

and therefore the cone has two series of parallel circular sections,  $z = k$  and  $2abx + (b^2 + c^2 - a^2)z - 2bc^2 = p^2$  (5583). (Frost and Wolstenholme.)

### CONICOIDS.

**5599** DEFS.—A *conicoid* is a surface every plane section of which is a conic.

The varieties are the *ellipsoid*, the *one-fold* and *two-fold hyperboloids*, the *elliptic* and *hyperbolic paraboloids*, the *spheroid of revolution*, the *cone*, and the *cylinder*.

In any of the following equations of a conicoid, by making one of the variables constant, the equation of a section parallel to a coordinate plane is obtained, and the equation of the surface is by that means verified. Thus, in the equations of (5600) or (5617), Figs. (172) and (173), if  $z$  be put  $= ON$ , we get the equation of the elliptic section  $RPQ$ , the semi-axes of which are  $NQ = \frac{a}{c} \sqrt{(c^2 - ON^2)}$  and  $NR = \frac{b}{c} \sqrt{(c^2 - ON^2)}$ ,  $a, b, c$  being the principal semi-axes of the conicoid; that is,  $OA, OB, OC$  in the figure.

### THE ELLIPSOID.

**5600** The equation referred to the principal axes of the figure is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad (\text{Fig. 172})$$

**5601** There are two planes of circular section whose equations are

$$x^2 \left( \frac{1}{b^2} - \frac{1}{a^2} \right) - z^2 \left( \frac{1}{c^2} - \frac{1}{b^2} \right) = 0,$$

with  $a > b > c$ .

PROOF: 
$$x^2 \left( \frac{1}{a^2} - \frac{1}{r^2} \right) + y^2 \left( \frac{1}{b^2} - \frac{1}{r^2} \right) + z^2 \left( \frac{1}{c^2} - \frac{1}{r^2} \right) = 0$$

is a cone having a common section with the conicoid and a sphere of radius  $r$ . If the common section be plane, one of the three terms must vanish in order that the rest may be resolved into two factors.

Since  $a > b > c$ , the only possible solution for real factors is got by making  $r = b$ .

**5602** Sections by planes parallel to the above are also circles, and any other sections are ellipses.

**5603** The umbilici of the ellipsoid (see 5777) are the points whose coordinates are

$$x = \pm a \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, \quad y = 0, \quad z = \pm c \sqrt{\frac{b^2 - c^2}{a^2 - c^2}}.$$

PROOF.—The points of intersection of the planes (5601) and the ellipsoid (5600) on the  $xz$  plane are given by  $x' = \pm a \sqrt{\frac{b^2 - c^2}{a^2 - c^2}}$ ,  $z' = \pm c \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}$ . Since, by (5602) the vanishing circular sections are at the points in the  $xz$  plane conjugate to  $x'$  and  $z'$ , we have, by (4352),  $x = -\frac{a}{c} z'$ ,  $z = \frac{c}{a} x'$ .

**5604** If  $a = b$ , in (5600), the figure becomes a spheroid, and every plane parallel to  $xy$  makes a circular section. Hence the spheroid is a surface of revolution. It is called *prolate* or *oblate* according as the ellipse is made to revolve about its major or minor axis.

### THE HYPERBOLOID.

**5605** The equation of a one-fold hyperboloid referred to its principal axes is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \quad (\text{Fig. 173})$$

**5606** The planes of circular section, when  $a > b > c$ , are all parallel to one or other of the planes whose equations are

$$y^2 \left( \frac{1}{b^2} - \frac{1}{a^2} \right) - z^2 \left( \frac{1}{c^2} + \frac{1}{a^2} \right) = 0.$$

PROOF.—As in (5601), putting  $r = a$ .

**5607** The generating lines of this surface belong to two parallel systems (i.) and (ii.) below, with all values of  $\theta$ .

**5608**

$$\left. \begin{aligned} \frac{x}{a} &= \cos \theta + \frac{z}{c} \sin \theta \\ \frac{y}{b} &= \sin \theta - \frac{z}{c} \cos \theta \end{aligned} \right\} \dots \text{(i.)}, \quad \left. \begin{aligned} \frac{x}{a} &= \cos \theta - \frac{z}{c} \sin \theta \\ \frac{y}{b} &= \sin \theta + \frac{z}{c} \cos \theta \end{aligned} \right\} \dots \text{(ii.)}.$$

For the coordinates which satisfy either pair of equations, (i.) or (ii.), satisfy also the equation of the surface. The equations may also be put in the forms

$$\mathbf{5610} \quad \frac{x - a \cos \theta}{a \sin \theta} = \frac{y - b \sin \theta}{-b \cos \theta} = \pm \frac{z}{c}.$$

**5612** If  $z = 0$ ,  $x = a \cos \theta$  and  $y = b \sin \theta$ . Hence  $\theta$  is the eccentric angle of the point in which the lines (i.) and (ii.) intersect in the  $xy$  plane.

**5613** Any two generating lines of opposite systems intersect, but no two of the same system do.



**5614** If two generating lines of opposite systems be drawn through the two points in the principal elliptic section whose eccentric angles are  $\theta + a$ ,  $\theta - a$ ,  $a$  being constant, the coordinates of the point of intersection will be

$x = a \cos \theta \sec a$ ,  $y = b \sin \theta \sec a$ ,  $z = \pm c \tan a$ ,  
and the locus of the point, as  $\theta$  varies, will be the ellipse

**5615** 
$$\frac{x^2}{a^2 \sec^2 a} + \frac{y^2}{b^2 \sec^2 a} = 1; \quad z = \pm c \tan a.$$

PROOF.—From (i.) and (ii.), putting  $\theta \pm a$  for  $\theta$ .\*

**5616** The asymptotic cone is the surface given in (5595).

PROOF.—Any plane through the  $z$  axis whose equation is  $y = mx$  cuts the hyperboloid and this cone in an hyperbola and its asymptotes respectively.

**5617** The equation of a two-fold hyperboloid is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \quad (\text{Fig. 174})$$

and the equation of its asymptotic cone is

**5618** 
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

PROOF.—Any plane through the  $z$  axis, whose equation is  $y = mz$ , cuts the hyperboloid and this cone in an hyperbola and its asymptotes respectively.

There are two surfaces, one the image of the other with regard to the plane of  $yz$ . One only of these is shown in the diagram.

**5619** The planes of circular section when  $b$  is  $> c$  are all parallel to one or other of the planes whose joint equation is

$$x^2 \left( \frac{1}{a^2} + \frac{1}{b^2} \right) - z^2 \left( \frac{1}{c^2} - \frac{1}{b^2} \right) = 0.$$

PROOF.—As in (5601), putting  $r^2 = -b^2$ .

**5620** If  $b = c$ , the figure becomes an hyperboloid of revolution.

#### THE PARABOLOID.

**5621** This surface is generated by a parabola which moves with its vertex always on another parabola; the axes of the two curves being parallel and their planes at right angles.

\* The surface of a one-fold hyperboloid, as generated by right lines, may frequently be seen in the foot-stool or work-basket constructed entirely of straight rods of cane or wicker.

The paraboloid is *elliptic* or *hyperbolic* according as the axes of the two parabolas extend in the same or opposite directions.

**5622** The equation of the elliptic paraboloid is

$$\frac{y^2}{b} + \frac{z^2}{c} = x, \quad (\text{Fig. 175})$$

$b$  and  $c$  being the *latera recta* of the two parabolas.

PROOF:  $QM^2 = b \cdot OM$ ;  $PN^2 = c \cdot QN$ ;  $\therefore \frac{QM^2}{b} + \frac{PN^2}{c} = OM + QN = x$ .

If  $b = c$ , the figure becomes the *paraboloid of revolution*.

**5623** Similarly the equation of the hyperbolic paraboloid is

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = x. \quad (\text{Fig. 176})^*$$

**5624** The equations of the generating lines of this surface

are  $\frac{y}{\sqrt{b}} \pm \frac{z}{\sqrt{c}} = m$  and  $\frac{y}{\sqrt{b}} \mp \frac{z}{\sqrt{c}} = \frac{x}{m}$ ,

the upper signs giving one system of generators and the lower signs another system.

**5625** The equations of the asymptotic planes are

$$\frac{y}{\sqrt{b}} \pm \frac{z}{\sqrt{c}} = 0.$$

## CENTRAL QUADRIC SURFACE.

### TANGENT AND DIAMETRAL PLANES.

**5626** Taking the equation of a central quadric  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$  to include both the ellipsoid and the two hyperboloids

\* The curvature of this surface is *anticlastic*, a sort of curvature which may be seen in the saddle of a mountain; for instance, on the smooth sward of some parts of the Malvern Hills, Worcestershire.

according to the signs of  $b^2$  and  $c^2$ , the equation of the tangent plane at  $xyz$  is

$$\frac{\xi x}{a^2} + \frac{\eta y}{b^2} + \frac{\zeta z}{c^2} = 1. \quad \text{By (5679).}$$

**5627** If  $p$  be the length of the perpendicular from the origin upon the tangent plane at  $xyz$ ,

$$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}.$$

PROOF.—From (5549) applied to (5626).

**5628** The length of the perpendicular let fall from any point  $\xi\eta\zeta$  upon the tangent plane at  $xyz$  is

$$p \left( \frac{\xi x}{a^2} + \frac{\eta y}{b^2} + \frac{\zeta z}{c^2} - 1 \right). \quad (5554 \text{ \& } 5627)$$

**5629** Direction cosines of the normal of the tangent plane

at  $xyz$ , 
$$l = \frac{px}{a^2}, \quad m = \frac{py}{b^2}, \quad n = \frac{pz}{c^2}.$$

PROOF.—By (5548) applied to (5626) and the value in (5627).

**5630** If  $l, m, n$  are the direction cosines of  $p$ ,

$$p = lx + my + nz \quad \text{and} \quad p^2 = a^2 l^2 + b^2 m^2 + c^2 n^2.$$

PROOF.—(5630) By projecting the three coordinates  $x, y, z$  upon  $p$ .

(5631) By substituting the values of  $x, y, z$ , obtained from (5629), in (5630).

**5632** The equation of the normal at  $xyz$  is

$$(\xi - x) \frac{a^2}{x} = (\eta - y) \frac{b^2}{y} = (\zeta - z) \frac{c^2}{z},$$

since the dir-cos. are the same as those of the tangent plane at (5626).

**5633** Each term of the above equations

$$= p \sqrt{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2}$$

or  $p$  multiplied into the length of the normal.

PROOF.—Each term squared  $= \frac{(\xi-x)^2}{\frac{x^2}{a^4}} = \frac{(\eta-y)^2}{\frac{y^2}{b^4}} = \frac{(\zeta-z)^2}{\frac{z^2}{c^4}}.$

Add numerators and denominators, and employ (5627).

**5634** Equation (5631) is the condition that the plane  $lx+my+nz=p$  may touch the conicoid; and if  $p=0$ , we have for the condition of the plane  $lx+my+nz=0$  touching the cone  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$ ,

**5635**  $a^2l^2 + b^2m^2 + c^2n^2 = 0.$

**5636** The section of the quadric made by a diametral plane conjugate to the diameter through the point  $xyz$  has for its

equation  $\frac{\xi x}{a^2} + \frac{\eta y}{b^2} + \frac{\zeta z}{c^2} = 0.$  By (5688).

**5637** Hence the relation between the direction cosines of two conjugate diameters is

$$\frac{ll'}{a^2} + \frac{mm'}{b^2} + \frac{nn'}{c^2} = 0.$$

#### ECCENTRIC VALUES OF THE COORDINATES.

**5638** These are defined to be

$$x = a\lambda, \quad y = b\mu, \quad z = c\nu, \quad \text{with} \quad \lambda^2 + \mu^2 + \nu^2 = 1.$$

**5640**  $\lambda, \mu, \nu$  are the dir-cos. of a line called the *eccentric line*; and  $\xi = r\lambda, \eta = r\mu, \zeta = r\nu$  are the coordinates of the corresponding point upon an auxiliary sphere of radius  $r$ .

**5641** The eccentric lines of two conjugate semi-diameters are at right angles. By (5637).

**5642** The sum of the squares of three conjugate semi-diameters is constant and  $= a^2 + b^2 + c^2$ .

PROOF.—Let  $a', b', c'$  be the semi-diameters, and  $x_1y_1z_1, x_2y_2z_2, x_3y_3z_3$  their extremities. Put the eccentric values in the equations  $x_1^2 + y_1^2 + z_1^2 = a'^2$ , &c., and add. By (5641),  $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$ , &c.

**5643** The sum of the squares of the reciprocals of the same is also constant.

PROOF.—Put  $r_1 \cos \alpha_1$ ,  $r_1 \cos \beta_1$ ,  $r_1 \cos \gamma_1$  for  $x_1, y_1, z_1$  in the equation of the quadric. So for  $x_2, y_2, z_2$  and  $x_3, y_3, z_3$ . Divide by  $r_1, r_2, r_3$ , and add the results.

**5644** The sum of squares of reciprocals of perpendiculars on three conjugate tangent planes is constant.

PROOF.—For each perpendicular take (5627), and substitute the eccentric values as in (5642).

**5645** The sum of the squares of the areas of three conjugate parallelograms is constant.

PROOF.—By the constant volume of the parallelepiped  $p_1 A_1 = p_2 A_2 = p_3 A_3$  (5648) and by (5644).

**5646** The sum of the squares of the projections of three conjugate semi-diameters upon a fixed line or plane is constant.

PROOF.—With the same notation as in (5642), let  $(lmn)$  be the given line. Substitute the eccentric values (5638) in  $(lx_1 + my_1 + nz_1)^2 + (lx_2 + my_2 + nz_2)^2 + (lx_3 + my_3 + nz_3)^2$ . In the case of the plane we shall have

$$a'^2 - (lx_1 + my_1 + nz_1)^2 + \&c.$$

**5647** COR.—The extremities of three conjugate semi-diameters being  $x_1 y_1 z_1, x_2 y_2 z_2, x_3 y_3 z_3$ , it follows that, by projecting upon each axis in turn,

$$x_1^2 + x_2^2 + x_3^2 = a^2; \quad y_1^2 + y_2^2 + y_3^2 = b^2; \quad z_1^2 + z_2^2 + z_3^2 = c^2.$$

**5648** The parallelepiped contained by three conjugate semi-diameters is of constant volume  $= abc$ .

PROOF.—By (5568), the volume  $= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = abc \begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix}$

by the eccentric values (5638). But the last determinant  $= 1$  by (584, I.).

**5649** COR.—If  $a', b', c'$  are the semi-conjugate diameters,  $\omega$  the angle between  $a'$  and  $b'$ , and  $p$  the perpendicular from the origin upon the tangent plane parallel to  $a'b'$ , the volume of the parallelepiped is  $pa'b' \sin \omega = abc$ .

**5650** Hence the area of a central section in the plane of  $a'b'$

$$= \pi a'b' \sin \omega = \pi \frac{abc}{p}.$$

**5651** Quadratic for the semi-axis of a central section of the quadric  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$  made by the plane  $lx + my + nz = 0$ :

$$\frac{a^2 l^2}{a^2 - r^2} + \frac{b^2 m^2}{b^2 - r^2} + \frac{c^2 n^2}{c^2 - r^2} = 0.$$

PROOF.—The equation is the condition, by (5635), that the plane  $lx + my + nz = 0$  may touch the cone

$$x^2 \left( \frac{1}{a^2} - \frac{1}{r^2} \right) + y^2 \left( \frac{1}{b^2} - \frac{1}{r^2} \right) + z^2 \left( \frac{1}{c^2} - \frac{1}{r^2} \right) = 0,$$

as in the Proof of (5600). For another method, see (1863).

**5652** When the equation of the quadric is presented in the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1,$$

the quadratic for  $r^2$  takes the form of the determinant equation annexed. Or, by expanding, and writing  $\Delta'$  for the same determinant, with the fraction  $\frac{1}{r^2}$  erased, the

equation becomes

$$\begin{vmatrix} a - \frac{1}{r^2} & h & g & l \\ h & b - \frac{1}{r^2} & f & m \\ g & f & c - \frac{1}{r^2} & n \\ l & m & n & \end{vmatrix} = 0.$$

$$\Delta' r^4 + \{ (b+c)l^2 + (c+a)m^2 + (a+b)n^2 - 2fmn - 2gnl - 2hlm \} r^2 - l^2 - m^2 - n^2 = 0.$$

PROOF.—The equation of the cone of intersection of the sphere and quadric now becomes

$$\left( a - \frac{1}{r^2} \right) x^2 + \left( b - \frac{1}{r^2} \right) y^2 + \left( c - \frac{1}{r^2} \right) z^2 + 2fyz + 2gzx + 2hxy = 0,$$

and the condition of touching (5700) produces the determinant equation.



**5654** To find the axes of a non-central section of the quadric  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ .

Let  $PNQ$  (Fig. 177) be the cutting plane. Take a parallel central section  $BOC$ , axes  $OB, OC$ , and draw  $NP, NQ$  parallel to them. These will be the axes of the section  $PNQ$ , and  $NQ$  will be found from the equation  $\frac{ON^2}{OA^2} + \frac{NQ^2}{OC^2} = 1$ .

**5655** The area of the same section

$$= \frac{\pi abc}{p} \left(1 - \frac{p'^2}{p^2}\right),$$

where  $p'$  and  $p$  are the perpendiculars from  $O$  upon the cutting plane and the parallel tangent plane.

$$\begin{aligned} \text{PROOF.}—\text{The area} &= \pi NP \cdot NQ = \pi \frac{NQ^2}{OC^2} \cdot OB \cdot OC \\ &= \pi \left(1 - \frac{ON^2}{OA^2}\right) OB \cdot OC = \frac{\pi abc}{p} \left(1 - \frac{p'^2}{p^2}\right), \text{ by (5650).} \end{aligned}$$

### SPHERO-CONICS.

DEF.—A *sphero-conic* is the curve of intersection of the surface of a sphere with any conical surface of the second degree whose vertex is the centre of the sphere.

Properties of cones of the second degree may be investigated by sphero-conics, and are analogous to the properties of conics.

A collection of formulæ will be found at page 562 of Routh's *Rigid Dynamics*, 3rd edition.

### CONFOCAL QUADRICS.

**5656** DEFINITION.—The two quadrics whose equations are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{and} \quad \frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} = 1,$$

are confocal. We shall assume  $a > b > c$ .

**5657** As  $\lambda$  decreases from being large and positive, the third axis of the confocal ellipsoid diminishes relatively to the

others until  $\lambda = -c^2$ , when the surface merges into the focal ellipse on the  $xy$  plane,

$$\frac{x^2}{a^2 - c^2} + \frac{y^2}{b^2 - c^2} = 1.$$

$\lambda$  still diminishing, a series of one-fold hyperboloids appear until  $\lambda = -b^2$ , when the surface coincides with the focal hyperbola on the  $zx$  plane,

$$\frac{x^2}{a^2 - b^2} - \frac{z^2}{b^2 - c^2} = 1.$$

The surface afterwards develops into a series of two-fold hyperboloids until  $\lambda = -a^2$ , when it becomes an imaginary focal ellipse on the  $yz$  plane.

**5658** Through any point  $xyz$  three confocal quadrics can be drawn according to the three values of  $\lambda$  furnished by the second equation in (5656). That equation, cleared of fractions, becomes

$$\begin{aligned} \mathbf{5659} \quad & \lambda^3 + (a^2 + b^2 + c^2 - x^2 - y^2 - z^2) \lambda^2 \\ & \{b^2 c^2 + c^2 a^2 + a^2 b^2 - (b^2 + c^2) x^2 - (c^2 + a^2) y^2 - (a^2 + b^2) z^2\} \lambda \\ & + a^2 b^2 c^2 - b^2 c^2 x^2 - c^2 a^2 y^2 - a^2 b^2 z^2 = 0. \end{aligned}$$

These three confocals are respectively an ellipsoid, a one-fold hyperboloid, and a two-fold hyperboloid. See Figure (178);  $P$  is the point  $xyz$ ; the lines of intersection of the ellipsoid with the two hyperboloids are  $DPE$  and  $FPG$ , and the two hyperboloids themselves intersect in  $HPK$ .

**PROOF.**—Substitute for  $\lambda$  successively in (5659)  $a^2, b^2, c^2, -\infty$ ; and the left member of the equation will be found to take the signs  $+, -, +, -$  accordingly. Consequently there are real roots between  $a^2$  and  $b^2, b^2$  and  $c^2, c^2$  and  $-\infty$ .

**5660** Two confocal quadrics of different species cut each other everywhere at right angles.

**PROOF.**—Let  $a, b, c; a', b', c'$  be the semi-axes of the two quadrics; then, at the line of intersection of the surfaces, we shall have

$$x^2 \left( \frac{1}{a^2} - \frac{1}{a'^2} \right) + y^2 \left( \frac{1}{b^2} - \frac{1}{b'^2} \right) + z^2 \left( \frac{1}{c^2} - \frac{1}{c'^2} \right) = 0,$$

which, since  $a^2 - a'^2 = b^2 - b'^2 = c^2 - c'^2 = \lambda$ , becomes the condition of perpendicularity of the normals by the values in (5629). Thus, in (Fig. 178), the tangents at  $P$  to the three lines of intersection of the surfaces are mutually at right angles.

**5661** If  $P$  be the point of intersection of three quadrics  $a_1b_1c_1$ ,  $a_2b_2c_2$ ,  $a_3b_3c_3$  confocal with the quadric  $abc$ ; the squares of the semi-axes,  $d_2$ ,  $d_3$ , of the diametral section conjugate to  $P$  in the first quadric are (considering  $a_1 > a_2 > a_3$ , and writing the suffixes in circular order)

$$d_3^2 = a_1^2 \sim a_2^2 \quad d_2^2 = a_1^2 \sim a_3^2,$$

$$\text{In the second, } d_1^2 = a_2^2 \sim a_3^2 \quad d_3^2 = a_2^2 \sim a_1^2,$$

$$\text{in the third, } d_2^2 = a_3^2 \sim a_1^2 \quad d_1^2 = a_3^2 \sim a_2^2.$$

Or, if for  $a_1^2$ ,  $a_2^2$ ,  $a_3^2$  we put  $a^2 + \lambda_1$ ,  $a^2 + \lambda_2$ ,  $a^2 + \lambda_3$ , the above values may be read with  $\lambda$  in the place of  $a$  and the same suffixes.

PROOF.—Put  $a_1^2 - a_3^2 = \mu$ ; then

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} + \frac{z^2}{c_1^2} = 1 \quad \text{and} \quad \frac{x^2}{a_1^2 - \mu} + \frac{y^2}{b_1^2 - \mu} + \frac{z^2}{c_1^2 - \mu} = 1$$

are confocal quadrics. Take the difference of the two equations, and we obtain, at a common point  $x'y'z'$ ,  $\frac{x'^2}{a^2(a^2 - \mu)} + \&c. = 0$ . Comparing this with (5651), the quadratic for the axes of the section of the quadric by the plane  $lx + my + nz = 0$ , we see that, if  $l, m, n$  have the values  $\frac{x'}{a^2}$ ,  $\&c.$ ,  $\mu$  is identical with  $r^2$ ; the plane is the diametral plane of  $P$ ; and the two values of  $\mu$  are the squares of its axes. Let  $d_1^2$ ,  $d_2^2$  be these values; then, since there are but three confocals, the two values of  $\mu$  must give the remaining confocals, i.e.,  $a_1^2 - d_3^2 = a_2^2$  and  $a_1^2 - d_2^2 = a_3^2$ .

The six axes of the sections are situated as shown in the diagram (Fig. 179). Either axis of any of the three sections is equal to one of the axes in one of the other sections, but the equal axes are not those which coincide.  $O$  is supposed to be the centre of the conicoids, and the three lines are drawn from  $O$  parallel to the three tangents at  $P$  to the lines of intersection.

**5662** Coordinates of the point of intersection of three confocal quadrics in terms of the semi-axes:

$$x^2 = \frac{a_1^2 a_2^2 a_3^2}{(a_1^2 - b_1^2)(a_1^2 - c_1^2)}, \quad y^2 = \frac{b_1^2 b_2^2 b_3^2}{(a_1^2 - b_1^2)(b_1^2 - c_1^2)},$$

$$z^2 = \frac{c_1^2 c_2^2 c_3^2}{(a_1^2 - c_1^2)(b_1^2 - c_1^2)}.$$

The denominators may be in terms of any of the confocals since  $a_1^2 - b_1^2 = a_2^2 - b_2^2 = a_3^2 - b_3^2$ , &c.

PROOF.—The equation of a confocal may be written  $\frac{x^2}{a_1^2} + \frac{y^2}{a_1^2 - h^2} + \frac{z^2}{a_1^2 - k^2} = 1$ , producing a cubic in  $a_1^2$ , the product of whose roots  $a_1^2, a_2^2, a_3^2$  gives  $x^2$ .

**5663** The perpendiculars from the origin upon the tangent planes of the three confocal quadrics being  $p_1, p_2, p_3$ :

$$p_1^2 = \frac{a_1^2 b_1^2 c_1^2}{(a_1^2 - a_2^2)(a_1^2 - a_3^2)}, \quad p_2^2 = \frac{a_2^2 b_2^2 c_2^2}{(a_1^2 - a_2^2)(a_2^2 - a_3^2)},$$

$$p_3^2 = \frac{a_3^2 b_3^2 c_3^2}{(a_1^2 - a_3^2)(a_2^2 - a_3^2)}.$$

PROOF.—By (5649),  $p_1 d_1 d_3 = a_1 b_1 c_1$ ; then by the values in (5661).

#### RECIPROCAL AND ENVELOPING CONES.

**5664** DEF.—A right line drawn through a fixed point always perpendicular to the tangent plane of a cone generates the reciprocal cone.

The enveloping cone of a quadric is the locus of all tangents to the surface which pass through a fixed point called the vertex.

**5665** The equations of a cone and its reciprocal are respectively

$$Ax^2 + By^2 + Cz^2 = 0 \dots\dots (i.), \quad \text{and} \quad \frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = 0 \dots\dots (ii.).$$

PROOF.—The equations of the tangent plane of (i.) at any point  $xyz$ , and of the perpendicular to it from the origin, are

$$Ax\xi + By\eta + Cz\zeta = 0 \dots\dots\dots (iii.), \quad \text{and} \quad \frac{\xi}{Ax} = \frac{\eta}{By} = \frac{\zeta}{Cz} \dots\dots\dots (iv.).$$

Eliminate  $x, y, z$  between (i.), (iii.), and (iv.).

**5667** The reciprocals of confocal cones are concyclic; that

is, have the same circular section; and the reciprocals of con-cyclic cones are confocal.

PROOF.—A series of con-cyclic cones is given by

$$Ax^2 + By^2 + Cz^2 + \lambda(x^2 + y^2 + z^2) = 0$$

by varying  $\lambda$ ; and the reciprocal cone is

$$\frac{x^2}{A+\lambda} + \frac{y^2}{B+\lambda} + \frac{z^2}{C+\lambda} = 0. \quad (5665)$$

**5668** The reciprocals of the enveloping cones of the series of confocal quadrics  $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1$ , with  $fgh$  for the common vertex,  $P$ , of the cones, are given by the equation

$$a^2x^2 + b^2y^2 + c^2z^2 - (fx + gy + hz)^2 + \lambda(x^2 + y^2 + z^2) = 0.$$

PROOF.—Let  $lmn$  be the direction of the perpendicular  $p$  from the origin upon the tangent plane drawn from  $P$  to the quadric. Equate the ordinary value of  $p^2$  at (5631) with that found by projecting  $OP$  upon  $p$ ; thus

$$(a^2 + \lambda)l^2 + (b^2 + \lambda)m^2 + (c^2 + \lambda)n^2 = (fl + gm + hn)^2.$$

Now  $p$  generates with vertex  $O$  a cone similar and similarly situated to the reciprocal cone with vertex  $P$ , and  $l, m, n$  are proportional to  $x, y, z$ , the coordinates of any point on the former cone. Therefore, by transferring the origin to  $P$ , the equation of the reciprocal cone is as stated.

**5669** COR.—These reciprocal cones are con-cyclic; and therefore the enveloping cones are confocal (5667).

**5670** The reciprocal cones in (5668) are all coaxal.

PROOF.—Transform the cone given by the terms in (5668) without  $\lambda$  to its principal axes; and its equation becomes  $Ax^2 + By^2 + Cz^2 = 0$ . Now, if the whole equation, including terms in  $\lambda$ , be so transformed,  $x^2 + y^2 + z^2$  will not be altered. Therefore we shall obtain

$$(A + \lambda)x^2 + (B + \lambda)y^2 + (C + \lambda)z^2 = 0,$$

a series of coaxal cones.

**5671** The axes of the enveloping cone are the three normals to the three confocals passing through its vertex.

PROOF.—The enveloping cone becomes the tangent plane at  $P$  for a con-focal through  $P$ , and one axis in this case is the normal through  $P$ . Also this axis is common to all the enveloping cones with the same vertex, by (5670). But there are three confocals through  $P$  (5658), and therefore three normals which must be the three axes of the enveloping cone.

**5672** The equation of the enveloping cone of the quadric

$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1$  is, when transformed to its principal axes,

$$\frac{x^2}{\lambda-\lambda_1} + \frac{y^2}{\lambda-\lambda_2} + \frac{z^2}{\lambda-\lambda_3} = 0 \quad \text{or} \quad \frac{x^2}{\lambda'} + \frac{y^2}{\lambda'+d_3^2} + \frac{z^2}{\lambda'+d_2^2} = 0,$$

where  $\lambda_1, \lambda_2, \lambda_3$  are the values of  $\lambda$  for the three confocals through  $P$ , the vertex, and  $d_2, d_3$  are the semi-axes of the diametral section of  $P$  in the first confocal (5661).

PROOF.—Transform equation (5668) of the reciprocal of the enveloping cone to its principal axes, as in (5670). Let  $\lambda_1, \lambda_2, \lambda_3$  be the values of  $\lambda$  which make the quadric become in turn the three confocal quadrics through  $P$ . Then the reciprocal  $(A+\lambda)x^2 + (B+\lambda)y^2 + (C+\lambda)z^2 = 0$  must become a right line in each case because the enveloping cone becomes a plane. Therefore one coefficient of  $x^2, y^2$ , or  $z^2$  must vanish. Hence  $A+\lambda_1 = 0, B+\lambda_2 = 0, C+\lambda_3 = 0$ . Therefore the reciprocal cone becomes

$$(\lambda-\lambda_1)x^2 + (\lambda-\lambda_2)y^2 + (\lambda-\lambda_3)z^2 = 0,$$

and therefore the enveloping cone is

$$\frac{x^2}{\lambda-\lambda_1} + \frac{y^2}{\lambda-\lambda_2} + \frac{z^2}{\lambda-\lambda_3} = 0.$$

## THE GENERAL EQUATION OF A QUADRIC.

**5673** This equation will be referred to as  $f(x, y, z) = 0$  or  $U = 0$ , and, written in full, is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2px + 2qy + 2rz + d = 0.$$

By introducing a fourth quasi variable  $t = 1$ , the equation may be put in the homogeneous form

$$\begin{aligned} \mathbf{5674} \quad & ax^2 + by^2 + cz^2 + dw^2 + 2fyz + 2gzx + 2hxy \\ & + 2p.xt + 2q.yt + 2r.zt = 0, \end{aligned}$$

abbreviated into

$$(a, b, c, d, f, g, h, p, q, r \chi x, y, z, t)^2 = 0,$$

as in (1620).

Transforming to an origin  $x'y'z'$  and coordinate axes parallel to the original ones, by substituting  $x' + \xi, y' + \eta, z' + \zeta$  for  $x, y$ , and  $z$ , the equation becomes, by (1514),



$$5675 \quad a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta \\ + \xi U_x + \eta U_y + \zeta U_z + U = 0,$$

where  $U = f(x', y', z')$  (omitting the accents).

**5676** The quadratic for  $r$ , the intercept between the point  $x'y'z'$  and the quadric surface measured on a right line drawn from  $x'y'z'$  in the direction  $lmn$ , is

$$r^2 (al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm) \\ + r (lU_x + mU_y + nU_z) + U = 0.$$

Obtained by putting  $\xi = rl$ ,  $\eta = rm$ ,  $\zeta = rn$  in (5674).

**5677** The tangents from any external point to a quadric are proportional to the diameters parallel to them.

PROOF.—From (5676), as in (1215) and (4317).

**5678** The equation of the tangent plane at a point  $xyz$  on the quadric is

$$(\xi - x) U_x + (\eta - y) U_y + (\zeta - z) U_z = 0$$

$$5679 \quad \text{or} \quad \xi U_x + \eta U_y + \zeta U_z + \tau U_t = 0,$$

with  $\tau$  and  $t$  made equal to unity after differentiating.

PROOF.—From (5676). Since  $xyz$  is a point on the surface, one root of the quadratic vanishes. In order that the line may now *touch* the surface, the other root must also vanish; therefore  $lU_x + mU_y + nU_z = 0$ . Put  $rl = \xi - x$ ,  $rm = \eta - y$ ,  $rn = \zeta - z$ ;  $\xi\eta\zeta$  being now a variable point on the line, and therefore on the tangent plane.

$$5680 \quad \text{Again,} \quad xU_x + yU_y + zU_z + tU_t = 2U, \text{ by (1624),} \\ \text{therefore} \quad xU_x + yU_y + zU_z = -tU_t,$$

which establishes the second form (5679).

**5681** Equation (5679) also represents the polar plane of any point  $xyz$  not lying on the quadric surface. Written in full it becomes

$$\begin{array}{ll} \xi (ax + hy + gz + p) & \text{or } x(a\xi + h\eta + g\zeta + p) \\ + \eta (hx + by + fz + q) & + y(h\xi + b\eta + f\zeta + q) \\ + \zeta (gx + fy + cz + r) & + z(g\xi + f\eta + c\zeta + r) \\ + px + qy + rz + d = 0, & + p\xi + q\eta + r\zeta + d = 0. \end{array}$$

**5683** That is, the forms

$$\xi U_x + \eta U_y + \zeta U_z + U = 0 \quad \text{and} \quad x U_\xi + y U_\eta + z U_\zeta + U = 0$$

are convertible,  $U$  standing for  $f(x, y, z)$  in the first, and for  $f(\xi, \eta, \zeta)$  in the second.

**5685** The intersection of the polar planes of two points is called the *polar line* of the points.

**5686** The polar plane of the vertex is the plane of contact of the tangent cone.

PROOF.—If  $\xi\eta\zeta$  be the vertex and  $xyz$  the point of contact, equation (5683) is satisfied. If  $x, y, z$  be the variables and  $\xi\eta\zeta$  constant, the second form of that equation shows that the points of contact all lie on the polar plane of the point  $\xi\eta\zeta$ .

**5687** Every line through the vertex is divided harmonically by the quadric and the polar plane.

PROOF.—In equation (5684) put  $x = \xi + Rl$ ,  $y = \eta + Rm$ ,  $z = \zeta + Rn$  to determine  $R$ , the distance from the vertex to the polar plane. This gives

$$R = \frac{-2U}{lU_\xi + mU_\eta + nU_\zeta}, \text{ employing (5680).}$$

Now, if  $r, r'$  are the roots of the quadratic (5676), with  $\xi, \eta, \zeta$  written for  $x, y, z$ , it appears that  $\frac{2rr'}{r+r'} = R$ , which proves the theorem.

**5688** Every line ( $lmn$ ) drawn through a point  $xyz$  parallel to the polar plane of that point is bisected at the point, and the condition of bisection is

$$lU_x + mU_y + nU_z = 0.$$

PROOF.—The equation is the condition for equal roots of opposite signs in the quadratic (5676). Since  $l, m, n$  are the dir. cos. of the line and  $U_x, U_y, U_z$  those of the normal of the polar plane (5683), the equation shows that the line and the normal are at right angles (5532).

**5689** The last, when  $x, y, z$  are the variables, is also the equation of the diametral plane conjugate to the direction  $lmn$ . Expanded it becomes

$$(al + hm + gn)x + (hl + bm + fn)y + (gl + fm + cn)z \\ + pl + qm + rn = 0.$$

For the point  $xyz$  moves, when  $x, y, z$  are variable, so that every diameter drawn through it parallel to  $lmn$  is bisected by it, and the locus is, by the form of the equation, a plane.

If the origin be at the centre of the quadric,  $p, q$ , and  $r$  of course vanish.

**5690** The coordinates of the centre of the general quadric  $U = 0$  (5673) are

$$x = \frac{\Delta'_p}{2\Delta}, \quad y = \frac{\Delta'_q}{2\Delta}, \quad z = \frac{\Delta'_r}{2\Delta}.$$

PROOF.—Every line through  $xyz$ , the centre, is bisected by it. The condition for this, in (5688), is  $U_x = 0$ ,  $U_y = 0$ , and  $U_z = 0$ , in order to be independent of  $lmn$ . The three equations in full are

$$\left. \begin{aligned} ax + hy + gz + p &= 0 \\ hx + by + fz + q &= 0 \\ gx + fy + cz + r &= 0 \end{aligned} \right\}; \text{ and } \Delta' = \begin{vmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & d \end{vmatrix}; \quad \Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

Solve by (582).

**5691** The quadric transformed to the centre becomes

$$a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta + \frac{\Delta'}{\Delta} = 0.$$

PROOF.—By the last theorem, the terms involving  $\xi, \eta, \zeta$  in (5675) vanish. The value of  $U$  or  $f(x, y, z)$ , when  $xyz$  is the centre, appears as follows:—

$$U = \frac{1}{2}U_t \text{ (5680)} = px + qy + rz + d = \frac{p\Delta'_p + q\Delta'_q + r\Delta'_r}{2\Delta} + d \text{ (5690)} = \frac{\Delta'}{\Delta} \text{ (1647)}.$$

The last equation, being again transformed by turning the axes so as to remove the terms involving products of coordinates, becomes

**5692** 
$$ax^2 + \beta y^2 + \gamma z^2 + \frac{\Delta'}{\Delta} = 0,$$

**5693** where  $a, \beta, \gamma$  are the roots of the discriminating cubic

$$R^3 - R^2(a + b + c) + R(bc + ca + ab - f^2 - g^2 - h^2) - \Delta = 0,$$

$$\text{or } (R - a)(R - b)(R - c) - (R - a)f^2 - (R - b)g^2 - (R - c)h^2 - 2fgh = 0.$$

PROOF.—It has been shown, in (1847-9), that the roots of the discriminating cubic (multiplied in this case by  $-\frac{\Delta}{\Delta'}$ ) are the reciprocals of the maximum and minimum values of  $x^2 + y^2 + z^2$ . But such values are evidently

the squares of the axes of the quadric surface. Let the central equation of the surface be  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ . Therefore  $\frac{1}{a^2} = -\frac{\Delta_a}{\Delta'}$ , &c., producing the equation above.

**5694** The equations of the new axis of  $x$  referred to the old axes of  $\xi, \eta, \zeta$  are

$$(F+af)x = (G+ag)y = (H+ah)z;$$

and similar equations with  $\beta$  and  $\gamma$  for the  $y$  and  $z$  axes.

PROOF.—When  $lmn$ , in (5689), is a principal diameter of the quadric, the diametral plane becomes perpendicular to it, and therefore the coefficients of  $x, y, z$  must be proportional to  $l, m, n$ . Putting them equal to  $Rl, Rm, Rn$  respectively, we have the equations

$$\left. \begin{aligned} (a-R)l + hm + gn &= 0 \dots\dots (1) \\ hl + (b-R)m + fn &= 0 \dots\dots (2) \\ gl + fm + (c-R)n &= 0 \dots\dots (3) \end{aligned} \right\} \begin{array}{l} \text{The eliminant of these equations is} \\ \text{the discriminating cubic in } R \text{ al-} \\ \text{ready obtained in (5693).} \end{array}$$

From (1) and (2),  $l : m = hf - g(b-R) : gh - f(a-R)$ ,  
and from (2) and (3),  $m : n = fg - h(c-R) : hf - g(b-R)$ ;  
therefore  $(gh - af + Rf)l = (hf - bg + Rg)m = (fg - ch + Rh)n$ ,  
which establish the equations, since  $x : y : z = l : m : n$  and  $F = gh - af$ ,  
&c., as in (4665).

**5695** The direction cosines of the axes of the quadric.

If the discriminating cubic be denoted by  $\phi(R) = 0$ , and its roots by  $a, \beta, \gamma$ ; the direction cosines of the first axis are

$$\sqrt{-\frac{\phi_a(a)}{\phi_a(a)}}, \quad \sqrt{-\frac{\phi_b(a)}{\phi_a(a)}}, \quad \sqrt{-\frac{\phi_c(a)}{\phi_a(a)}}.$$

For the second and third axes write  $\beta$  and  $\gamma$  in the place of  $a$ .

PROOF.—Let  $F+af=L, G+ag=M, H+ah=N \dots\dots\dots$  (i.),  
 $(a-b)(a-c)-f^2=\lambda, (a-c)(a-a)-g^2=\mu, (a-a)(a-b)-h^2=\nu \dots\dots$  (ii.).  
Then the equation  $\phi(a)=0$  may be put in either of the forms

$$L^2 = \mu\nu, \quad M^2 = \nu\lambda, \quad N^2 = \lambda\mu \dots\dots\dots$$
 (iii.).

Now the dir. cos. of the first axis are, by (5694), proportional to

$$\frac{1}{L} : \frac{1}{M} : \frac{1}{N} = \sqrt{\lambda} : \sqrt{\mu} : \sqrt{\nu}, \text{ by (iii.).}$$

Their values are, therefore,

$$\frac{\sqrt{\lambda}}{\sqrt{(\lambda+\mu+\nu)}}, \quad \frac{\sqrt{\mu}}{\sqrt{(\lambda+\mu+\nu)}}, \quad \frac{\sqrt{\nu}}{\sqrt{(\lambda+\mu+\nu)}}.$$

But  $\lambda = -\frac{d\phi(a)}{da}$  and  $\lambda+\mu+\nu = \frac{d\phi(a)}{da}$ , by actual differentiation of the cubic in (5693).

**5696** Cauchy's proof that the roots of the discriminating cubic (5693) are all real will be found at (1850).

**5697** The equation of the enveloping cone, vertex  $xyz$ , of the general quadric surface  $U = 0$  (5673) is

$$4(abc fgh \chi lmn)^2 U = (lU_x + mU_y + nU_z)^2,$$

with  $\xi = x$ ,  $\eta = y$ ,  $\zeta = z$  substituted for  $l, m, n$ .

PROOF.—The generating line through  $xyz$  moves so as to touch the quadric. Hence the quadratic in  $r$  (5676) must have equal roots. The equation admits of some reduction.

**5698** When  $U$  takes the form  $ax^2 + by^2 + cz^2 = 1$ , equation (5697) becomes

$$(al^2 + bm^2 + cn^2)(ax^2 + by^2 + cz^2 - 1) = (alx + bmy + cnz)^2.$$

**5699** The condition that the general quadric equation may represent a cone is  $\Delta' = 0$ ; that is, the discriminant of the quaternary quadric, (5674) or (1644), must vanish.

PROOF.—By (5692). Otherwise  $\Delta' = 0$  is the eliminant of the four equations  $U_x = 0$ ,  $U_y = 0$ ,  $U_z = 0$ ,  $U = 0$ , the condition that equation (5675) may represent a cone.

**5700** The condition that the plane  $lx + my + nz = 0$  may touch the cone  $(abc fgh \chi xyz)^2 = 0$  is the determinant equation on the right.

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & \end{vmatrix} = 0.$$

PROOF.—Equate the coefficients  $l, m, n$  to those of the tangent plane (5681),  $p, q, r$  being zero, and  $xyz$  the point of contact. A fourth equation is  $lx + my + nz = 0$ , which holds at the point of contact. The eliminant of the four equations is the determinant above.

**5701** The condition that the plane  $lx + my + nz + t = 0$  may touch the quadric  $(abcd fgh pqr \chi xyz 1)^2 = 0$  (5673) is the determinant equation on the right.

$$\begin{vmatrix} a & h & g & p & l \\ h & b & f & q & m \\ g & f & c & r & n \\ p & q & r & d & t \\ l & m & n & t & \end{vmatrix} = 0.$$

PROOF.—As in (5700).

**5702** If the origin is at the centre,  $p = q = r = 0$ . In that case, transposing the last two rows and last two columns, the determinant becomes

$$\begin{vmatrix} a & h & g & l & 0 \\ h & b & f & m & 0 \\ g & f & c & n & 0 \\ l & m & n & 0 & t \\ 0 & 0 & 0 & t & d \end{vmatrix} = 0, \quad \text{or,} \quad d \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & \end{vmatrix} = t^2 \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

**5703** The condition that the line of intersection of the planes

$$lx + my + nz + t = 0 \dots (i.) \quad \text{and} \quad l'x + m'y + n'z + t' = 0 \dots (ii.)$$

may touch the general quadric  $(abcdfghpqr \chi xyz1)^2 = 0$ , is the determinant equation deduced below.

Multiply equation (i.) by  $\xi$  and (ii.) by  $\eta$  to obtain the plane

$$(l\xi + l'\eta)x + (m\xi + m'\eta)y + (n\xi + n'\eta)z + t\xi + t'\eta = 0 \dots\dots\dots (iii.),$$

passing through the intersection of (i.) and (ii.). The line of intersection will touch the quadric if (iii.) coincides with the tangent plane at a point  $xyz$ , and if  $xyz$  be also on (i.) and (ii.). Therefore, equating coefficients of (iii.) and the tangent plane at  $xyz$  (5681), we get the six following equations, the eliminant of which furnishes the required condition,

$$\left. \begin{aligned} ax + hy + gz + pw &= l\xi + l'\eta \\ hx + by + fz + qw &= m\xi + m'\eta \\ gx + fy + cz + rw &= n\xi + n'\eta \\ px + qy + rz + dw &= t\xi + t'\eta \\ lx + my + nz + tw &= 0 \\ l'x + m'y + n'z + t'w &= 0 \end{aligned} \right\}, \quad \therefore \begin{vmatrix} a & h & g & p & l & l' \\ h & b & f & q & m & m' \\ g & f & c & r & n & n' \\ p & q & r & d & t & t' \\ l & m & n & t & & \\ l' & m' & n' & t' & & \end{vmatrix} = 0.$$

## RECIPROCAL POLARS.

**5704** The method of reciprocal polars explained at page 665 is equally applicable to geometry of three dimensions.

Taking poles and polar planes with respect to a sphere of reciprocation, we have the following rules analogous to those on page 666.



## RULES FOR RECIPROCATING.

**5705** *A plane becomes a point.*

**5706** *A plane at infinity becomes the origin.*

**5707** *Several points on a straight line become as many planes passing through another straight line. These lines are called reciprocal lines.*

**5708** *Points lying on a plane become planes passing through a point, the pole of the plane.*

**5709** *Points lying on a surface become planes enveloping the reciprocal surface.*

**5710** *Therefore, by rules (5708) and (5709), the points in the intersection of the plane and a surface become planes passing through the pole of the plane and enveloped both by the reciprocal surface and by its tangent cone.*

**5711** *When the intersecting plane is at infinity, the vertex of the tangent cone is the origin.*

**5712** *Therefore the asymptotic cone of any surface is orthogonal to the tangent cone drawn from the origin to the reciprocal surface. The cones are therefore reciprocal.*

**5713** *The reciprocal surface of the quadric is a hyperboloid, an ellipsoid, or a paraboloid, according as the origin is without, within, or upon the quadric surface.*

**5714** *The angle subtended at the origin by two points is equal to the angle between their corresponding planes.*

**5715** *The reciprocal of a sphere is a surface of revolution of the second order.*

**5716** *The shortest distance between two reciprocal lines passes through the origin.*

**5717** *The reciprocal surface of the general quadric  $(abcdfghpqr\chi xyz1)^2 = 0$  (5674), the auxiliary sphere being  $x^2 + y^2 + z^2 = k^2$ , is*

$$\begin{vmatrix} a & h & g & p & \xi \\ h & b & f & q & \eta \\ g & f & c & r & \zeta \\ p & q & r & d & -k^2 \\ \xi & \eta & \zeta & -k^2 & \end{vmatrix} = 0, \quad \text{or, if } p = q = r = 0, \quad d \begin{vmatrix} a & h & g & \xi \\ h & b & f & \eta \\ g & f & c & \zeta \\ \xi & \eta & \zeta & 0 \end{vmatrix} - k^4 \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

PROOF.—The polar plane of the point  $\xi\eta\zeta$  with respect to the sphere is  $\xi x + \eta y + \zeta z - k^2 = 0$ . This must touch the given surface, and the condition is given in (5701).

**5718** The reciprocal surface of the central quadric  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ , when the origin of reciprocation is the point  $x'y'z'$ , is

$$(\xi x' + \eta y' + \zeta z' - k^2)^2 = a^2 \xi^2 + b^2 \eta^2 + c^2 \zeta^2,$$

or, with the origin at the centre,

**5719** 
$$a^2 \xi^2 + b^2 \eta^2 + c^2 \zeta^2 = k^4.$$

PROOF.—Let  $p$  be the perpendicular from  $x'y'z'$  upon a tangent plane of the quadric, and  $\xi\eta\zeta$  the point where  $p$  produced, intersects the reciprocal surface at a distance  $\rho$  from  $x'y'z'$ . Then

$$k^2 \rho^{-1} = p = lx' + my' + nz' - \sqrt{(a^2 l^2 + b^2 m^2 + c^2 n^2)}. \quad (5630)$$

Multiplying by  $\rho$  produces the desired equation.

## THEORY OF TORTUOUS CURVES.

**5721** DEFINITIONS.—The *osculating plane* at any point of a curve of double curvature, or *tortuous curve*,\* is the plane containing either two consecutive tangents or three consecutive points.

**5722** The *principal normal* is the normal in the osculating plane. The radius of *circular curvature* coincides with this normal in direction.

**5723** The *binormal* is the normal perpendicular both to the tangent and principal normal at the point.

**5724** The *osculating circle* is the circle of curvature in the osculating plane, and its centre, which is the centre of circular curvature, is the point in which the osculating plane intersects two consecutive normal planes of the curve.

**5725** The *angle of contingence*,  $d\psi$ , is the angle between two consecutive tangents or principal normals. The *angle of torsion*,  $d\tau$ , is the angle between two consecutive osculating planes.

\* Otherwise named "space curve."

**5726** The *rectifying plane* at any point on the curve is perpendicular to the principal normal; and the intersection of two consecutive rectifying planes is the *rectifying line* and axis of the osculating cone.

**5727** The *osculating cone* is a circular cone touching three consecutive osculating planes and having its vertex at their point of intersection.

The *rectifying developable* is the envelope of the rectifying planes, and is so named because the curve, being a geodesic on this surface, would become a straight line if the surface were developed into a plane.

**5728** The *polar developable* is the envelope of the normal planes, being the locus of the line of intersection of two consecutive normal planes. Three consecutive normal planes intersect in a point which is the centre of *spherical curvature*: for a sphere having that centre may be described passing through four consecutive points of the curve.

**5729** The *edge of regression* is the locus of the centre of spherical curvature.

**5730** The *rectifying surface* is the surface of centres (5773) of the polar developable.

**5731** An *evolute* of a curve is a geodesic line on the polar developable. It is the line in which a free string would lie if stretched between two points, one on the curve and one anywhere on the smooth surface of the polar developable.

**5732** In Figure (180)  $A, A', A'', A'''$  are consecutive points on a curve. The normal planes drawn through  $A$  and  $A'$  intersect in  $CE$ ; those through  $A'$  and  $A''$  in  $C'E'$ , and those through  $A''$  and  $A'''$  in  $C''E''$ .  $CE$  meets  $C'E'$  in  $E$ , and  $C'E'$  meets  $C''E''$  in  $E'$ . The principal normals in the normal planes are  $AC, A'C', A''C''$ , and these are also the radii of curvature at  $A, A', A''$ , while  $C, C', C''$  are the centres of curvature.  $\angle ACA' = d\psi$  and  $CA'C' = dr$ .

The surface  $ECC'C''E'$  is the polar developable,  $CC'C''$  being the locus of the centres of curvature, and  $EE'E''$  is the edge of regression.

$EA$  is the radius and  $E$  the centre of spherical curvature for the point  $A$ .  $hH, H'H', H''H''$  are elemental chords of an evolute of the curve,  $AhH$  being a normal at  $A$ , and  $A'H'H'$  a normal at  $A'$ , and so on. The first normal drawn is arbitrary, but it determines the position of all the rest.

## PROPERTIES OF A TORTUOUS CURVE.

**5733** The equation of the osculating plane at a point  $xyz$  on the curve is

$$(\xi - x)\lambda + (\eta - y)\mu + (\zeta - z)\nu = 0.$$

**5734**  $\lambda, \mu, \nu$  are the direction cosines of the binormal, and their complete values are

$$\rho(y_s z_{2s} - y_{2s} z_s), \quad \rho(z_s x_{2s} - z_{2s} x_s), \quad \rho(x_s y_{2s} - x_{2s} y_s).$$

**5735** The angle of contingence

$$d\psi = \sqrt{\{(y_s z_{2s} - y_{2s} z_s)^2 + (z_s x_{2s} - z_{2s} x_s)^2 + (x_s y_{2s} - x_{2s} y_s)^2\}} ds.$$

PROOF.—Let the direction of a tangent be  $lmn$ , and that of a consecutive tangent  $l+dl, m+dm, n+dn$ . Since the normal of the plane must be perpendicular to both these lines, we shall have, by (5532),

$$l\lambda + m\mu + n\nu = 0 \quad \text{and} \quad (l+dl)\lambda + (m+dm)\mu + (n+dn)\nu = 0,$$

therefore  $\lambda : \mu : \nu = mdn - ndm : ndl - ldn : ldm - mdl$ ,

and the denominator in the complete values of  $\lambda, \mu, \nu$  is

$$\sqrt{\{(mdn - ndm)^2 + \&c.\}} = \sin d\psi,$$

by (5521); that is,  $= d\psi$ . Also  $l, m, n = x_s, y_s, z_s$  and  $dl = x_{2s} ds$ , &c.

Therefore  $\lambda = (y_s z_{2s} - y_{2s} z_s) \frac{ds}{d\psi}$ . Similarly,  $\mu$  and  $\nu$ ; and  $s_s = \rho$ , by (5146).

**5736** The radius of curvature  $\rho$  at a point  $xyz$ .

$$\frac{1}{\rho^2} = x_{2s}^2 + y_{2s}^2 + z_{2s}^2 = \frac{x_{2t}^2 + y_{2t}^2 + z_{2t}^2 - s_{2t}^2}{s_t^4}.$$

PROOF:  $d\psi = \sqrt{\{(y_s z_{2s} - y_{2s} z_s)^2 + \&c.\}} ds$ , in (5735),

therefore  $\psi_s = \sqrt{\{(x_s^2 + y_s^2 + z_s^2)(x_{2s}^2 + y_{2s}^2 + z_{2s}^2) - (x_s x_{2s} + y_s y_{2s} + z_s z_{2s})^2\}}$   
 $= \sqrt{(x_{2s}^2 + y_{2s}^2 + z_{2s}^2)}$ ; since  $x_s^2 + y_s^2 + z_s^2 = 1$ ;

and differentiating this equation makes  $x_s x_{2s} + \&c. = 0$ .

Otherwise, geometrically, precisely as in the proof of (5141), we find the direction cosines of the principal normal to be

**5737**  $\cos \alpha = \rho x_{2s}, \quad \cos \beta = \rho y_{2s}, \quad \cos \gamma = \rho z_{2s}.$

Therefore  $\rho^2 (x_{2s}^2 + y_{2s}^2 + z_{2s}^2) = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$

The change to the independent variable  $t$  is made by (1762).

**5738** The angle of torsion, in terms of  $\lambda, \mu, \nu$  of (5734), is

$$d\tau = \sqrt{(\lambda_s^2 + \mu_s^2 + \nu_s^2)} ds = (\lambda x_{3s} + \mu y_{3s} + \nu z_{3s}) \rho ds$$

$$= \sqrt{\{(\mu\nu_s - \mu_s\nu)^2 + (\nu\lambda_s - \nu_s\lambda)^2 + (\lambda\mu_s - \lambda_s\mu)^2\}}.$$

5 H

PROOF.—By (5745), we have  $(d\tau)^2 = (d\lambda)^2 + (d\mu)^2 + (d\nu)^2$  ..... (i), which gives the first form. The third reduces to this by the method in (5736). For the second form put  $u = y_1 z_2 - y_2 z_1$ , &c., then

$$\frac{\lambda}{u} = \frac{\mu}{v} = \frac{\nu}{w} = \frac{1}{K} = \frac{ds}{d\psi} \quad (5734), \quad d\lambda = \frac{du}{K} - \frac{u dK}{K^2}, \text{ \&c.}$$

Substitute in (i.), reducing by  $K^2 = u^2 + v^2 + w^2$  and  $K dK = u du + v dv + w dw$ .

### CURVATURE AND TORTUOSITY.

**5739** Radius of curv.,  $\rho = \frac{ds}{d\psi}$ ; Curvature  $= \frac{1}{\rho} = \frac{d\psi}{ds}$ .

Radius of torsion,  $\sigma = \frac{ds}{d\tau}$ ; Tortuosity  $= \frac{1}{\sigma} = \frac{d\tau}{ds}$ .

If  $\tau$ , changes sign while passing through the values zero or infinity, there is a point of *inflected torsion* or a *cuspidal point*, respectively. If  $\tau$ , without changing sign, passes through zero or infinity, there is a point of *suspended torsion* or *infinite torsion* respectively.

If  $\tau$ , is zero, identically, the curve is plane.

**5740** The radius of spherical curvature,

$$R = \sqrt{(\rho^2 + \rho_\tau^2)}.$$

PROOF.—In Fig. (180)  $R^2 = \rho^2 + EC^2$  and  $EC = \rho_\tau$  by analogy with  $q = p_\phi$  in a plane curve (see proof of 5147).

**5741** The element of arc of the locus of centres of circular curvature is

$$ds' = R d\tau, \text{ and therefore } R = s'_\tau.$$

PROOF.—In Fig. (180)  $ds' = CC' = \rho d\tau \sec \phi = R d\tau$ .

**5742** The radius of curvature of the edge of regression

$$= S'' = R R_\rho = \rho + \rho_2,$$

$S''$  being the arc of the edge of regression.

PROOF.—An inspection of Figure (180) shows that  $R$  and  $\rho$  stand in the same relation to the edge of regression that  $r$  and  $p$  occupy with regard to a curve in the standard formula. In fact we may substitute  $R$  for  $r$ ,  $\rho$  for  $p$ ,

$\phi$  for  $\theta$ ,  $\tau$  for  $\psi$ , and  $\phi$  remains  $\phi$ . The chosen line of reference  $AB$  being always parallel to the tangent  $EC$ , then  $AEC = BAE = \phi$ . Also the angle of contingence  $CEC' = CAC' = d\tau$ , by the right angles at  $C$  and  $C'$ . Accordingly, we have the formula  $\rho = s_{\phi} = rr_p = p + p_{\phi}$  from (5146-8), and the values above corresponding to them.

**5743** *A method of estimating the variation in direction of a right line whose position is given as depending upon the form of a tortuous curve at every point.*

Let  $x, y, z$  be the direction cosines of the line referred to a fixed principal normal, tangent, and binormal of the curve [ $x, y, z$  may either be constants with respect to the varying principal normal, tangent, and binormal, or they may be functions of the angle between the binormal and the spherical radius].

**5744** The complete changes in  $x, y, z$ , with respect to the fixed origin and axes, will be

$$\delta x = dx + y d\psi - z d\tau,$$

$$\delta y = dy - x d\psi,$$

$$\delta z = dz + x d\tau.$$

PROOF.—In Figure (180)  $AC, AB$  are the fixed axes of  $x$  and  $z$ . Let a line  $AL$  of unit length be drawn always parallel to the line in question; then, if  $x, y, z$  be the coordinates of  $L$ ,  $x, y, z$  will also be the direction cosines of  $AL$ , and therefore of the given line.

Now, suppose  $A$  to move to  $A'$ , and consequently  $AL$  to take the position  $A'L'$ . Then the changes in  $x, y, z$  will be the changes  $dx, dy, dz$  relatively to the moving axes, plus the changes due to the rotations  $d\psi$  round the binormal and  $d\tau$  round the tangent. With the usual notation, we shall have

$$\delta x = dx + \omega_2 z - \omega_3 y, \quad \delta y = dy + \omega_3 x - \omega_1 z, \quad \delta z = dz + \omega_1 y - \omega_2 x,$$

with  $\omega_1 = 0$ ,  $\omega_2 = -d\tau$ ,  $\omega_3 = -d\psi$ .

**5745** If  $d\chi$  be the angular change in the direction of the right line,

$$d\chi = \sqrt{(\delta x)^2 + (\delta y)^2 + (\delta z)^2}.$$

For  $d\chi = LL'$  since  $AL$  is a unit length.

#### EXAMPLES.

**5746** The angle between two consecutive radii of circular curvature being  $d\epsilon$ ,

$$(d\epsilon)^2 = (d\psi)^2 + (d\tau)^2.$$



PROOF.—Here, in (5744),  $x = 1$ ,  $y = 0$ ,  $z = 0$ , therefore  $\delta x = 0$ ,  $\delta y = -d\psi$ ,  $\delta z = d\tau$ . Substitute these values in (5745).

**5747** The angle,  $d\eta$ , between two consecutive radii of spherical curvature,  $\phi$  being the inclination to the binormal,

$$(d\eta)^2 = (d\psi \cdot \sin \phi)^2 + (d\phi - d\tau)^2.$$

PROOF.—In (5744) the direction cosines of  $R$  (Fig. 180) are  $x = \sin \phi$ ,  $y = 0$ ,  $z = \cos \phi$ , therefore  $\delta x = \cos \phi (d\phi - d\tau)$ ,  $\delta y = -d\psi \sin \phi$ ,  $\delta z = -\sin \phi (d\phi - d\tau)$ . Substitute in (5745).

**5748** The angle of contingence of the locus of the centres of circular curvature,

$$(d\chi)^2 = (d\psi \cdot \cos \phi)^2 + (d\phi + d\tau)^2.$$

PROOF.—The dir. cos. of the tangent at  $C$  to the locus (Fig. 177) are  $x = \cos \phi$ ,  $y = 0$ ,  $z = \sin \phi$ ; therefore  $\delta x = -\sin \phi (d\phi + d\tau)$ ,  $\delta y = -d\psi \cos \phi$ ,  $\delta z = \cos \phi (d\phi + d\tau)$ . Substitute in (5745).

**5749** The osculating plane of the same curve has its direction cosines in the ratios

$$\frac{d\psi}{d\chi} \sin \phi \cos \phi : -\left(\frac{d\phi}{d\chi} + \frac{d\tau}{d\chi}\right) : -\frac{d\psi}{d\chi} \cos^2 \phi.$$

PROOF.—As in the Proof of (5735), the dir. cos. of the normal to this plane are proportional to  $y\delta z - z\delta y$ ,  $z\delta x - x\delta z$ ,  $x\delta y - y\delta x$ . Substitute the values in last proof.

**5750** The angle of torsion of the same curve is found from (5745) and (5744) as above,  $x$ ,  $y$ ,  $z$  being in this case the dir. cos. of the normal of the osculating plane as given in (5749).

**5751** The direction cosines of the rectifying line are

$$0, \quad \frac{d\tau}{d\epsilon}, \quad \frac{d\psi}{d\epsilon}.$$

PROOF.—The rectifying plane at  $A'$  (Fig. 180) is perpendicular to the normal  $A'C'$ . Therefore its equation is  $x - yd\psi + zd\tau = 0$ . The ultimate intersection of this plane with the rectifying plane at  $A$  (that is, the plane of  $yz$ ) is the rectifying line. Hence the equation of the latter is  $y d\psi = z d\tau$ ; and the dir. cosines reduce to the above by (5746).

**5752** COR.—The vertical angle of the osculating cone

$$= 2 \tan^{-1} \frac{d\psi}{d\tau}.$$

**5753** The angle of torsion of the involute of the curve is

$$= \sqrt{(\psi_{2\epsilon}^2 + \tau_{2\epsilon}^2)} d\epsilon.$$

PROOF.—This angle is also the angle between two consecutive rectifying lines. Therefore, taking the dir. cosines from (5751), we must put in (5744)

$$x = 0, \quad y = \frac{d\tau}{d\epsilon}, \quad z = \frac{d\psi}{d\epsilon};$$

therefore  $\delta x = \frac{d\tau}{d\epsilon} d\psi - \frac{d\psi}{d\epsilon} d\tau = 0; \quad \delta y = \tau_{2\epsilon} d\epsilon; \quad \delta z = \psi_{2\epsilon} d\epsilon.$

**5754** The angle of torsion of an evolute of the curve

$$= d\psi \sin(\alpha - \tau).$$

PROOF.—(Fig. 180.) Let  $HH'H''$  be an evolute of the curve,  $AH$  the tangent to it in the normal plane of the original curve at  $A$ , and let  $\alpha = CAH$ , the inclination of  $AH$  to the principal normal. At any other point  $H''$  of the evolute, where its tangent is  $A'H'H''$ , let the corresponding angle be  $\theta = C'A'H''$ . Then  $\theta = \alpha - \tau$ ,  $\tau$  being the sum of the angles of torsion between  $A$  and  $A'$ , or the total amount of twist of the osculating plane. Now the normal of the osculating plane of the evolute at  $H''$  is perpendicular to  $HH'$  and  $H'H''$ , two consecutive tangents. Therefore its dir. cosines in (5744) must be

$$x = -\sin(\alpha - \tau), \quad y = 0, \quad z = \cos(\alpha - \tau);$$

therefore  $\delta x = \cos(\alpha - \tau) d\tau + 0 - \cos(\alpha - \tau) d\tau = 0,$

$$\delta y = \sin(\alpha - \tau) d\psi; \quad \delta z = \sin(\alpha - \tau) d\tau - \sin(\alpha - \tau) d\tau = 0.$$

Hence the angle required  $= \delta y = d\psi \sin(\alpha - \tau).$

**5755** Approximate values of the coordinates of a point on a tortuous curve near to the origin in terms of the arc, the axes of  $x, y, z$  being the principal normal, tangent, and binormal, and the arc  $s$  being measured from the origin:

$$x = \frac{s^2}{2\rho} - \frac{s^3\rho_s}{6\rho^2} - \frac{s^4}{24\rho^3} (1 - 2\rho_s^2 + \rho\rho_{2s}) + \&c.,$$

$$y = s - \frac{s^3}{6\rho^2} + \frac{s^4\rho_s}{8\rho^3} + \&c., \quad z = \frac{s^3}{6\rho\sigma} - \frac{s^4}{24} \left( \frac{\sigma_s}{\rho\sigma^2} + \frac{2\rho_s}{\rho^2\sigma} \right) + \&c.,$$

$\rho$  and  $\sigma$  being respectively the radii of circular curvature and torsion.

PROOF.—By Taylor's theorem (1500), since  $x, y, z, s$  are the same as  $dx, dy, dz, ds$  initially, we have  $x = x_s + \frac{1}{2}x_{ss}s^2 + \frac{1}{6}x_{sss}s^3 + \&c.$ , and similar expansions for  $y$  and  $z$ . The dir. cosines of the principal normal at the point  $xyz$  will be, from (5737),

$$\cos(-\psi) = \rho x_{ss}, \quad \cos\left(\frac{\pi}{2} + \psi\right) = \rho y_{ss}, \quad \cos\left(\frac{\pi}{2} - \tau\right) = \rho z_{ss};$$

$\psi \equiv d\psi$  and  $\tau \equiv d\tau$  being estimated positive as drawn in Figure (180) for positive values of  $x, y, z$ .

Differentiate these equations for  $s$ , and in the results put the initial values

$$x_s = z_s = 0, \quad y_s = 1, \quad \psi = \tau = 0, \quad \psi_s = \frac{1}{\rho}, \quad \tau_s = \frac{1}{\sigma}, \quad \&c.,$$

to determine the derivatives in the above expansions.

### THE HELIX.

**5756** The *helix* is a curve traced on a cylinder of radius  $a$ , so that its tangent preserves a constant inclination,  $= \frac{1}{2}\pi - a$ , to the axis. Taking the axis of the cylinder for the  $z$  axis of coordinates, the equations of the helix are

$$x = a \cos \theta, \quad y = a \sin \theta, \quad z = a\theta \tan a.$$

**5757** The radius of curvature  $\rho = a \sec^2 a$ .

**5758** The radius of torsion  $\sigma = 2a \operatorname{cosec} 2a$ .

PROOF.— $\rho$  from (5806); since  $\rho_1 = a$ ,  $\rho_2 = \infty$ , and  $\theta = a$  at every point. By (5739),  $\sigma = s_\tau$ . But  $dz = ds \sin a$  and  $ad\tau = dz \cos a$ .

**5759** The helix of closest contact with a given curve may be found as follows.

Determine the constants  $a$  and  $a$  from equations (5757–8), with the known values of  $\rho$  and  $\sigma$  for the given curve; then place the helix to have a common tangent with the curve at the point, and make the osculating planes coincide.

### GENERAL THEORY OF SURFACES.

**5770** DEFINITIONS.—A *tangent plane* passes through three consecutive points on a surface which are not in the same right line.

**5771** The *normal* at any point of a surface is perpendicular to the tangent plane.

**5772** A *normal plane* is any plane through the normal.

**5773** A *line of curvature* on a surface is a line along which consecutive normals to the surface intersect. At every point of a surface there are usually two lines of curvature at right angles to each other; and to these correspond two principal radii of curvature. The two lines of curvature coincide with the principal axes of the indicatrix at the point. See (5778).

**5774** The *surface of centres* is the locus of the centres of principal curvature. There are two such surfaces, for there are two centres on each normal, and the normal is a tangent to both surfaces. Either surface may be regarded as generated by the evolutes of the lines of principal curvature.

**5775** A *geodesic* is a line traced on a surface along which the osculating plane at every point contains the normal to the surface. See (5779).

**5776** The *radius of geodesic curvature* \* of a curve traced on a surface is measured by the ratio of the element of arc of the curve to the angle between consecutive normal sections of the surface drawn through consecutive tangents of the curve. *Geodesic curvature*, being the reciprocal of this, is therefore the rate of angular deviation of the normal section per unit length of the curve.

**5777** An *umbilicus* is a point on a surface where a section parallel to and close to the tangent plane is a circle; in other words, the indicatrix is a circle.

For a definition of *Indicatrix*, see (5795).

**5778** In Figure (182)  $OCD$  is the normal at  $O$  to a curved surface;  $AOA'$ ,  $BOB'$  are the lines of curvature, therefore the normals to the surface at  $A$  and  $O$  intersect in the centre of curvature radius  $\rho_1$  (5773), and the normals at  $B$  and  $O$ , in the centre, radius  $\rho_2$ . The normals to the line of curvature  $BOB'$  at  $B$  and  $O$ , drawn in the osculating plane  $BOB'$ , intersect in  $K$ , and those at  $B'$  and  $O$  intersect in  $H$ .  $HOD$  is the angle between the osculating plane of the line of curvature and the plane of normal section. Similarly for the line of curvature  $AOA'$ .

**5779** If  $POP'$  be a geodesic, its osculating plane  $POP'$  contains  $OD$  the normal to the surface at  $O$ , and therefore  $\rho = OD$ , the radius of curvature of this section at  $O$ ; but  $PE$ , the normal to the surface at  $P$ , does not intersect  $OD$ , the consecutive normal at  $O$ , unless the geodesic coincides with one of the lines of curvature,  $OA$  or  $OB$ . The angle  $DPE$  is the angle of torsion which vanishes in the latter case.

\* Not to be confounded with the radius of curvature of a geodesic.

## GENERAL EQUATION OF A SURFACE.

**5780** Let the general equation of a surface be represented by  $\phi(x, y, z) = 0$ .

**5781** The equations of any tangent at a point  $xyz$  are

$$\frac{\xi-x}{l} = \frac{\eta-y}{m} = \frac{\zeta-z}{n}, \quad \text{with} \quad l\phi_x + m\phi_y + n\phi_z = 0.$$

PROOF.—At an adjacent point  $x+rl, y+rm, z+rn$ , we have

$$\phi(x+rl, y+rm, z+rn) = 0,$$

therefore, by (1514),  $\phi(x, y, z) + r(l\phi_x + m\phi_y + n\phi_z) = 0$ ,

the rest vanishing in the limit. But  $\phi(x, y, z) = 0$ , therefore

$$l\phi_x + m\phi_y + n\phi_z = 0.$$

But  $l, m, n$  are the direction cosines of the line joining the two points, which becomes a tangent in the limit; and if  $\xi\eta\zeta$  be any point on this line distant  $\rho$  from  $xyz$ ,  $\xi-x = \rho l$ ,  $\eta-y = \rho m$ ,  $\zeta-z = \rho n$ , &c.

**5782** The equation of the tangent plane at  $xyz$  is

$$(\xi-x)\phi_x + (\eta-y)\phi_y + (\zeta-z)\phi_z = 0.$$

PROOF.—Eliminate  $l, m, n$  from  $l\phi_x + m\phi_y + n\phi_z = 0$  by  $\xi-x = \rho l$ , &c., as above.

## TANGENT LINE AND CONE AT A SINGULAR POINT.

**5783** If, in the expansion in (5781) by Taylor's theorem, all the derivatives of  $\phi(x, y, z)$  of an order up to  $n$  inclusive vanish, we have

$$\phi(x+rl, y+rm, z+rn) = \phi(x, y, z) + \frac{r^{n+1}}{n+1} (ld_x + md_y + nd_z)^{n+1} \phi(x, y, z) = 0.$$

There are in this case  $n+2$  coincident points at  $xyz$  in the direction  $lmn$ , and since the equation  $(ld_x + md_y + nd_z)^{n+1} \phi(x, y, z) = 0$  is of the  $n+1^{\text{th}}$  degree in  $l, m, n$ ;  $n+1$  tangents to the surface at  $xyz$  can, in general, be drawn in any given plane through that point. This equation now takes the place of the conditional equation in (5781).

**5784** Equation (5782) is now replaced by

$$\{(\xi-x)d_x + (\eta-y)d_y + (\zeta-z)d_z\}^n \phi(x, y, z) = 0,$$

the equation of the locus of all tangents at the point  $xyz$ , and representing a conical surface generated by the motion of those tangents.

**5785** The equation of the normal at  $xyz$  is

$$\frac{\xi-x}{\phi_x} = \frac{\eta-y}{\phi_y} = \frac{\zeta-z}{\phi_z}. \quad (5782)$$

**5786** The equation of the tangent at a point  $x'y'z'$  on the curve of intersection of the tangent plane at  $xyz$  with the surface is

$$\frac{\xi - x'}{\lambda} = \frac{\eta - y'}{\mu} = \frac{\zeta - z'}{\nu},$$

with the two conditions

$$\lambda\phi_x + \mu\phi_y + \nu\phi_z = 0, \quad \lambda\phi_{x'} + \mu\phi_{y'} + \nu\phi_{z'} = 0.$$

For these are the conditions of perpendicularity to the normals of the tangent planes at  $xyz$  and  $x'y'z'$  respectively.

There are three exceptional cases in which the ratios  $\lambda : \mu : \nu$  have more than one set of values; namely—

**5787** I.—When  $\phi_x, \phi_y, \phi_z$  vanish simultaneously, there is a tangent cone at  $xyz$ .

**5788** II.—When  $\phi_{x'}, \phi_{y'}, \phi_{z'}$  vanish simultaneously,  $x'y'z'$  is a singular point on the surface.

**5789** III.—When  $\frac{\phi_x}{\phi_{x'}} = \frac{\phi_y}{\phi_{y'}} = \frac{\phi_z}{\phi_{z'}}$ . In this case the point  $x'y'z'$  coincides with  $xyz$ , and the tangent there meets the curve in more than two coincident points, the condition for which is

$$(\lambda d_x + \mu d_y + \nu d_z)^2 \phi(x, y, z) = 0 \dots\dots\dots (i.),$$

$$\text{with} \quad \lambda\phi_x + \mu\phi_y + \nu\phi_z = 0 \dots\dots\dots (ii.).$$

These equations furnish two sets of values of the ratios  $\lambda : \mu : \nu$ , giving thereby the directions of two *inflexional tangents* (tangents to the curve of intersection) at  $xyz$ , each meeting the surface in three coincident points. If all the derivatives of an order less than  $n$  vanish at  $xyz$ , equation (i.) will be replaced by  $(\lambda d_x + \mu d_y + \nu d_z)^n \phi(x, y, z) = 0$ , which, together with (ii.), will determine  $n$  inflexional tangents at the point.

**5790** The polar equation of the tangent plane at the point  $r\theta\phi$ ,  $r', \theta', \phi'$  being the variables, is, writing  $u$  for  $r^{-1}$ ,

$$u' = (u \cos \theta - u_\theta \sin \theta) \cos \theta' + (u \sin \theta + u_\theta \cos \theta) \cos (\phi' - \phi) \sin \theta' \\ + u_\phi \operatorname{cosec} \theta \sin (\phi' - \phi) \sin \theta'.$$



PROOF.—Write the polar equation of the plane through  $pa\beta$ , the foot of the perpendicular on the plane from the origin; thus

$$pu = \cos \theta \cos \alpha + \sin \theta \sin \alpha \cos (\phi - \beta).$$

Differentiate for  $\theta$  and  $\phi$  to find  $pu_\theta$  and  $pu_\phi$ , and eliminate  $p$ ,  $\alpha$ , and  $\beta$ . This elimination is troublesome.

**5791** The length of the perpendicular from the origin upon the tangent plane at  $xyz$ ,

$$p = \frac{x\phi_x + y\phi_y + z\phi_z}{\sqrt{\{\phi_x^2 + \phi_y^2 + \phi_z^2\}}} \quad \text{or} \quad \frac{nc}{\sqrt{\{\phi_x^2 + \phi_y^2 + \phi_z^2\}}}, \quad (5782, 5549)$$

the second form being the value of  $p$  when the equation of the surface is  $\phi(x, y, z) = c$ , a constant, and when  $\phi$  is a homogeneous function of the  $n^{\text{th}}$  degree (1624).

**5793** In polar coordinates,

$$\frac{1}{p^2} = u^2 + u_\theta^2 + u_\phi^2 \operatorname{cosec}^2 \theta = \frac{r^2 + r_\theta^2 + r_\phi^2 \operatorname{cosec}^2 \theta}{r^4}.$$

PROOF.—Add together the values of the squares of  $pu$ ,  $pu_\theta$ , and  $pu_\phi$  found in (5790).

For a geometrical proof, see *Frost and Wolstenholme*, Art. (314).

### THE INDICATRIX CONIC.

**5795** DEF.—The indicatrix at any point of a surface is the curve in which the surface is intersected by a plane drawn parallel to the tangent plane at that point and infinitely near to it.

**5796** The following abbreviations will be employed—

The derivatives of  $\phi(x, y, z)$ ,  $\phi_{2x}$ ,  $\phi_{2y}$ ,  $\phi_{2z}$ ,  $\phi_{yz}$ ,  $\phi_{zx}$ ,  $\phi_{xy}$ ,  $\phi_x$ ,  $\phi_y$ ,  $\phi_z$ , will be denoted by  $a$ ,  $b$ ,  $c$ ,  $f$ ,  $g$ ,  $h$ ,  $l$ ,  $m$ ,  $n$ .

**5797** PROP.—The *indicatrix* at a point  $xyz$  of a surface  $\phi(x, y, z) = 0$  is the conic in which the elementary quadric surface

**5798** I.

$$a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta = -\frac{R^2}{\rho} \sqrt{l^2 + m^2 + n^2} \equiv N$$

is intersected by the tangent plane at  $xyz$ , whose equation is

**5799** II.  $l\xi + m\eta + n\zeta + \frac{1}{2}N = 0.$

The origin of coordinates is the point  $xyz$  in both equations.  $R$  is an indefinitely small radius from the centre of the quadric (I.) to a point  $\xi\eta\zeta$  on the indicatrix, and  $\rho$  is the radius of curvature of the section of the surface  $\phi$  by a normal plane drawn through  $R$ ; the ratio  $R^2 : \rho$  being constant for all such planes.

PROOF.—Let  $O$ , in Fig. (181), be the point  $xyz$  on the surface  $\phi$ ;  $x+\xi$ ,  $y+\eta$ ,  $z+\zeta$  an adjacent point  $P$ . Then

$$\phi(x+\xi, y+\eta, z+\zeta) = \phi(x, y, z) + l\xi + m\eta + n\zeta + \frac{1}{2}(a\xi^2 + \dots + 2h\xi\eta) + \&c.$$

With  $xyz$  for origin, draw the quadric surface

$$a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta = N \dots\dots\dots (i.)$$

and the plane

$$l\xi + m\eta + n\zeta + \frac{1}{2}N = 0 \dots\dots\dots (ii.).$$

Since  $\xi$ ,  $\eta$ ,  $\zeta$  are very small,  $N$  is likewise. Also the unwritten terms in the above expansion may be neglected in the limit. Hence, any point  $\xi\eta\zeta$  lying on the intersection of the quadric (i.) and the plane (ii.) will also lie on the original surface  $\phi(x+\xi, y+\eta, z+\zeta) = 0$ .

To determine  $N$ , we have the perpendicular from  $xyz$  upon the plane (ii.),

$p = \frac{-N}{2\sqrt{l^2+m^2+n^2}}$  (5549). The radius of curvature of the section of the surface  $\phi$  made by a normal plane at  $O$  drawn through  $P$  being  $\rho$ , we have  $\rho = \frac{R^2}{2p}$ , and therefore  $N = -\frac{R^2}{\rho}\sqrt{l^2+m^2+n^2}$ .

In the Figure,  $R = OP$ ,  $p = OL$ , and the intersection of (i.) and (ii.) is the conic  $PSQ$ . Since  $p$  is indefinitely small, we may put  $N = 0$  in equation (ii.). This amounts to taking the parallel section of the quadric by the tangent plane at  $O$  instead of the section  $PSQ$ . But these two will be equal in all respects, since the section of the quadric is a *central* one.

**5800** If  $m = 0$ , equation (II.) becomes  $l\xi + n\zeta = 0$ , and if the inclination of the indicatrix plane to the plane of  $xy$  be  $a$ ,  $\tan a = -\frac{l}{n}$ . To obtain, in this case, the equation of the indicatrix in its own plane, put  $\xi = \xi' \cos a$ ,  $\zeta = \zeta' \sin a$ , and  $\eta = \eta'$ , in equation (I.).

**5801** When none of the three constants  $l$ ,  $m$ ,  $n$  are zero, the quadric (I.) simplifies as follows—

From (II.) we have  $l\xi + m\eta = -n\zeta$  and two similar equations. Square these, and by the results eliminate the terms in  $\eta\zeta$ ,  $\zeta\xi$ ,  $\xi\eta$  from (I.), which then becomes

$$\mathbf{5802 \quad III.} \quad H\xi^2 + K\eta^2 + L\zeta^2 = N,$$

$$\text{where} \quad H = a + \frac{l}{mn}(lf - mg - nh), \quad K = b + \frac{m}{nl}(mg - nh - lf),$$

$$L = c + \frac{n}{lm}(nh - lf - mg).$$

This is the equation of another quadric intersecting the plane (II.) in the indicatrix.

**5803** The equation of a surface for points near an origin  $O$  (Fig. 182), the normal at  $O$  being taken for  $z$  axis, is

$$\frac{x^2}{\rho_1} + \frac{y^2}{\rho_2} = 2z,$$

where  $\rho_1, \rho_2$  are the radii of curvature of the normal sections through the  $x$  and  $y$  axes, and those sections will be proved to be the lines of curvature at  $O$ .

PROOF.—Let  $AC = a$  and  $BC = b$  be the semi-axes of the indicatrix conic at a small distance  $z$  from  $O$  (5795). The equation of the conic will therefore be  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ; but  $\frac{a^2}{z} = 2\rho_1$  and  $\frac{b^2}{z} = 2\rho_2$ , giving the equation required.

Secondly, on a line of curvature, the normal to the surface at a point  $xyz$  will intersect the  $z$  axis (5773). The condition for this, by (5533) [with  $xyz$  for  $abc$ , the origin for  $a'b'c'$ ,

$L, M, N = \phi_x, \phi_y, \phi_z$  (5785)  $= \frac{2x}{\rho_1}, \frac{2y}{\rho_2}, -2$ , and  $L', M', N' = 0, 0, 1$ ], gives  $xy \left( \frac{1}{\rho_2} - \frac{1}{\rho_1} \right) = 0$ , therefore  $x = 0$  or  $y = 0$  on a line of curvature. Q. E. D.

**5804** If the equation of the surface with the same axes be

$$z = ax^2 + 2hxy + by^2 + 2fyz + 2gzx + cz^2 + \text{higher powers},$$

then 
$$\rho_1 = \frac{1}{2a}, \quad \rho_2 = \frac{1}{2b}.$$

PROOF.—Put  $y = 0$  and divide by  $z$ , therefore  $1 = a \frac{x^2}{z} + 2gx + cz + \&c$ . When  $x$  and  $z$  vanish, we have  $1 = 2a\rho_1$ .

**5805** For a normal section making an angle  $\theta$  with  $AC$ ,

$$\frac{1}{\rho} = 2(a \cos^2 \theta + 2h \sin \theta \cos \theta + b \sin^2 \theta).$$

PROOF.—Turning the axes in (5804) through the angle  $\theta$  by (4049), the coefficient of  $x'^2$  becomes  $a \cos^2 \theta +$  as above.

**5806** *Euler's Theorem.*—If  $\rho$  be the radius of curvature of

any other normal section at  $O$ , making an angle  $ACP = \theta$  with  $AC$  (Fig. 182),

$$\frac{1}{\rho} = \frac{\cos^2 \theta}{\rho_1} + \frac{\sin^2 \theta}{\rho_2}.$$

PROOF.—Let  $r = CP$ ; then  $x = r \cos \theta$ ,  $y = r \sin \theta$ , and  $r^2 = 2\rho z$ , which substitute in (5793).

**5807** COR.—The sum of the curvatures of two normal sections at right angles to each other is constant; or, if  $\rho, \rho'$  be the radii of curvature for those sections, and  $\rho_a, \rho_b$  the radii for the principal sections,

$$\frac{1}{\rho} + \frac{1}{\rho'} = \frac{1}{\rho_a} + \frac{1}{\rho_b}.$$

**5808** The radius of curvature of a normal section varies as the square of the radius of the indicatrix in that section.

PROOF.—From  $r^2 = 2\rho z$ , in Figure (182).

**5809** *Meunier's Theorem*.—The radius of curvature of an oblique section of a surface is equal to the radius of curvature of the normal section through the same tangent multiplied by the cosine of the inclination of the planes.

PROOF.—(Fig. 183.)  $\rho' = \frac{PN^2}{NC}$ ,  $\rho = \frac{PN^2}{NO}$ , therefore  $\frac{\rho'}{\rho} = \frac{NO}{NC} = \cos \phi$ , when  $NO$  and  $NC$  vanish.

**5810** Quadratic for  $y_x$  at a point on the surface  $z = \phi(x, y)$  giving the direction of the principal normal sections, and, therefore, of the lines of curvature (notation 1815).

$$\{pqt - (1 + q^2)s\} y_x^2 + \{(1 + p^2)t - (1 + q^2)r\} y_x + \{(1 + p^2)s - pqr\} = 0.$$

PROOF.—(i.) The equations of the normals at the consecutive points  $xyz$  and  $x+dx, y+dy, z+dz$  of the surface  $\phi(x, y, z) = 0$  are

$$\frac{\xi - x}{\phi_x} = \frac{\eta - y}{\phi_y} = \frac{\zeta - z}{\phi_z} \quad \text{and} \quad \frac{\xi - (x+dx)}{\phi_x + d\phi_x} = \frac{\eta - (y+dy)}{\phi_y + d\phi_y} = \frac{\zeta - (z+dz)}{\phi_z + d\phi_z}.$$

**5811** The condition of intersection is, by (5533),

$$\begin{vmatrix} dx & dy & dz \\ \phi_x & \phi_y & \phi_z \\ d\phi_x & d\phi_y & d\phi_z \end{vmatrix} = 0, \quad \text{or} \quad \begin{vmatrix} 1 & y_x & p + qy_x \\ p & q & -1 \\ r + sy_x & s + ty_x & 0 \end{vmatrix} = 0,$$

by dividing the first row by  $dx$ , and putting  $z_x = \phi_x + \phi_y y_x$ ,  $d\phi_x = \phi_{xx} + \phi_{xy} y_x$  &c. The form of  $\phi(x, y, z)$  being, in this case,  $\phi(x, y) - z$ ,  $\phi_z$  becomes  $-1$ , and  $d\phi_z$  becomes zero. The determinant equation produces the quadratic.

(ii.) *Otherwise.*—Consider  $\xi\eta\zeta$  the point of intersection of consecutive normals. The equations of a normal being

$$\frac{\xi-x}{p} = \frac{\eta-y}{q} = \frac{\zeta-z}{-1} \quad \text{or} \quad \xi-x = p(z-\zeta) \quad \text{and} \quad \eta-y = q(z-\zeta).$$

Differentiate both equations for  $x$ , considering  $\xi, \eta, \zeta$  constant and  $p, q$  functions of  $x$  and  $y$ ; the results are

$$1 + (r + sy_x)(z - \zeta) + p(p + qy_x) = 0 \quad \text{and} \quad y_x + (s + ty_x)(z - \zeta) + q(p + qy_x) = 0.$$

Eliminate  $z - \zeta$  to obtain the quadratic in  $y_x$ .

**5812** If the equation of the surface be in the form  $\phi(x, y, z) = 0$ , the quadratic for  $y_x$  may be obtained in the same way. The requisite substitutions in the first determinant are found from  $\phi_x + \phi_y y_x + \phi_z z_x = 0$ , giving  $z_x$ ;  $d\phi_x = \phi_{xx} + \phi_{xy} y_x + \phi_{xz} z_x$ , &c., and with the notation of (5796) the determinant equation and quadratic for  $y_x$  becomes

$$\begin{vmatrix} n & ny_x & -(l + my_x) \\ l & m & n \\ an - gl + (hn - gm)y_x & hn - fl + (bn - fm)y_x & gn - cl + (fn - cm)y_x \end{vmatrix} = 0.$$

**5813** The above determinant, or the corresponding one in (5810), is the differential equation of the lines of curvature.

**5814** The radii of curvature of the principal normal sections of the surface  $\phi(x, y, z) = 0$  at a point  $xyz$  are given by the following quadratic, in which  $\Delta'$  is the bordered determinant in (5700), and the notation is that of (5796) and (1620).

$$\Delta' \rho^2 + \{ (a+b+c)(l^2 + m^2 + n^2) - (abcfgh \chi lmn)^2 \} k\rho - k^4 = 0,$$

where  $k^2 = l^2 + m^2 + n^2$ .

PROOF.—The quadratic in (5653) applied to a section of the quadric (I.) (5798) by the plane (II.), becomes

$$\Delta' R^4 + \{ (b+c)l^2 + (c+a)m^2 + (a+b)n^2 - 2fmn - 2gnl - 2hlm \} NR^2 - (l^2 + m^2 + n^2) N^2 = 0,$$

whose roots, being the two values of  $R^2$ , are the squares of the principal semi-axes of the indicatrix. Put  $R^2 = \frac{N\rho}{k}$ , as in the Proof of (5797).

**5815** Otherwise, the quadratic in (5651) might be applied to a section of the quadric (III.) (5802) by the plane (I.).

**5816** If the equation of the surface be given in the form

$z = \phi(x, y)$ , the quadratic becomes [writing, as in (1815),  $p, q, r, s, t$  for  $z_x, z_y, z_{xx}, z_{xy}, z_{yy}$ ],

$$(rt - s^2) \rho^2 - \{ (1 + p^2) t - 2pqs + (1 + q^2) r \} k\rho + k^4 = 0,$$

where  $k^2 = p^2 + q^2 + 1$ .

Otherwise, this equation may be found from the two equations obtained in the second proof of (5810), by eliminating  $y_x$  instead of  $z - \zeta$ .

**5817** The radius of curvature at a point  $xyz$  on the surface  $\phi(x, y, z) = 0$  of the normal section whose tangent has the direction cosines  $\lambda, \mu, \nu$  is, with the notation of (5796) and (1620),

$$\rho = \frac{\sqrt{(l^2 + m^2 + n^2)}}{(abcfgh)(\lambda\mu\nu)^2}.$$

PROOF.—From equation (I.) (5798), since  $\xi, \eta, \zeta$  are respectively equal to  $R\lambda, R\mu$ , and  $R\nu$ .

**5818** The curvature at any point of a surface  $\phi(x, y, z) = 0$  is termed *elliptic* or *synclastic*, *hyperbolic* or *anticlastic*, and *parabolic* or *cylindrical*, according as the indicatrix is an ellipse, hyperbola, or two parallel right lines, or according as the principal curvatures have the same signs, opposite signs, or one of them vanishes; and this will be according as the determinant  $\Delta'$ , in (5814), or  $s^2 - rt$ , in (5816), is *negative*, *positive*, or *zero*.

PROOF.—The rule follows at once from the consideration that the two values of  $\rho$  in the quadratic of (5814) must have the same sign in the first case, different signs in the second, and that one value must be infinite in the third case.

**5819** The condition for an umbilicus is that the indicatrix must be a circle; therefore, either (III.) (5802) must be a sphere, or, if it be a quadric surface, the plane (II.) must make a circular section of it, and therefore either  $l, m$ , or  $n$  must vanish.

**5820** Otherwise, the quadratic in (5814) or (5816) must have equal roots.

**5821** Otherwise, the conditions for an umbilicus on the surface  $\phi(x, y, z) = 0$  are the two equations

$$\frac{bn^2 + cm^2 - 2fmn}{m^2 + n^2} = \frac{cl^2 + an^2 - 2gnl}{n^2 + l^2} = \frac{am^2 + bl^2 - 2hlm}{l^2 + m^2}.$$



PROOF.—The radius of the indicatrix, and therefore also  $\rho$  in (5817), is constant for all values of  $\lambda, \mu, \nu$ . Now, by (5817),

$$a\lambda^2 + \&c. = \frac{k}{\rho};$$

$$\therefore \left(a - \frac{k}{\rho}\right)\lambda^2 + \left(b - \frac{k}{\rho}\right)\mu^2 + \left(c - \frac{k}{\rho}\right)\nu^2 + 2f\mu\nu + 2g\nu\lambda + 2h\lambda\mu = 0,$$

and  $l\lambda + m\mu + n\nu = 0$ , since  $\lambda\mu\nu$  is always tangential, and  $lmn$  is normal to the surface. As these equations are true for all values of  $\lambda, \mu, \nu$ , the second expression must be a factor of the first. The quotient, by division, is there-

fore 
$$\left(a - \frac{k}{\rho}\right) \frac{\lambda}{l} + \left(b - \frac{k}{\rho}\right) \frac{\mu}{m} + \left(c - \frac{k}{\rho}\right) \frac{\nu}{n}.$$

Equating to zero each of the three coefficients of the remainder, and eliminating  $\rho$ , we obtain the above conditions.

**5822** If a common factor of the three fractions in (5821) exists, that factor equated to zero is the differential equation of a *line of spherical curvature* at every point of which there is an umbilicus. If the fractions are identically equal, the surface has an umbilicus at every point, and must therefore be a sphere.

**5823** The number of umbilici on a surface of the  $n^{\text{th}}$  degree cannot exceed  $n(10n^2 - 25n + 16)$ . *Salmon*, p. 208.

**5824** The condition that the indicatrix may be a rectangular hyperbola is

$$(a+b+c)(l^2+m^2+n^2) = (abcfgh \times lmn)^2.$$

PROOF.—The quadratic in (5814) must have equal roots of opposite signs.

Similarly, when  $z = \phi(x, y)$  is the equation of the quadric, the condition becomes

$$(1+p^2)t - 2pqs + (1+q^2)r = 0. \quad (5816)$$

**5825** The condition that the indicatrix may become two coinciding lines.

Here equation I. (5798) must represent a cone, and the plane (II.) must touch it. Hence  $N = 0$ , and, if  $\zeta$  be eliminated, the quadratic for the ratio  $\xi : \eta$  obtained is

$$(an^2 + cl^2 - 2gnl)\xi^2 + 2(clm - fnl - gmn + hn^2)\xi\eta + (bn^2 + cm^2 - 2fmn)\eta^2 = 0,$$

and this must have equal roots.

#### CURVATURE OF A SURFACE.

**5826** DEFS.—*Integral curvature* of a closed surface is equal to the area of that part of the surface of a sphere of unit radius which is intercepted by radii drawn parallel to the

normals at all points of the given surface. This area also measures the solid angle of the cone generated by the radii. The curve on the sphere is called the *horograph* of the curve on the original surface. In other words, *integral curvature* of a closed surface is the area of the horograph of its boundary.

**5827** *Average curvature* is the integral curvature divided by the area of the surface.

*Specific curvature* is the average curvature of a small element at the point; i.e.,  $\frac{(ds)^2}{\rho_1 \rho_2} \div (ds)^2 = \frac{1}{\rho_1 \rho_2}$ .

**5828** The last is the usual measure of curvature at a point, and its value in coordinates of the point is given by

$$\frac{1}{\rho_1 \rho_2} = -\frac{\Delta'}{(l^2 + m^2 + n^2)^2} \quad \text{or} \quad \frac{rt - s^2}{(1 + p^2 + q^2)^2}, \quad (5796)$$

according as  $\phi(x, y, z) = 0$  or  $z = \phi(x, y)$  is the form of the equation to the surface.

PROOF.—From the product of roots of the quadratics in (5814) and (5816).

**5829** In a plane curve *integral curvature* is the plane angle contained by the terminal normals, and *average curvature* is the integral curvature divided by the length of the curve.

**5830** Another measure of curvature at a given point of a surface is the ratio of the area of the indicatrix to the area of the indicatrix cut off by the same plane on a sphere of unit radius which touches the surface internally at the point. This measure is  $= \sqrt{\rho_1 \rho_2}$ .

PROOF.—Putting  $AC = R_1$ ,  $BC = R_2$ , in Fig. (182), and  $OC = z$ , the area of the indicatrix of the surface is  $\pi R_1 R_2$  at an ellipsoidal point. But  $R_1^2 = 2\rho_1 z$  and  $R_2^2 = 2\rho_2 z$ , therefore  $\pi R_1 R_2 = 2\pi z \sqrt{(\rho_1 \rho_2)}$ . Also the indicatrix of the sphere  $= 2\pi z$  since  $\rho_1 = \rho_2 = 1$  for the sphere.

**5831** The radius of curvature of any normal section at a point  $P$  of an ellipsoid (Fig. 184) is equal to the square of the semi-diameter parallel to the tangent of that section,

divided by the perpendicular from  $P$  upon the diametral plane conjugate to  $OP$ .

PROOF.—Let  $AOB$  be the plane parallel to the tangent plane at  $P$ ;  $OA = d$ , the semi-diameter in it parallel to the given tangent  $PT$ . Draw  $PR$  perpendicular to  $OA$  and  $PN \equiv p$  perpendicular to the plane  $AOB$ . The radius of curvature at  $P$  of the elliptic section  $PA = \frac{d^2}{PR}$  (4536). Therefore, by (5809), the radius of curvature of the normal section through the same tangent  $PT$ , will be  $\rho = \frac{d^2}{PR} \times \frac{PR}{PN} = \frac{d^2}{p}$ .

**5832** The principal radii of curvature at  $P$ , viz.  $\rho_1, \rho_2$ , are found from their sum and product thus: putting  $\gamma$  for  $OP$ , and  $a, b, c$  for the semi-axes of the ellipsoid,

$$\rho_1 + \rho_2 = \frac{a^2 + b^2 + c^2 - \gamma^2}{p}, \quad \rho_1 \rho_2 = \frac{a^2 b^2 c^2}{p^3}.$$

PROOF.—Let  $\alpha, \beta$  be the semi-axes of the section  $AOB$  (Fig. 184), then  $\alpha^2 + \beta^2 + \gamma^2 = a^2 + b^2 + c^2$  (5642) and  $\alpha\beta\gamma = abc$  (5648). By these values eliminate  $\alpha, \beta$  from  $\rho_1 + \rho_2 = \frac{\alpha^2 + \beta^2}{p}$  and  $\rho_1 \rho_2 = \frac{\alpha^2 \beta^2}{p^3}$  (5831).

**5833** The lines of curvature on a quadric surface are its intersections with the confocal quadrics.

PROOF.—Let the quadric and confocal be the ellipsoid and one-fold hyperboloid in (Fig. 178) intersecting in the line  $DPE$ , and let their equations be, as in (5656),

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \dots\dots (i.) \quad \text{and} \quad \frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} = 1 \dots\dots (ii.).$$

At any point  $P$  on the line of intersection  $x, y, z$  satisfy the three following equations:—

$$\text{First, the differential of (ii.), } \frac{x dx}{a^2 + \lambda} + \frac{y dy}{b^2 + \lambda} + \frac{z dz}{c^2 + \lambda} = 0.$$

Second, the difference of (i.) and (ii.),

$$\frac{x^2}{a^2(a^2 + \lambda)} + \frac{y^2}{b^2(b^2 + \lambda)} + \frac{z^2}{c^2(c^2 + \lambda)} = 0.$$

Third, the difference of their differentials,

$$\frac{x dx}{a^2(a^2 + \lambda)} + \frac{y dy}{b^2(b^2 + \lambda)} + \frac{z dz}{c^2(c^2 + \lambda)} = 0.$$

The eliminant of these equations in  $x, y, z$  produces the determinant equation here annexed, which, by (5811), is the condition for the intersection of consecutive normals. Hence this condition holds for every point of the line of intersection of (i.) and (ii.).

$$\begin{vmatrix} \frac{dx}{a^2} & \frac{dy}{b^2} & \frac{dz}{c^2} \\ \frac{x}{a^2} & \frac{y}{b^2} & \frac{z}{c^2} \\ \frac{dx}{a^2} & \frac{dy}{b^2} & \frac{dz}{c^2} \end{vmatrix} = 0.$$

The general method of determining the lines of curvature of a surface from the differential equation in (5811) is here exemplified in the case of an ellipsoid.

**5834** The determinant just written gives for the differential equation of the lines of curvature

$$(b^2 - c^2) x dy dz + (c^2 - a^2) y dz dx + (a^2 - b^2) z dx dy = 0 \dots\dots\dots (i.).$$

To solve this, multiply by  $\frac{z}{c^2}$  and substitute for  $z$  and  $dz$  from the equation of the quadric. The result is of the form

$$Axy y_x^2 + (x^2 - Ay^2 - B) y_x - xy = 0,$$

in which  $A = \frac{a^2(b^2 - c^2)}{b^2(a^2 - c^2)}$ ,  $B = \frac{a^2(a^2 - b^2)}{a^2 - c^2}$ ; or, multiplying by  $\frac{y}{x}$ ,

$$A \frac{yy_x}{x} (xy y_x - y^2) - B \frac{yy_x}{x} + (xy y_x - y^2) = 0,$$

which is of the form in (3236). Solving by that method, we find that the two equations  $\frac{yy_x}{x} = \alpha$  and  $xy y_x - y^2 = \beta$  have the common primitive  $\alpha x^2 - y^2 = \beta$ , which, with the relation  $A\alpha\beta - B\alpha + \beta = 0$ , constitutes the solution. The result is that the projections of the lines of curvature upon the  $xy$  plane are a series of conics coaxial with the principal section of the ellipsoid, and having their axes  $a, b$  varying according to the equation

$$\frac{a^2(a^2 - c^2)}{a^2(a^2 - b^2)} + \frac{b^2(b^2 - c^2)}{b^2(b^2 - a^2)} = 1.$$

At an umbilicus  $y = 0$ , therefore, equation (i.) becomes  $[(b^2 - c^2) x dz + (a^2 - b^2) z dx] dy = 0$ . Here  $dy = 0$ , being a solution, gives  $y = C = 0$ , showing that the plane of  $zx$  contains a line of curvature. The other factor, equated to zero, taken with the differential equation of the curve  $c^2 x dz + a^2 z dx = 0$ , gives the coordinates of the umbilicus, as in (5603).

#### OSCULATING PLANE OF A LINE OF CURVATURE.

**5835** Let  $\phi$  be the angle between the osculating plane and the normal section through the same line of curvature,  $ds$  an element of the other line of curvature, and  $\rho, \rho'$  their radii of curvature respectively: then

$$\tan \phi = \frac{d\rho}{ds} \cdot \frac{\rho'}{\rho' - \rho}.$$

PROOF.—Fig. (185). Let  $OA, OB$  be the lines of curvature;  $OP, AP$  consecutive normals along  $OA$ ; and  $OS, BS$  the same along  $OB$ . Also, let  $BQ, CQ$  be consecutive normals along the line of curvature  $BC$ . Then, ultimately,  $OP = \rho$ ,  $OS = \rho'$ ,  $BQ = \rho + d\rho$ . Also, let  $QP$  produced meet the osculating plane of  $AO$  in  $R$ . Join  $RO$  and  $RA$ , and draw  $QN$  at right angles to  $PS$ . Since the tangent to  $AO$  at  $O$  is perpendicular to the plane  $OBQP$  and that at  $A$  to  $ACQP$ , it follows that both tangents are perpendicular to  $QP$ , which must therefore be perpendicular to the osculating plane  $ARO$ . Hence  $\phi$  or  $ROP = PQN$ .

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$$\text{Now } \frac{NQ}{ds} = \frac{SQ}{SB} = \frac{\rho' - \rho - d\rho}{\rho'}, \quad \therefore \tan \phi = \frac{NP}{NQ} = \frac{d\rho}{ds} \cdot \frac{\rho'}{\rho' - \rho} \text{ ultimately.}$$


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**5836** At every point on a line of curvature of a central conicoid  $pd$  is constant, where  $d$  is the semi-diameter parallel to the tangent at the point and  $p$  is the perpendicular from the centre upon the tangent plane.

PROOF.—Let the first and third confocals in (5661) be fixed, and therefore  $a_1$  and  $a_3$  constant. Draw the second confocal through the point of contact  $P$  of the tangent plane (Fig. 178). Then, by (5663),  $p_1d_3$  and  $p_3d_1$  are constant along the line of intersection of the first and third surface, because, by (5661),  $d_3^2 = a_1^2 - a_2^2$  and  $d_1^2 = a_2^2 - a_3^2$ .

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### GEODESIC LINES.

**5837** The equations of a geodesic on the surface  $\phi(x, y, z) = 0$

are

$$\frac{x_{2s}}{\phi_x} = \frac{y_{2s}}{\phi_y} = \frac{z_{2s}}{\phi_z}.$$

PROOF.—The osculating plane of the curve contains the normal to the surface (5775); therefore, by (5737) and (5785).

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**5838** A geodesic is a line of maximum or minimum distance along the surface between two points.

PROOF.—The curve drawn in the osculating plane from one point to a contiguous point is shorter than any other by Meunier's theorem (5809), for any oblique section has a shorter radius of curvature and therefore a longer arc. A succession of minimum arcs, however, may constitute a maximum curve distance between the extreme points; for example, two points on a sphere can be joined by either of two arcs of a great circle, the one being a minimum and the other a maximum geodesic.

**5839** A surface of revolution such as the terrestrial globe affords a good illustration. A meridian and a parallel of latitude drawn through a point near the pole are the two lines of curvature at the point. The meridian is also a geodesic, but the parallel is evidently not, for its plane does not contain the normal to the surface.

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**5840** A geodesic is the line in which a string would lie if stretched over the convex side of a smooth surface between two fixed points.

PROOF.—Any small arc of the string  $POP'$  (Fig. 182) is acted upon by tensions along the tangents at  $P$  and  $P'$ , and by the normal reaction of the surface at  $O$ . But these three forces act in the osculating plane (5775); therefore the string will rest in equilibrium on the surface in that plane.



**COR.**—Two equal geodesics drawn from a point and indefinitely near to each other are at right angles to the line which joins their extremities.

**5841** If a geodesic has a constant inclination to a fixed line, the normals along it will be at right angles to that line.

**PROOF.**—Let  $lmn$  be the fixed line and  $\alpha$  the constant angle; then

$$lx_1 + my_1 + nz_1 = \cos \alpha, \text{ and therefore } lx_2 + my_2 + nz_2 = 0.$$

Therefore, by (5837), the principal normal is at right angles to  $lmn$ .

**EXAMPLE.**—The helix, the axis being the fixed line.

**5842** On any central conicoid  $pd$  is constant along a geodesic, where  $p$  is the perpendicular from the centre upon the tangent plane and  $d$  is the semi-diameter parallel to the tangent of the geodesic.

**PROOF.**—(Fig. 186.) Let  $AT$ ,  $BT$  be the tangents at the two extremities of a small geodesic arc  $AB$ , and let the tangent planes at  $A$  and  $B$  be  $ADC$  and  $BCD$ .  $AT$  and  $BT$  make equal angles with  $CD$ , by the property of shortest distance, for if the plane  $BCD$  be turned about  $CD$  until it coincides with the plane  $ADC$ ,  $ATB$  will become a straight line, and therefore  $\angle ATD = \angle BTC = i$ , say.

Let  $\omega$  be the angle between the tangent planes; let the perpendiculars upon those planes from  $A$ ,  $B$  be  $AM = q$ ,  $BN = q'$ , and from the centre of the quadric  $p$ ,  $p'$ ; and let  $xyz$  and  $x'y'z'$  be the points  $A$ ,  $B$ . Then

$$q = AT \sin i \sin \omega, \quad q' = BT \sin i \sin \omega, \quad \therefore q : q' = AT : BT \dots\dots (i),$$

$$q = p' \left( \frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} - 1 \right) \text{ (5628); } \quad q' = p \left( \frac{x'x}{a^2} + \frac{y'y}{b^2} + \frac{z'z}{c^2} - 1 \right),$$

therefore

$$q : q' = p' : p \dots\dots\dots (ii).$$

Again, let  $d$ ,  $d'$  be the semi-diameters parallel to  $AT$  and  $BT$ . Then, by (5677),  $AT : BT = d : d'$ ; therefore  $p' : p = d : d'$  or  $pd = p'd'$ ; that is,  $pd$  is constant.

**5843** If a line of curvature be plane, that plane makes a constant angle with the tangent plane to the surface.

**PROOF.**—Let  $PQ$ ,  $QR$ ,  $RS$  be equal consecutive elements of the line of curvature. The consecutive normals to the surface bisect  $PQ$  and  $QR$  and meet in a point. Therefore they are equally inclined to the plane  $PQR$ . Similarly the second and third normals are equally inclined to the plane  $QRS$ , and so on. Hence, if the curve be plane, all the normals are equally inclined to its plane. Hence also the following theorem.

**5844** *Lancret's Theorem.*—The variation in the angle between the tangent plane and the osculating plane of a line of



curvature is equal to the angle between consecutive osculating planes.

**5845** COR.—If a geodesic be either a line of curvature or a plane curve, it is both, but a plane line of curvature, as in (5839), is not necessarily a geodesic.

### GEODESIC CURVATURE.

*Theorem.*—The square of the curvature at any point of a curve traced on a surface is equal to the sum of the squares of the normal and geodesic curvatures (5776), or

$$\mathbf{5846} \quad \frac{1}{\rho^2} = \frac{1}{\rho'^2} + \frac{1}{\rho''^2},$$

where  $\rho'$  is the radius of curvature of the normal section and  $\rho''$  the radius of geodesic curvature. Also, if  $\phi$  be the angle between the plane of normal section and the osculating plane,

$$\mathbf{5847} \quad \rho = \rho'' \sin \phi = \rho' \cos \phi.$$

PROOF.—Let  $PQ = QR$  (Fig. 187) be consecutive elements of any curve traced on a surface. Produce  $PQ$  to  $S$ , making  $QS = PQ$ . Let  $QT = PQ$  be the consecutive elements of the section of the surface drawn through  $PQS$  and the normal at  $Q$ . Join  $RS$ ,  $ST$ ,  $TR$ .  $PQSR$  is the osculating plane of the curve  $PQR$ .  $PQST$  is the plane of normal section, and therefore  $PQT$  is a geodesic.  $QRT$  is the tangent plane, and  $STR$  is a right angle.

Then, putting  $SQR = d\psi$ ,  $SQT = d\psi'$ ,  $RQT = d\psi''$ ,  $RST = \phi$ , we have

$$\rho = \frac{ds}{d\psi}, \quad \rho' = \frac{ds}{d\psi'}, \quad \rho'' = \frac{ds}{d\psi''}. \quad (5776)$$

Therefore 
$$\frac{\rho}{\rho''} = \frac{ds \cdot d\psi''}{ds \cdot d\psi} = \frac{RT}{RS} = \sin \phi.$$

Also 
$$\frac{\rho}{\rho'} = \frac{ds \cdot d\psi'}{ds \cdot d\psi} = \frac{ST}{SR} = \cos \phi, \text{ as in (5809).}$$

Thus both theorems are proved. Note that  $\rho'$  is the radius of curvature of the geodesic  $PQT$ , while  $\rho''$  is the radius of geodesic curvature of  $PQR$ .

### RADIUS OF TORSION OF A GEODESIC.

**5848** If  $\theta$  be the angle between the geodesic and one of the lines of curvature;  $\rho_1$ ,  $\rho_2$  the principal radii of normal curvature, and  $\sigma$  the radius of torsion,

$$\frac{1}{\sigma} = \left( \frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \sin \theta \cos \theta.$$

PROOF.—(Fig. 182.) Let  $OP = ds$  be the geodesic,  $OA, OB$  the lines of curvature, and  $\theta = ACP$ . The angle of torsion  $d\tau$  measures the rotation of the normal to the surface round  $OP = ds$ . But this angle is equal to the sum of the rotations of the normal round  $OA$  and  $OB$  resolved along  $ds$ . For, in travelling along each of the lines  $CN$  and  $NP$ , which are in the directions of the lines of curvature, the normal rotates only about the other. Therefore, if  $\omega_1, \omega_2$  be the rotations round  $OA, OB$ ,  $d\tau = \omega_1 \cos \theta + \omega_2 \sin \theta$ .

But  $\omega_1 = -\frac{ds \sin \theta}{\rho_2}$ ,  $\omega_2 = \frac{ds \cos \theta}{\rho_1}$ ;  $\therefore \frac{1}{\sigma} = \frac{d\tau}{ds} = \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) \sin \theta \cos \theta$ .

**5849** The product  $pd$  has the same value for all geodesics which touch the same line of curvature.

PROOF.—By theorems (5836) and (5842), since the product where they touch it must be the same as that for the line of curvature.

**5850** The product  $pd$  has the same value for all geodesics drawn through any umbilicus on a conicoid.

PROOF.—The semi-diameter  $d$ , in this case, is the radius of a circular section, and therefore equal to the mean semi-axis  $b$  for all the geodesics; and  $p$  is the same for all.

**5851** The geodesics drawn through any point on a conicoid to two umbilici make equal angles with either line of curvature through the point.

PROOF.— $pd$  is the same for each geodesic, by the last, and  $p$  is the same for each; therefore  $d$  is the same, that is, the diameters parallel to the two geodesics at the point are equal; therefore they are equally inclined to each axis of their section; but these axes are parallel to the lines of curvature (5803); therefore, &c.

**5852** Hence the geodesics joining any point to two opposite umbilici lying on the same diameter are continuations of each other.

**5853** The sum of the distances of any point on a line of curvature from two interior umbilici is constant; and the difference of the distances from one interior and one exterior umbilicus is constant.

PROOF.—Geometrically, as in the analogous theorem for the focal distances in a conic, if  $r, r'$  are the distances and  $r+dr, r'+dr'$  the distances for a consecutive point on the line of curvature, it follows from (5851) that  $dr = -dr'$  for interior umbilici and  $dr = dr'$  for exterior ones.

**5854** A system of lines of curvature and the umbilici on a

quadric surface has therefore analogous properties with a system of confocal conics and their foci in a plane, the geodesics corresponding to straight lines.

**5855** In the same way, every surface has a geodesic geometry proper to itself; spherical trigonometry, for instance, being the geodesic geometry of the sphere.

## INVARIANTS.

### INVARIANTS OF A SINGLE FUNCTION.

**5856** The constancy of the ratio  $R^2 : \rho$  in equation (5798) gives rise to the following invariant forms. Since the quadric surface I and the tangent plane II are the same for all positions of the coordinate axes, they have been called respectively the *invariable quadric* and the *invariable plane*. As a consequence,

$$\mathbf{5857} \quad \phi_x^2 + \phi_y^2 + \phi_z^2$$

is an invariant of  $\phi(x, y, z)$ .

PROOF.—By (5791), since the perpendicular from the origin upon the invariable plane is constant. Also, the coefficients of the discriminating cubic (5693) of the invariable quadric will not be altered by transformation of axes. Therefore the following are also invariant forms :—

$$\mathbf{5858} \quad \phi_{2x} + \phi_{2y} + \phi_{2z},$$

$$\mathbf{5859} \quad \phi_{2y}\phi_{2z} + \phi_{2z}\phi_{2x} + \phi_{2x}\phi_{2y} - \phi_{yz}^2 - \phi_{zx}^2 - \phi_{xy}^2,$$

$$\mathbf{5860} \quad \phi_{2x}\phi_{2y}\phi_{2z} + 2\phi_{yz}\phi_{zx}\phi_{xy} - \phi_{2x}\phi_{yz}^2 - \phi_{2y}\phi_{zx}^2 - \phi_{2z}\phi_{xy}^2.$$

**5861** A similar theorem applied to a function  $\phi(x, y)$  of two variables gives the *invariable conic* and *invariable line*; namely,

$$\xi^2\phi_{2x} + 2\xi\eta\phi_{xy} + \eta^2\phi_{2y} = 1 \quad \text{and} \quad \xi\phi_x + \eta\phi_y = 1;$$

and from these the invariants,

$$\mathbf{5863} \quad \phi_x^2 + \phi_y^2, \quad \phi_{2x} + \phi_{2y}, \quad \phi_{2x}\phi_{2y} - \phi_{xy}^2,$$

$$\mathbf{5866} \quad x\phi_y - y\phi_x, \quad x\phi_y + y\phi_x.$$

PROOF.—The last two invariants are obtained from the cosine of the angle between the invariable line (5862) and the fixed line  $y\xi - x\eta = 0$ , joining the point  $xy$  with the origin, or the fixed line  $x\xi + y\eta = 0$ .

### INVARIANTS OF TWO FUNCTIONS.

**5868** An invariant of the two functions  $\phi(x, y)$ ,  $\psi(x, y)$  is

$$\phi_x \psi_x + \phi_y \psi_y.$$

PROOF.—Form the cosine of the angle between the invariable lines  $\xi\phi_x + \eta\phi_y = 1$  and  $\xi\psi_x + \eta\psi_y = 1$ , observing (5863).

Also the two following expressions are invariants,

**5869**  $\phi_{2y} \psi_{2x} + \phi_{2x} \psi_{2y} - 2\phi_{xy} \psi_{xy},$

**5870**  $\phi_{2x} \psi_{2x} + \phi_{2y} \psi_{2y} + 2\phi_{xy} \psi_{xy}.$

PROOF.—From the invariable conics (5861) of  $\phi$  and  $\psi$ , we get

$$(\phi_{2x} + \lambda\psi_{2x}) \xi^2 + 2(\phi_{xy} + \lambda\psi_{xy}) \xi\eta + (\phi_{2y} + \lambda\psi_{2y}) \eta^2$$

invariable for any value of  $\lambda$ . Hence the coefficients of the several powers of  $\lambda$  in the invariant

$$(\phi_{2x} + \lambda\psi_{2x})(\phi_{2y} + \lambda\psi_{2y}) - (\phi_{xy} + \lambda\psi_{xy})^2$$

are also invariants. This gives (5869). Subtracting the latter from the invariant  $(\phi_{2x} + \phi_{2y})(\psi_{2x} + \psi_{2y})$  produces (5870).

### INTEGRALS FOR VOLUMES AND SURFACES.

**5871** If  $V$  be the volume included between the surface  $z = \phi(x, y)$ , three rectangular coordinate planes, the cylindrical surface  $y = \psi(x)$ , and the plane  $x = a$ , Fig. of (1906)

**5872** 
$$V = \iiint dx dy dz = \iint z dx dy.$$

For the limits and demonstration, see (1906).

**5874** The area of the surface  $\phi(x, y, z) = 0$  or  $z = f(x, y)$  will be

$$S = \iint \frac{\sqrt{(\phi_x^2 + \phi_y^2 + \phi_z^2)}}{\phi_z} dx dy \quad \text{or} \quad S = \iint \sqrt{(1 + z_x^2 + z_y^2)} dx dy.$$

PROOF.—The area of the element whose projection is  $dx dy$  will be  $dx dy \sec \gamma$ , where  $\gamma$  is its inclination to the plane of  $xy$ , and therefore the angle between the normal and the  $z$  axis. Therefore

$$\sec \gamma = \sqrt{(\phi_x^2 + \phi_y^2 + \phi_z^2)} \div \phi_z = \sqrt{(1 + z_x^2 + z_y^2)}, \text{ by (1708).}$$

**5875** Let the equation of a surface  $APB$  (Fig. 188) in polar coordinates be  $r = f(\theta, \phi)$ , and let  $V$  be the volume of the sector contained by the planes  $AOB$ ,  $AOP$ , including an angle  $\phi = PHC$ , the given surface  $APB$ , and the portion  $OPB$  of the surface of a right cone whose vertex is  $O$ , axis  $OA$ , and semi-vertical angle  $\theta = AOB$  or  $AOP$ ; then

$$V = \frac{1}{3} \int_0^\phi \int_0^\theta r^3 \sin \theta d\theta d\phi.$$

PROOF.—Through  $P$ , any point on the surface, describe a spherical surface  $PCD$ , with centre  $O$  and radius  $r = OP$ . The volume of the elemental pyramid, vertex  $O$ , base  $Pe$ ,  $= \frac{1}{3} r \cdot Pf \cdot Pg = \frac{1}{3} r \cdot r d\theta \cdot r \sin \theta d\phi$ . Here the error of the small portions, like  $PE$ , ultimately disappears in the summation, since the volume of  $PE$ , being equal to  $\frac{1}{3} dr \cdot r d\theta \cdot r \sin \theta d\phi$ , is of the third order of small quantities; and so in similar instances.

**5876** The area of the same surface  $APB$  (Fig. 188) is

$$S = \int_0^\phi \int_0^\theta r \sqrt{\{(r^2 + r_\theta^2) \sin^2 \theta + r_\phi^2\}} d\theta d\phi.$$

PROOF.—Let the perpendicular from  $O$  upon the tangent plane at  $P$  to the given surface be  $ON = p$ . The element of

$$\text{area } PE = \text{area } Pe \cdot \frac{OP}{ON} = r d\theta \cdot r \sin \theta d\phi \cdot \frac{r}{p} = \frac{r^3 \sin \theta}{p} d\theta d\phi.$$

Substitute the value of  $p$  in (5793).

#### SURFACE OF REVOLUTION.

If  $y = f(x)$  (Fig. 90) be the generating curve, and the  $x$  axis the axis of revolution,  $V$  the volume, and  $S$  the surface included between the planes  $x = a$ ,  $x = b$ ;

$$\mathbf{5877} \quad V = \int_a^b \pi y^2 dx, \quad S = \int_a^b 2\pi y \sqrt{(1 + y_x^2)} dx.$$

PROOF.—The volume of the elemental cylinder of radius  $y$  and height  $dx$  is  $\pi y^2 dx$ . The element of the surface of revolution is

$$2\pi y ds = 2\pi y s_x dx = 2\pi y \sqrt{(1 + y_x^2)} dx. \quad (5113)$$

*Guldin's Rules.*—When the generating curve of a surface of revolution is a closed curve, and does not cut the axis of revolution, a solid annulus, or ring, is formed.

**5879** RULE I.—*The volume of the solid ring is equal to the area of the generating curve multiplied by the circumference of the circle described by the centroid\* of the area.*

**5880** RULE II. — *The surface of the ring is equal to the perimeter of the generating curve multiplied by the circumference described by the centroid of the perimeter.*

PROOF.—Let  $A$  be the area of the closed curve, and  $dA$  any element of  $A$  at a distance  $y$  from the axis of revolution. The volume generated

$$= \int 2\pi y dA = 2\pi \int y dA = 2\pi \bar{y} A,$$

by the definition of the centroid (5885),  $\bar{y}$  being its distance from the axis. Similarly, if  $P$  be the perimeter, writing  $P$  instead of  $A$ .

Quadrature of surfaces bounded by lines of constant gradient.

**5881** Defining the curve ( $\gamma$ ) as the locus of a point on the given surface at which the normal has the constant inclination  $\gamma$  to the  $z$  axis; let  $F(\gamma)$  be the projection of the area bounded by the curve ( $\gamma$ ) upon the  $xy$  plane; then the area itself will be found from the formula,

$$S = \int_0^\gamma \sec \gamma F'(\gamma) d\gamma.$$

PROOF.—The element of area between two consecutive curves ( $\gamma$ ) and ( $\gamma + d\gamma$ ) projected on the  $xy$  plane will be  $dF(\gamma) = F'(\gamma) d\gamma$ ; and, since the slope is the same throughout the curve ( $\gamma$ ), this projected element must be equal to the corresponding element of the surface multiplied by  $\cos \gamma$ .

**5882** RULE. — *Equate coefficients of the equation of the tangent plane with those of  $l\xi + m\eta + n\zeta = p$ , and eliminate  $l$  and  $m$  from  $l^2 + m^2 + n^2 = 1$ . The result will be an equation in  $x, y$  and  $n = \cos \gamma$ , representing the projection of the curve ( $\gamma$ ) upon the  $xy$  plane. From this  $F(\gamma)$  must be found.*

**5883** Ex.—Taking the elliptic paraboloid  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z$ ; the tangent plane at  $xyz$  is  $\frac{\xi x}{a^2} + \frac{\eta y}{b^2} - \zeta = z$ . Equating coefficients of the last with  $l\xi + m\eta + n\zeta = p$ , and substituting for  $l$  and  $m$  in  $l^2 + m^2 + n^2 = 1$ , we obtain for the projection on the  $xy$  plane,  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \tan^2 \gamma$ . The area of this ellipse

\* Centre of mass, or gravity.



is  $F(\gamma) = \pi ab \tan^3 \gamma$ , and therefore  $F'(\gamma) = 2\pi ab \tan \gamma \sec^2 \gamma$ . Consequently, by (5881),  $S = 2\pi ab \int_0^\gamma \tan \gamma \sec^2 \gamma d\gamma = \frac{2}{3}\pi ab (\sec^3 \gamma - 1)$ .

## CENTRE OF MASS.

**5884** DEFINITIONS.—The *moment* of a body with respect to a plane is the sum of the products of each element of mass of the body and the distance of the element from the plane.

**5885** The distance (denoted by  $\bar{x}$ ) of the *centre of mass*\* from the same plane is equal to the *moment* of the body divided by its *mass*.

**5886** NOTE.—If the body be of uniform density, as is supposed to be the case in all the following examples, assume unity for the density, and read *volume* instead of *mass* in the above definitions.

The definition gives the following formulæ for the position of the centre of mass of a uniform body :

**5887** For a *plane curve*,

$$\bar{x} = \frac{\int x ds}{\int ds} = \frac{\int x \sqrt{(1+y_x^2)} dx}{\int \sqrt{(1+y_x^2)} dx} = \frac{\int r \cos \theta \sqrt{(r^2+r_\theta^2)} d\theta}{\int \sqrt{(r^2+r_\theta^2)} d\theta}. \quad (5116)$$

For  $\bar{y}$ , change  $x$  into  $y$  and  $\cos \theta$  into  $\sin \theta$ ; but observe that in all cases, if the body be symmetrical about the axis of  $x$ ,  $\bar{y}$  vanishes. The formula gives the centre of volume of the portion of the curve included between the limits of integration.

For a *plane area*,

$$\bar{x} = \frac{\iint x dx dy}{\iint dx dy} = \frac{\int xy dx}{\int y dx}.$$

The area is bounded by the curve, the  $x$  axis, and the ordinates  $x=a$ ,  $x=b$ , if such be the limits of integration.

For a *plane sectorial area* bounded by two radii  $SP = r$ ,  $SP' = r'$  (Fig. 28) and the curve  $r = F(\theta)$ ;

\* Also called *centre of gravity* or *inertia*, and more recently *centroid*.

$$5892 \quad \bar{x} = \frac{\iint r^2 \cos \theta d\theta dr}{\iint r d\theta dr} = \frac{\frac{2}{3} \int r^3 \cos \theta d\theta}{\int r^2 d\theta}.$$

$$5894 \quad \bar{y} = \frac{\iint r^2 \sin \theta d\theta dr}{\iint r d\theta dr} = \frac{\frac{2}{3} \int r^3 \sin \theta d\theta}{\int r^2 d\theta}.$$

The second forms for  $\bar{x}$  and  $\bar{y}$  give the centroid of an area like  $SPP'$  (Fig. 28). The double integrals applied to that figure require the limits of the integration for  $r$  to be from 0 to  $F(\theta)$ , and afterwards for  $\theta$ , from  $\theta_1 = ASP$  to  $\theta_2 = ASP'$ . But, if applied to the area in (Fig. 109), the order of integration must be reversed, as explained in (5209).

For a *surface of revolution* round the  $x$  axis,

$$5896 \quad \bar{x} = \frac{\int xy \sqrt{(1+y_x^2)} dx}{\int y \sqrt{(1+y_x^2)} dx} = \frac{\int r^2 \sin \theta \cos \theta \sqrt{(r^2+r_\theta^2)} d\theta}{\int r \sin \theta \sqrt{(r^2+r_\theta^2)} d\theta}.$$

PROOF.—By (5885), for the *moment*  $= \int x \cdot 2\pi y ds$  and the *area*  $= \int 2\pi y ds$ ; the second form by (5116). If  $x=a$ ,  $x=b$  are the limits of integration, the surface is bounded by the parallel planes  $x=a$ ,  $x=b$ ; and in the second form, the corresponding values of  $\theta$  are the limits defining the same parallel planes.

For any surface,

$$5898 \quad \bar{x} = \frac{\iint x \sqrt{(1+z_x^2+z_y^2)} dx dy}{\iint \sqrt{(1+z_x^2+z_y^2)} dx dy}. \quad (5874)$$

For a *solid of revolution* round the  $x$  axis,

$$5899 \quad \bar{x} = \frac{\int xy^2 dx}{\int y^2 dx} = \frac{\iint r^3 \sin \theta \cos \theta d\theta dr}{\iint r^2 \sin \theta d\theta dr}.$$

PROOF.—By (5885), for the *moment*  $= \int x \cdot \pi y^2 dx$  and the *volume*  $= \int \pi y^2 dx$ . The limits as in (5896).

**5901** For *any solid figure* bounded as described in (5871), the coordinates of the centroid are given by

$$V\bar{x} = \iiint x dx dy dz = \iint xz dx dy,$$

$$V\bar{y} = \iiint y dx dy dz = \iint yz dx dy,$$

$$V\bar{z} = \iiint z dx dy dz = \frac{1}{2} \iint z^2 dx dy,$$

where  $V = \iiint dx dy dz = \iint z dx dy,$

as in (5872-3), and the limits are as defined in (1906).

**5902** For the *wedge shaped solid* ( $OAPB$ , Fig. 188) defined by the polar coordinates  $r, \theta, \phi$ , as in (5875),

$$V\bar{x} = \frac{1}{4} \iint r^4 \sin^2 \theta \cos \phi d\theta d\phi,$$

$$V\bar{y} = \frac{1}{4} \iint r^4 \sin^2 \theta \sin \phi d\theta d\phi,$$

$$V\bar{z} = \frac{1}{4} \iint r^4 \sin \theta \cos \theta d\theta d\phi,$$

where  $V = \frac{1}{3} \iint r^3 \sin \theta d\theta d\phi.$

PROOF.—By (5875); multiplying the elementary pyramid  $\frac{1}{3}r^3 \sin \theta d\theta d\phi$  separately by the distances of its centroid from the coordinate planes; viz.,  $\frac{3}{4}r \sin \theta \cos \phi$ ,  $\frac{3}{4}r \sin \theta \sin \phi$ , and  $\frac{3}{4}r \cos \theta$ .

## MOMENTS AND PRODUCTS OF INERTIA.

**5903** DEFINITIONS.—The *moment of inertia* of a body about a given right line or axis is the sum of the products of each element of mass and the square of its distance from the line.

**5904** The square of the *radius of gyration* of the body about the given line is equal to the *moment of inertia* of the body divided by its mass.

**5905** The moment of inertia of a body with respect to a plane or point is the sum of the products of each element of mass and the square of its distance from the plane or point.

**5906** The *product of inertia* of a body with respect to two rectangular coordinate planes is the sum of the products of each element of mass and its distances from the two planes.

**5907** Let  $A, B, C$  be the moments of inertia of a body about three rectangular axes;  $A', B', C'$  the moments of inertia with respect to the three planes of  $yz, zx$ , and  $xy$ ; and

$F, G, H$  the products of inertia with respect to the second and third planes, the third and first, and the first and second respectively.  $F, G, H$  are more frequently called the products of inertia about the axes of  $yz, zx,$  and  $xy$  respectively.

By the definitions we have the values

$$\begin{aligned} 5908 \quad A &= \Sigma m (y^2 + z^2), & F &= \Sigma m yz, \\ B &= \Sigma m (z^2 + x^2), & G &= \Sigma m zx, \\ C &= \Sigma m (x^2 + y^2), & H &= \Sigma m xy. \end{aligned}$$

$$\begin{aligned} 5914 \quad \left. \begin{aligned} A' &= \Sigma m x^2 = S - A \\ B' &= \Sigma m y^2 = S - B \\ C' &= \Sigma m z^2 = S - C \end{aligned} \right\} \quad \text{where } S &= \frac{A + B + C}{2}, \\ &= \Sigma m (x^2 + y^2 + z^2), \\ &= \Sigma m r^2. \end{aligned}$$

**5920 Theorem I.**—The M. I. of a lamina about an axis perpendicular to its plane is equal to the sum of the two M. I. about any two axes in its plane drawn through the foot of the perpendicular axis and at right angles to each other.

PROOF.—By the definition (5903), and *Eucl. I. 47*.

**5921 Theorem II.**—The M. I. of a body about a given axis, plane, or point is equal to the M. I. about a parallel axis or plane through the centroid, or about the centroid itself respectively, plus the M. I. of the whole mass, supposed collected at the centroid, about the given axis, plane, or point.

PROOF.—In the figure, p. 168, let the given axis be perpendicular to the paper at  $B$ ; let  $A$  be the centroid, and  $m$  an element of mass at  $C$ ; then, for every thin section of the solid parallel to the paper,

$$\begin{aligned} \text{M. I.} &= \Sigma m \cdot BC^2 = \Sigma m (AC^2 + AB^2 - 2AB \cdot AD) \\ &= \Sigma m \cdot AC^2 + \Sigma m \cdot AB^2 - 2AB \cdot \Sigma m \cdot AD. \end{aligned}$$

But  $\Sigma m \cdot AD = 0$ , by (5885), since  $A$  is the centroid of the body, which proves the proposition. Similarly for the plane or point.

**COR. I.**—Hence, if the M. I. about any axis is known, that about any parallel axis can be found without integration. For, let  $I_1$  be the M. I. about a given axis, whose distance from the centroid is  $a$ , and let  $I_2$  be the required M. I. about an axis whose distance from the centroid is  $b$ ; then, by Theorem

II., 
$$I_2 = I_1 - m(a^2 - b^2).$$

COR. II.—The M. I. has the same value for all parallel axes at the same distance from the centroid.

**5922** *Theorem III.*—The product of inertia for two assigned axes is equal to the product for two parallel axes through the centroid of the body plus the product taken for the whole mass collected at the centroid with respect to the assigned axes.

PROOF.—Let  $x = \bar{x} + x'$ ,  $y = \bar{y} + y'$  be the coordinates of an element of the body with respect to the assigned axes;  $\bar{x}$ ,  $\bar{y}$  being the coordinates of the centroid, and  $x'$ ,  $y'$  the coordinates of the same element with respect to parallel axes through the centroid, all axes being parallel to  $z$ . Then

$$\begin{aligned}\Sigma mxy &= \Sigma m(\bar{x} + x')(\bar{y} + y') = \bar{x}\bar{y}\Sigma m + \Sigma mx'y' + \bar{x}\Sigma my' + \bar{y}\Sigma mx' \\ &= \bar{x}\bar{y}\Sigma m + \Sigma mx'y'.\end{aligned}$$

Since  $\Sigma mx'$  and  $\Sigma my'$  vanish by the definition of the centroid.

**5923** The M. I. of a body with respect to a point is equal to the M. I. for any plane through the point plus the M. I. about the normal to the plane through the point.

PROOF.—For the origin and  $yz$  plane,

$$\Sigma mx^2 + \Sigma m(y^2 + z^2) = \Sigma mr^2. \quad (5908, '14, '19)$$

**5924** Given the moments and products of inertia,  $A$ ,  $B$ ,  $C$ ,  $F$ ,  $G$ ,  $H$ , as above, about three rectangular axes, the moment of inertia of the body about a line through the origin, whose direction cosines are  $l$ ,  $m$ ,  $n$ , will be

$$I = Al^2 + Bm^2 + Cn^2 - 2Fmn + 2Gnl + 2Hlm.$$

PROOF.—(Fig. 11.) Let  $xyz$  be a point  $P$  of the body,  $OM$  the line  $lmn$ , and  $PM$  the perpendicular upon it. Then the M. I. about  $OM$

$$= \Sigma m(OP^2 - OM^2) = \Sigma m\{(x^2 + y^2 + z^2)(l^2 + m^2 + n^2) - (lx + my + nz)^2\} \quad (5530)$$

producing the above result, by (5908-13).

#### ELLIPSOIDS OF INERTIA.

**5925** The equation of the Momental Ellipsoid is

$$Ax^2 + By^2 + Cz^2 - 2Fyz - 2Gzx - 2Hxy = M\epsilon^4,$$

obtained by putting  $Ir^2 = M\epsilon^4$ .  $M$  being the mass of the body, and  $\epsilon^4$  a constant to make the equation homogeneous. Hence the square of the radius of the momental ellipsoid for any point varies inversely as the moment of inertia of the body about that radius.

**5926** If the products of inertia vanish, the axes are called the *principal axes* of the body.

**5927** At every point of a body there are always three principal rectangular axes.

PROOF.—These are evidently the principal axes of the momental ellipsoid of the point; for if the coordinate axes be made to coincide with the former,  $F, G, H$  will vanish.

**5928** The equation of the momental ellipsoid referred to its principal axes will be

$$Ax^2 + By^2 + Cz^2 = Me^4.$$

**5929** The moment of inertia about a line  $lmn$  will now be

$$I = Al^2 + Bm^2 + Cn^2.$$

#### THE ELLIPSOID OF GYRATION.

**5930** The equation of the Ellipsoid of Gyration referred to principal axes is

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = \frac{1}{M}.$$

It is the reciprocal surface of the momental ellipsoid (5719), and its property is—

**5931** The moment of inertia about the perpendicular from the origin upon the tangent plane varies as the square of the perpendicular.

**5932** For any other rectangular axes through the point, the equation of the ellipsoid of gyration is, by (5717),

$$\begin{vmatrix} A & -H & -G & x \\ -H & B & -F & y \\ -G & -F & C & z \\ x & y & z & \frac{1}{M} \end{vmatrix} = 0,$$
 being the reciprocal surface of the momental ellipsoid,  
 $(A, B, C, -F, -G, -H \propto xyz)^2 = M,$   
 with the radius of the sphere of reciprocation = 1. The equation when expanded becomes

**5933**  $(BC - F^2)x^2 + \dots + 2(FG + CH)xy = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} \frac{1}{M}.$



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 LEGENDRE'S EQUI-MOMENTAL ELLIPSOID.

**5934** The equation is

$$\frac{x^2}{A'} + \frac{y^2}{B'} + \frac{z^2}{C'} = \frac{5}{M},$$

with the values in (5914).

**5935** The mass of this ellipsoid is taken equal to that of the body, and it has the same principal moments of inertia.

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## THE MOMENTAL ELLIPSOID FOR A PLANE.

**5936** If  $A', B', C'$  be the moments of inertia for the three coordinate planes, as in (5914), the M. I. for a plane through the origin whose dir. cos. are  $l, m, n$ , will be

$$I' = A'l^2 + B'm^2 + C'n^2 + 2Fmn + 2Gnl + 2Hlm.$$

PROOF:  $I' = \Sigma m (lx + my + nz)^2 = \Sigma mx^2 \cdot l^2 + \&c. = A'l^2 + \&c.$

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**5937** The momental ellipsoid for this plane will be

$$A'x^2 + B'y^2 + C'z^2 + 2Fyz + 2Gzx + 2Hxy = M\epsilon^2,$$

and its property is—

**5938** *The M. I. for any plane passing through the centre of the ellipsoid is equal to the inverse square of the radius perpendicular to the plane.*

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**5939** If  $r$  be a radius of this ellipsoid, and  $a, b, c$  its semi-axes, the M. I. about  $r$

$$= \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - \frac{1}{r^2}.$$

PROOF.—(Fig. 11.) M. I. about  $r$ , plus M. I. for the plane  $OM$  perpendicular to  $r$

$$= \Sigma m OP^2 = \Sigma mx^2 + \Sigma my^2 + \Sigma mz^2 = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}, \text{ by (5938).}$$


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## EQUI-MOMENTAL CONE.

**5940** The equation of the equi-momental cone at any point of a body, referred to principal axes of the body at the point,

is  $(A-I)x^2 + (B-I)y^2 + (C-I)z^2 = 0$ ,

its property being that

**5941** *The generating line passes through the given point, and moves so that the M. I. about it is a constant = I.*

PROOF.—Let  $lmn$  be the generating line in one position, then

$$Al^2 + Bm^2 + Cn^2 = I(l^2 + m^2 + n^2). \quad \text{Therefore, \&c.}$$

**5942** *Theorem.*—If two systems have the same mass, the same centroid, principal axes and principal moments of inertia at the centroid, they have equal moments of inertia about any right line whatever, and are termed *equi-momental*. By (5906) and (5929).

**5943** If two bodies are equi-momental, their projections are equi-momental.

PROOF.—If the projection be from the  $xy$  plane in the ratio  $1 : n$ , the coordinates  $x, y, z$  of a particle become  $x, y, nz$ , and the mass  $m$  becomes  $nm$ . The conditions in (5942) will then be fulfilled.

#### MOMENT OF INERTIA OF A TRIANGLE.

**5944** The M. I. of a triangle  $ABD$  (Fig. 190) about a side  $BD$ , distant  $p$  from the opposite vertex  $A$ , is

$$I = \frac{mp^2}{6}.$$

PROOF.—Let  $BD \equiv a$  and  $EF \equiv y$ ;  $I = \int_0^p \frac{a(p-y)}{p} y^2 dy = \frac{ap^3}{12} = m \frac{p^2}{6}$

**5945** The M. I. of a triangle  $ABC$  (Fig. 190) about a straight line  $BD$  passing through a vertex  $B$ , and distant  $p$  and  $q$  from the vertices  $A$  and  $C$ , is

$$I = m \frac{p^2 + pq + q^2}{6}.$$

PROOF.—By (5944), taking difference of M. I. of the triangles  $ABD, CBD$ .

**5946** The M. I. of a triangle  $ABC$  about an axis through its centroid parallel to  $BD$ , is

$$I = m \frac{p^2 - pq + q^2}{18}. \quad \text{By (5921)}$$

**5947** COR.—If the triangle be isosceles, so that  $p = q$ , the last two moments of inertia become

$$\frac{mp^2}{2} \quad \text{and} \quad \frac{mp^2}{18}.$$

**5949** The M. I. of the triangle about axes perpendicular to  $ABC$  through  $B$  and through the centroid, respectively, are

$$m \frac{3(c^2 + a^2) - b^2}{12} \quad \text{and} \quad m \frac{(a^2 + b^2 + c^2)}{36}. \quad (5920)$$

**5951** The M. I. about  $GF$  of the trapezoid  $ACGF$  (Fig. 190), is

$$m \frac{p^2 + q^2}{6}.$$

**5952** The moments and products of inertia of a triangle about any axes are the same for three equal particles, each one-third of the mass of the triangle, placed at the mid-points of its sides.

PROOF.—(Fig. 190.) The M. I. of the three particles at the mid-points of  $AB$ ,  $BC$ ,  $CA$  about  $BD$ , any line through a vertex, will be

$$\frac{M}{3} \left\{ \frac{(p+q)^2}{4} + \frac{p^2}{4} + \frac{q^2}{4} \right\},$$

which is equal to that of the triangle, by (5945).

#### MOMENTAL ELLIPSE.

**5953** If  $\alpha$ ,  $\beta$  be the radii of gyration of a plane area to principal axes  $Ox$ ,  $Oy$ , where  $O$  is the centroid, the equation of the momental ellipse for the point  $O$  will be

$$\alpha^2 x^2 + \beta^2 y^2 = 2\alpha^2 \beta^2.$$

**5954** Also the area is equi-momental with three equal particles, each one one-third of its mass placed anywhere on the ellipse so that  $O$  may be their centroid.

PROOF.—Let  $xy$ ,  $x'y'$ ,  $x''y''$  be the coordinates of three equi-momental particles: then

$$\frac{m}{3} (x^2 + x'^2 + x''^2) = m\beta^2; \quad \frac{m}{3} (y^2 + y'^2 + y''^2) = m\alpha^2; \quad xy + x'y' + x''y'' = 0;$$

and the two systems have the same centroid; therefore

$$x+x'+x''=0 \quad \text{and} \quad y'+y''+y'''=0.$$

Eliminating  $x', y', x'', y''$  between the five equations, we find the equation of (5953) for the locus of  $xy$ .

**5955** The momental ellipse for the centroid of a triangle is the inscribed ellipse touching the sides at their mid-points.

PROOF.—(Fig. 189.) The inscribed ellipse, which touches two sides at their mid-points, also touches the third side at its mid-point, by Carnot's theorem (4779). Now  $DF$  is parallel to  $AC$ , the tangent at  $E$ ; therefore  $BE$ , which bisects  $DF$ , passes through the centre  $O$  of the ellipse. Similarly,  $AD$  passes through it; therefore  $O$  is the centroid of the triangle.

Let  $OE \equiv a'$ , and let  $b'$  be the semi-diameter parallel to  $AC$ ; then  $\frac{ON^2}{a'^2} + \frac{FN^2}{b'^2} = 1$ . But  $ON = \frac{a'}{2}$ , therefore  $FN^2 = \frac{3}{4}b'^2$ . The M. I. about  $OE$ , by (5954),  $= \frac{2}{3}m \frac{3}{4}b'^2 \sin^2 \omega = m \frac{a'^2 b'^2}{2a'^2}$ , where  $a, b$  are the semi-axes.

Hence the M. I. about  $OD, OE, OF$  varies inversely as the squares of those lines, and therefore the ellipse in the diagram is a momental ellipse, since it has six points which fulfil the requirements.

**5956** The projections of a plane area and its momental ellipse form another plane area and its momental ellipse. (5943)

**5957** The M. I. of a tetrahedron  $ABCD$  about any plane through  $A$  is

$$\frac{m}{10} (a^2 + \beta^2 + \gamma^2 + \beta\gamma + \gamma a + a\beta),$$

where  $a, \beta, \gamma$  are the perpendiculars on the plane from  $B, C, D$ .

**5958** The tetrahedron is also equi-momental with four particles, each one-twentieth of the mass, placed at the vertices, and a particle equal to the remaining mass placed at the centroid (5942).

**5959** The equi-momental ellipsoid of a tetrahedron has the same centroid, and touches each edge at its middle point.

PROOF.—By projecting a regular tetrahedron and its equi-momental sphere (for the centroid) of radius  $= \sqrt{3} \times$  radius of inscribed sphere.

**5960** To find the point, if it exists, in a given right line at which the line is a principal axis, and to find the other principal axes at the point.

Let  $O$  be a datum point in the line. Take this for origin, the given line for axis of  $z$ , and  $OX, OY$  for the other axes. Then, if  $h$  be the distance from  $O$  to the required point  $O'$ , and  $\theta$  the angle between  $OX$  and the principal axis  $O'X'$ ,

$$5961 \quad h = \frac{\Sigma myz}{\Sigma my} = \frac{\Sigma mzx}{\Sigma mx} \quad \text{and} \quad \tan 2\theta = \frac{2H}{A-B},$$

where  $A, B, H$  are the moments and product of inertia about  $OX, OY$ .

PROOF.—At the point  $0, 0, h$ ,  $\Sigma m(z-h)x = \Sigma m(z-h)y = 0$ , from which  $h$  is found; and the equation for  $\theta$  is that for determining the principal axes of the elliptic section of the momental ellipsoid, whose equation is  $Ax^2 + 2Hxy + By^2 = Me^2$ , as in (4408).

**5964** The equality of the two ratios in (5961) is the condition that the  $z$  axis should be a principal axis at *some* point of its length.

**5965** If an axis be a principal axis at more than one point of its length, it passes through the centroid of the system; and, conversely, if it be a principal axis at the centroid, it is so at every point of its length.

PROOF.—For  $h$  must be indeterminate in (5961). Therefore  $\Sigma myz = 0$ ,  $\Sigma my = 0$ ,  $\Sigma mzx = 0$ ,  $\Sigma mx = 0$ .

**5966** The principal axes  $O'X', O'Y'$  are parallel to the principal axes of the projection of the body in the original plane of  $xy$ . By (5962-3).

**5967** Given the principal axes of a body at its centroid, to find the principal axes and moments of inertia at any point in the principal plane of  $xy$ .

Let  $C$  in the Figure of (1171) be the centroid,  $CX, CY$  principal axes,  $A, B$  the M. I. about them, and  $P$  the given point. Find two points  $S, S'$ , called *foci of inertia*, such that the  $X$  and  $Y$  moments of inertia there are equal, and therefore

$$B + m.CS^2 = A; \quad \text{giving} \quad CS = CS' = \sqrt{\frac{A-B}{m}} \dots (i.).$$

The internal and external bisectors of the angle  $SPS'$  will be two of the principal axes at  $P$ , and the third will be the normal to the plane.

PROOF.—The  $X$  and  $Y$  principal moments being equal at  $S$ , the moment about every line through  $S$  in this plane is the same. [For  $I = Al^2 + Bm^2 + Cn^2$  and  $n = 0$  and  $A = B$ , therefore  $I = A$ .] Therefore the moments about  $SP$  and  $S'P$  are equal. Therefore the bisectors  $PT$ ,  $PG$  of the angles at  $P$  will be principal axes.

**5968** Let  $SY$ ,  $S'Y'$  be the perpendiculars on  $PT$ , and  $SZ$ ,  $S'Z'$  those upon  $PG$ ; then the M. I. about  $PT$  and  $PG$  will be respectively,

$$A + mSY \cdot S'Y' = B + m \left( \frac{SP + S'P}{2} \right)^2.$$

$$A - mSZ \cdot S'Z' = B + m \left( \frac{SP - S'P}{2} \right)^2.$$

PROOF.—Draw  $CR$  perpendicular to  $SY$ . The M. I. about  $CR$  ( $\theta \equiv SCR$ )  
 $= A \cos^2 \theta + B \sin^2 \theta$  (5929)  $= A - (A - B) \sin^2 \theta$   
 $= A - m CS^2 \sin^2 \theta$  (by i.)  $= A - mSR^2$ .

Therefore M. I. about  $PT = A - mSR^2 + mRY^2$  (5921)

$$= A + m(RY + SR)(RY - SR) = A + mSY \cdot S'Y'$$

$$= A + mBC^2$$
 (1178)  $= B + mAC^2$  (by i.)  $= B + m \left( \frac{SP + S'P}{2} \right)^2$ .

Similarly for the M. I. about  $PG$ .

**5969** Hence, if an ellipse or hyperbola be described with  $S$ ,  $S'$  for foci, the tangent and normal at any point of the curve are principal axes, and the M. I. about either is constant for that curve.

**5970** Similarly, for a point  $P$  in *any* plane through the centroid  $O$ , it may be shown that the same construction will give the axes  $PT$ ,  $PG$  about which the product of inertia vanishes,  $OX$ ,  $OY$  being the axes at  $O$  in the given plane about which the product of inertia vanishes.

**5971** The condition for the existence of a point in a body at which the M. I. about every axis through it shall be the same, is—

*There must be two principal axes of equal moment at the centroid, and the M. I. about each must be less than the third principal moment.*

Two such points will then exist situated on the axis of unequal moment, and equi-distant from the centroid.



**5972** Given the principal axes at the centroid of a body and the moments of inertia about them, to find the principal axes and moments at any other point.

[See (5975) for the result.]

Let  $A, B, C$  be the given principal moments, and let the mass of the body be unity. Then the ellipsoid of gyration at the centroid  $O$ , and a quadric confocal with it, will be

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = 1 \quad \text{and} \quad \frac{x^2}{A+\lambda} + \frac{y^2}{B+\lambda} + \frac{z^2}{C+\lambda} = 1.$$

**5973** PROP. I.—The M. I. is constant for all tangent planes of the confocal, and is equal to the

$$M. I. \text{ for the origin } O + \lambda = S + \lambda. \quad (5919)$$

PROOF.—Let  $l, m, n$  be the dir. cos. of the tangent plane of the confocal,  $p$  the perpendicular on the plane from  $O$ . The M. I. for this plane

$$= M. I. \text{ for a parallel plane through } O + p^2 \quad (5921)$$

$$= A'l^2 + B'm^2 + C'n^2 + p^2 \quad (5936)$$

$$= (S-A)l^2 + (S-B)m^2 + (S-C)n^2 + (A+\lambda)l^2 + (B+\lambda)m^2 + (C+\lambda)n^2 \\ (5914, 5631) = S + \lambda, \text{ which is independent of } l, m, n.$$

**5974** PROP. II.—All these planes are principal planes at their points of contact, and if the three confocals be drawn through any point  $P$ , the tangent planes at  $P$  to the confocal ellipsoid, two-fold hyperboloid, and one-fold hyperboloid, are respectively the principal planes of greatest, least, and mean moments of inertia. The normal to the confocal ellipsoid is the axis of least moment, and the normal to the two-fold hyperboloid is the axis of greatest moment.

PROOF.—Draw any other plane through  $P$ . The perpendicular on it from  $O$  is less than the perpendicular on the parallel tangent plane to the confocal ellipse, and greater in the case of the two-fold hyperbola. Then, by (5921).

The solution of the problem at (5972) is now given by Proposition III.

**5975** PROP. III.—The principal moments of inertia at  $P$  are  $OP^2 - \lambda_1$ ,  $OP^2 - \lambda_2$ ,  $OP^2 - \lambda_3$ , and the normals to the three confocals at  $P$  are the principal axes.

PROOF.—The M. I. about the  $x$  axis at  $P$

$$= M. I. \text{ for the origin } P - M. I. \text{ for the } yz \text{ plane}$$

$$= S + OP^2 - (S + \lambda_1) = OP^2 - \lambda_1 \quad (5921-73).$$

**5976** The principal moments of inertia above, expressed in terms of  $\lambda_1$  of the confocal ellipsoid and  $d_2, d_3$ , its principal semi-diameters conjugate to  $OP$ , will, by (5661), become

$$OP^2 - \lambda_1, \quad OP^2 - \lambda_1 + d_2^2, \quad OP^2 - \lambda_1 + d_3^2.$$

**5977** The condition that the line  $abc, lmn$ , referred to principal axes at the centroid, may itself be a principal axis at some point of its length, is

$$\frac{\frac{a}{l} - \frac{b}{m}}{A - B} = \frac{\frac{b}{m} - \frac{c}{n}}{B - C} = \frac{\frac{c}{n} - \frac{a}{l}}{C - A} = \frac{1}{p}.$$

Here  $abc$  is *any* point on the line, and if a confocal quadric of the ellipsoid of gyration at the centroid be drawn through the stated principal point of the given line,  $p$  is the perpendicular from the origin upon the tangent plane of the confocal at that point.

PROOF.—The given line  $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$  ..... (i.)

must be a normal to the confocal  $\frac{x^2}{A+\lambda} + \frac{y^2}{B+\lambda} + \frac{z^2}{C+\lambda} = 1$  ..... (ii.).

Therefore, by (5629),  $l = \frac{px}{A+\lambda}$ ,  $m = \frac{py}{B+\lambda}$ ,  $n = \frac{pz}{C+\lambda}$  ..... (iii.).

Eliminate  $x, y, z$  from (i.) by means of (iii.), and from the resulting equations eliminate  $p$ , and the condition above is obtained.

Also, by (5631),

$$p^2 = (A+\lambda)l^2 + (B+\lambda)m^2 + (C+\lambda)n^2 = Al^2 + Bm^2 + Cn^2 + \lambda \dots \text{(iv.)}.$$

The principal point  $xyz$  is now found by eliminating  $\lambda$  and  $p$  from equations (iii.), by means of (iv.) and (5977).

## INTEGRALS FOR MOMENTS OF INERTIA.

By the definition (5903), the following indefinite integrals for moments of inertia are obtained:—

**5978** For a plane curve,  $y = f(x)$ , the M. I. about the  $x$  and  $y$  coordinate axes are

$$\int y^2 ds \text{ and } \int x^2 ds; \text{ and therefore } \int (x^2 + y^2) ds = \int r^2 ds$$

is the M. I. about an axis perpendicular to both the former through the origin (5920).

**5980** Observe that  $ds$  may be changed into  $dx$ ,  $dy$ , or  $d\theta$  by the substitution formulæ (5113, '16).

**5981** For a plane area bounded by the coordinate axes, the ordinate  $y$  and the curve  $y = f(x)$ , the M. I. about the  $x$  and  $y$  axes are

$$\iint y^2 dx dy = \frac{1}{3} \int y^3 dx \quad \text{and} \quad \iint x^2 dx dy = \int x^2 y dx.$$

**5983** And the M. I. about an axis perpendicular to both the former drawn through the origin,

$$= \iint (x^2 + y^2) dx dy = \iint r^3 dr d\theta = \frac{1}{4} \int r^4 d\theta,$$

but in the last two integrals the area has the boundaries described in (5894).

**5986** The M. I. of a solid bounded by three rectangular coordinate planes and the surface  $z = f(x, y)$  about the  $z$  axis, will be

$$\iint (x^2 + y^2) z dx dy = \iiint r^4 \sin^3 \theta dr d\theta d\phi,$$

but in the last integral the solid is bounded as described in (5875).

**5988** The volume, which represents the mass in all these cases, has already been expressed (5205, 5871); and by dividing by the volume, the square of the radius of gyration of the solid is found (5904).

PROOFS.—Formulæ (5981–3) are directly obtainable by geometry from figures 90 and 91, and formulæ (5986–7) from figures 168 and 188. The transition to polar coordinates may also be effected by the formula of (2774).

**5989** In expressing moments of inertia, the factor  $m$  will stand for the mass of the body, and the remaining factor will therefore be the value of the square of the radius of gyration.

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PERIMETERS, AREAS, VOLUMES,  
CENTRES OF MASS, AND MOMENTS OF INERTIA  
OF VARIOUS FIGURES.

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RECTANGULAR LAMINA AND RIGHT SOLID.\*

For a *rectangle* whose sides are  $a, b$ , the *moments of inertia* about the sides, and an axis perpendicular to both where they meet, are respectively

$$6015 \quad m \frac{b^2}{3}, \quad m \frac{a^2}{3}, \quad m \frac{a^2 + b^2}{3}.$$

PROOF:  $\int_0^b ax^2 dx = \frac{ab^3}{3} = m \frac{b^2}{3}$ . The third by (5920).

6018 Hence, for a *right solid*, whose dimensions are  $2a, 2b, 2c$ ,

$$M. I. \text{ about the axis of figure } 2c = m \cdot \frac{a^2 + b^2}{3}.$$


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ARC OF A CIRCLE.

6019 Let  $AB$  (Fig. 191) be the arc of a circle whose centre is  $O$  and radius  $r$ . Let the angle  $AOB \equiv \theta$ ; then

$$\text{Length of arc } AB = r\theta. \quad (601)$$

6020 Huygens' Approximation.—RULE.—From 8 times the chord of half the arc take the chord of the whole arc, and divide the remainder by 3.

PROOF.—The rule gives  $\frac{r}{3} \left( 16 \sin \frac{\theta}{4} - 2 \sin \frac{\theta}{2} \right)$ .

Expand the sines by (764) as far as  $\theta^3$ , and the result is  $r\theta$ .

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6021 Taking an axis  $OX$  through the mid-point of the arc with origin  $O$ , the *centroid of the arc* is given by (5889)

$$\bar{x} = \frac{2r \sin \frac{1}{2}\theta}{\theta}. \quad \text{Hence for a semi-circle } \bar{x} = \frac{2r}{\pi}.$$


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\* For  $M. I.$  of a triangle, see (5944-52).

**6023** Also, for the centroid of  $BX$ ,  $\bar{y} = \frac{2r \sin^2 \frac{1}{2}a}{a}$ ,  
where  $a = \angle XOY$ .

**6025** The *M. I.* of the arc  $AB$  about  $OX$  and  $OY$  are

$$\frac{mr^2}{2} \left(1 - \frac{\sin \theta}{\theta}\right) \quad \text{and} \quad \frac{mr^2}{2} \left(1 + \frac{\sin \theta}{\theta}\right). \quad (5978)$$

**6027** *M. I.* about axes perpendicular to  $XOY$ , through  $O$  and  $X$  the mid-point of the arc respectively, are

$$mr^2 \quad \text{and} \quad m \cdot 2r^2 \left(1 - \frac{\sin \frac{1}{2}\theta}{\frac{1}{2}\theta}\right). \quad (5979)$$

**6029** COR.—The *M. I.* of a }  
circular ring about a diameter }  $= \frac{mr^2}{2}$ .

#### SECTOR OF CIRCLE.

$AOB$  (Fig. 191)

**6030**  $Area = \frac{r^2\theta}{2}, \quad \bar{x} = \frac{4r \sin \frac{1}{2}\theta}{3\theta}. \quad \text{For } XOB, \bar{y} = \frac{4r \sin^2 \frac{1}{2}a}{3a}.$

PROOF.— $\bar{x}, \bar{y}$  are respectively  $\frac{2}{3}$  of  $\bar{x}, \bar{y}$  in (6021, '3); since the centroid of each elemental sector is distant  $\frac{2}{3}r$  from  $O$ . Otherwise, by (5893, '5).

**6033** The *M. I.* about  $OX$  and  $OY$  are

$$\frac{mr^2}{4} \left(1 - \frac{\sin \theta}{\theta}\right) \quad \text{and} \quad \frac{mr^2}{4} \left(1 + \frac{\sin \theta}{\theta}\right).$$

PROOF.—By (5981-2); or integrate (6025-6) for  $r$  from 0 to  $r$ .

#### SEGMENT OF CIRCLE.

$ABX$  (Fig. 191)

**6035**  $Area = \frac{r^2}{2} (\theta - \sin \theta), \quad \bar{x} = \frac{4r \sin^3 \frac{1}{2}\theta}{3(\theta - \sin \theta)}.$

**6037** For  $CBX$ ,  $\bar{y} = \frac{r(2 - 3 \cos a + \cos^3 a)}{3(a - \sin a \cos a)}.$

PROOFS.—From the sector and triangle; otherwise, the centroid, by (5893, '5).

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**6038** *M. I.* about  $OX$  and  $OY$ , (5981-2)

$$\frac{r^4}{24} (3\theta - 4 \sin \theta + \sin \theta \cos \theta) \quad \text{and} \quad \frac{r^4}{8} (\theta - \sin \theta \cos \theta).$$


---

**6040** COR.—Hence, for a *semi-circle*,  $\bar{x} = \frac{4r}{3\pi}$ .

**6041** Also, the *M. I.* of a *circle* about a diameter, and about a central axis perpendicular to its plane, are respectively

$$\frac{mr^2}{4} \quad \text{and} \quad \frac{mr^2}{2}. \quad (5920)$$


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#### THE RIGHT CONE.

If  $h$  be the height,  $r$  the radius of the base, and  $l$  the slant,

**6043** *Curved surface*  $= \pi rl$ . *Volume*  $= \frac{1}{3} \pi r^2 h$ .

**6045** Distance of *centroid* from vertex  $= \frac{3}{4} h$ .

**6046** *M. I.* about axis of figure  $= m \frac{3}{10} r^2$ .

**6047** *M. I.* about cross axes through the vertex and centroid respectively.

$$m \frac{3}{20} (r^2 + 4h^2) \quad \text{and} \quad m \frac{3}{80} (4r^2 + h^2).$$


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#### FRUSTUM OF CYLINDER.

Let  $\theta$  be the inclination of the cutting plane to the base, and  $c$  the length of the axis intercepted.

**6048** The distance of the *centroid* from the axis is

$$\bar{x} = \frac{a^2 \tan \theta}{4c}.$$

**6049** The *M. I.* about the axis  $= m \frac{a^2}{2}$ , being the same as that of a cylinder of height  $c$ . Hence, by (5921) and the value of  $\bar{x}$  above, the *M. I.* about any line parallel to the axis can be found.



## SEGMENT OF SPHERICAL SURFACE. (Fig. 191)

Let  $O$  be the origin of coordinates;  $OA \equiv r$  the radius; and  $OC \equiv x$  the abscissa of  $AB$  the plane of section.

**6050** The curved area of  $AB = 2\pi r(r-x)$  = the area of its projection on the enveloping cylinder of the sphere.

PROOF: 
$$\text{Area} = \int_x^r 2\pi y \frac{r}{y} dx = 2\pi r(r-x). \quad (5878)$$

**6051** For centroid of surface,  $\bar{x} = \frac{1}{2}(r+x)$ .

**6052** The M. I. about the axes  $OX$ ,  $OY$  are

$$\frac{m}{3}(2r^2 - rx - x^2) \quad \text{and} \quad \frac{m}{6}(4r^2 + rx + x^2).$$

## HEMISPHERICAL SURFACE.

**6054**  $\text{Area} = 2\pi r^2, \quad \bar{x} = \frac{r}{2}. \quad (6050-1)$

**6056**  $M. I. \text{ about } OX \text{ or } OY = m \frac{2}{3}r^3. \quad (6052-3)$

## SEGMENT OF SPHERE.

**6057**  $\text{Volume} = \frac{\pi}{3}(2r+x)(r-x)^2, \quad \bar{x} = \frac{3(r+x)^2}{4(2r+x)}.$

**6059**  $M. I. \text{ about } OX = \frac{\pi}{30}(r-x)^3(8r^2 + 9rx + 3x^2).$

**6060**  $M. I. \text{ about } OY$

$$= \frac{\pi}{60}(r-x)^2(16r^3 + 17r^2x + 18rx^2 + 9x^3).$$

PROOF.—As in (6146-7); or put  $a = b = c$  in the results.

## HEMISPHERE.

**6061**  $\text{Volume} = \frac{2}{3}\pi r^3, \quad \bar{x} = \frac{3}{8}r. \quad (6064)$

PROOF.—Vol. = surface (6054)  $\times \frac{r}{3}$ , by elemental pyramids having their common vertex at the centre of the sphere. Otherwise, make  $x = 0$  in (6057).

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**6063** M. I. about  $OX$  or  $OY = m \frac{2}{3} r^3$ . (6059-60)

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## SECTOR OF SPHERE.

**6064**  $Volume = \frac{2}{3} \pi r^3 (r-h), \quad \bar{x} = \frac{3}{8} (r+h).$

PROOF.—Vol. = surface (6050)  $\times \frac{r}{3}$ .  $\bar{x} = \frac{3}{4}$  of  $\bar{x}$  in (6051), since the centroid of each elemental pyramid is distant  $\frac{3}{4}$ ths of  $r$  from the centre.

**6066** For the *M. I.* add together the *M. I.* of the cone and segment (6046, '59).

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THE PARABOLA,  $y^2 = 4ax$ . (Fig. of 1220)

**6067** Rad. of curv.  $\rho = \frac{2SP^2}{SY} = 2a \left(1 + \frac{x}{a}\right)^{\frac{3}{2}}$ . (4542)

**6069** Coordinates of centre of curvature

$$3x+2a, \quad -\frac{y^3}{4a^2}. \quad (4545)$$


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**6071**  $Arc AP \equiv s = \sqrt{(ax+x^2)} + a \log \frac{\sqrt{x} + \sqrt{(a+x)}}{\sqrt{a}}.$

**6072**  $= a [\cot \theta \operatorname{cosec} \theta + \log \cot (\frac{1}{2}\theta)].$

PROOF:  $s = \int \sqrt{\left(1 + \frac{a}{x}\right)} dx. \quad (5197, 4206)$

Substitute  $\sqrt{x}$ , and integrate by (1931).

**6073**  $Arc AL = a \sqrt{2} + a \log (1 + \sqrt{2}).$

---

Centroid of arc  $AP$  with above value of  $s$ .

**6074**  $s\bar{x} = \frac{2x+a}{4} \sqrt{(x^2+ax)} + \frac{a^2}{8} \log \frac{2x+a+2\sqrt{(x^2+ax)}}{a}.$

**6075**  $s\bar{y} = \frac{4}{3} \{ \sqrt{a(x+a)^3} - a^2 \}.$

**6076** For centroid of arc  $AL$ , putting  $x = a$ ,

$$\bar{x} = \frac{6\sqrt{2} + \log(3+2\sqrt{2})}{8\{\sqrt{2} + \log(1+\sqrt{2})\}} a, \quad \bar{y} = \frac{4}{3} \cdot \frac{(2\sqrt{2}-1)a}{\sqrt{2} + \log(1+\sqrt{2})}.$$

Half-segment of parabola  $ANP$ .

**6078**  $Area = \frac{2}{3}xy, \quad \bar{x} = \frac{3}{5}x, \quad \bar{y} = \frac{3}{8}y.$

**6081** The  $M. I.$  about the  $x$  and  $y$  axes are  
 $m \frac{4}{3}ax$  and  $m \frac{3}{7}x^2.$

#### THE ELLIPSE.

**6083** The equation being  $b^2x^2 + a^2y^2 = a^2b^2$ , the *length of the arc AP* (Fig. of 1205), putting  $\phi$  for the eccentric angle of  $P$ , is

$$s = a \int_0^\phi \sqrt{(1 - e^2 \cos^2 \phi)} d\phi.$$

PROOF. — In  $(ds)^2 = (dx)^2 + (dy)^2$  (5113), substitute  $dx = -a \sin \phi d\phi$ ,  $dy = b \cos \phi d\phi$ , by (4276), and use (4260).

**6084** The length of the *elliptic quadrant AB* is

$$\frac{\pi a}{2} \left\{ 1 - \frac{e^2}{4} - \frac{3e^4}{2! 2! 2^4} - \frac{3^3 \cdot 5e^6}{3! 3! 2^6} - \frac{3^3 \cdot 5^2 \cdot 7e^8}{4! 4! 2^8} - \&c. \right\}.$$

PROOF. — Expand the binomial surd above, and employ (2454) and (2472). Similarly, from (5887) and (5978) the three following values are found.

**6085** For the *centroid* of the same quadrant,

$$\bar{x} = \frac{2a}{\pi} \cdot \frac{1 - \frac{1}{8}e^2 - \frac{1}{15}e^4}{1 - \frac{1}{4}e^2 - \frac{3}{64}e^4} \quad \text{approximately.}$$

**6086** The  $M. I.$  about the  $x$  and  $y$  axes are approximately,

$$\frac{mb^2}{2} \cdot \frac{1 - \frac{1}{8}e^2 - \frac{1}{64}e^4}{1 - \frac{1}{4}e^2 - \frac{3}{64}e^4} \quad \text{and} \quad \frac{ma^2}{2} \cdot \frac{1 - \frac{3}{8}e^2 - \frac{5}{64}e^4}{1 - \frac{1}{4}e^2 - \frac{3}{64}e^4}.$$

**6088** *Fagnani's Theorem.* — (Fig. 192.) Let  $P$  be any point on the ellipse,  $CY$  the perpendicular on the tangent at  $P$ ;  $\angle ACY = \theta$ ;  $Q$  the point whose eccentric angle  $= \frac{1}{2}\pi - \theta$ . Then

**6089**  $PY + AP = a \int \sqrt{(1 - e^2 \sin^2 \theta)} d\theta = BQ;$

and in the hyperbola (Fig. 193)

**6090**  $PY - AP = a \int \sqrt{(1 - e^2 \sin^2 \theta)} d\theta.$

**6091** COR. — The difference between the lengths of the infinite curve and asymptote  $= a \int_0^a \sqrt{1 - e^2 \sin^2 \theta} d\theta$ , where  $\tan a = \frac{a}{b}$ .

PROOFS.—By (5203),

$$AP + PY \text{ or } s + q = \int_0^{\theta} p d\theta = a \int \sqrt{1 - e^2 \sin^2 \theta} d\theta = BQ, \text{ by (6083).}$$

In the hyperbola we have  $q - s = \int p d\theta$ .

**6092** Draw the tangent at  $Q$  and the perpendicular  $CU$  upon it. Let  $x, x'$  be the abscissæ of  $P, Q$ . The following relations subsist,

$$PY = \frac{e^2 x x'}{a} = QU, \quad CY \cdot CU = ab, \quad CP^2 + CU^2 = a^2 + b^2.$$

PROOF.—Let  $\phi$  = the eccentric angle of  $P$ , and let  $ACU = \theta'$ . Then

$$\tan \phi = \frac{ay}{bx} = \frac{b}{a} \tan \theta'. \quad (4276-80)$$

$$\text{Similarly for } Q, \quad \tan \left( \frac{\pi}{2} - \theta \right) = \cot \theta = \frac{b}{a} \tan \theta',$$

$$\text{therefore} \quad \tan \phi = \cot \theta' \quad \text{or} \quad \phi = \frac{\pi}{2} - \theta' \dots\dots\dots (i.).$$

The relation therefore between  $P$  and  $Q$  is reciprocal. Now  $PY = e^2 x \sin \theta$  (4295) and  $x' = a \sin \theta$ , therefore  $PY = \frac{e^2 x x'}{a} = QU$ , by the reciprocity.

$$\text{Again, } CU^2 = a^2 \cos^2 \theta' + b^2 \sin^2 \theta' \text{ (4372)} = a^2 \sin^2 \phi + b^2 \cos^2 \phi \dots\dots (ii.).$$

Put  $\phi$  in terms of  $\theta$  by the above, and we find

$$CU^2 = \frac{a^2 b^2}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} = \frac{a^2 b^2}{CY^2}.$$

$$\text{Lastly, } CP^2 + CU^2 = x^2 + y^2 + a^2 \sin^2 \phi + b^2 \cos^2 \phi, \text{ by (ii), } = a^2 + b^2 \text{ (4276-7).}$$

**6095** When  $P$  coincides with  $Q$ , the point is called “Fagnani’s point,”  $CY = \sqrt{ab}$ ,  $PY = a - b$ , and  $x = a^{\frac{3}{2}}(a + b)^{-\frac{1}{2}}$ .

**6096** *Griffiths’ Theorem.\**—If an ellipse of eccentricity  $e$ , and a hyperbola of eccentricity  $e^{-1}$ , be placed as in the figure of 1205 (the circle representing the ellipse),  $P, p$  being considered corresponding points; then, calling  $PQ$ , in (6088), a Fagnanian arc, we have the following theorem:—

\* J. Griffiths, M.A., *Proc. Lond. Math. Soc.*, Vol. v., p. 95.

The ratio of the difference of two Fagnanian arcs on the ellipse to the difference of the two corresponding arcs on the hyperbola is equal to the product of  $e^2$  and the four abscissæ of the points on the ellipse.

#### SECTOR AND SEGMENT OF ELLIPSE.

**6097** The formulæ for the sector and segment of a circle may be adapted to the ellipse by writing  $a$  for  $r$  and multiplying linear dimensions parallel to the minor axis by  $b : a$ . But  $a$  will then represent the eccentric angle of the semi-arc, and  $\theta$  twice that angle. Thus, in the figure of (1205), if  $ACP$  be the half sector,  $a = ACP$ ,  $\theta = 2ACP$ .

Sector of ellipse ( $2ACP$  in fig. of 1205) :

$$\mathbf{6098} \quad \text{Area} = \frac{ab\theta}{2}, \quad \bar{x} = \frac{4a \sin \frac{1}{2}\theta}{3\theta}, \quad \bar{y} = \frac{4b \sin^2 \frac{1}{2}\theta}{3a}, \quad (6030-2)$$

the last being for the half sector  $ACP$ . The M. I. about the  $x$  and  $y$  axes are

$$\mathbf{6101} \quad \frac{mb^3}{4} \left(1 - \frac{\sin \theta}{\theta}\right) \quad \text{and} \quad \frac{ma^3}{4} \left(1 + \frac{\sin \theta}{\theta}\right). \quad (6033)$$

Segment of ellipse ( $2ANP$  in same figure) :

$$\mathbf{6103} \quad \text{Area} = \frac{ab}{2} (\theta - \sin \theta), \quad \bar{x} = \frac{4a \sin^3 \frac{1}{2}\theta}{3(\theta - \sin \theta)}. \quad (6035-6)$$

**6105** For  $\bar{y}$  of the half segment  $ANP$ , and for the M. I. about the  $x$  and  $y$  axes, replace  $r$  by  $b$  in (6037-8) and by  $a$  in (6039).

$$\mathbf{6108} \quad \text{For the whole ellipse, the area} = \pi ab. \quad (6103)$$

$$\mathbf{6109} \quad \text{For the half ellipse, } \bar{x} = \frac{4a}{3\pi}. \quad (6104)$$

**6110** The M. I. about the  $x$  and  $y$  axes, and a third central axis perpendicular to both,

$$\frac{mb^3}{4}, \quad \frac{ma^3}{4}, \quad \text{and} \quad \frac{m(a^2 + b^2)}{4}. \quad (6041-2)$$

**6113** The area of the ellipse whose equation is

$$(a b c f g h \chi x y 1)^2 = 0, \text{ is } = \frac{\pi \Delta}{\sqrt{(ab-h^2)^3}} \text{ or } \frac{\pi \Delta}{\sqrt{C^3}}.$$

PROOF.—If  $\alpha, \beta$  be the semi-axes of the conic, the area  $\pi\alpha\beta$  takes this value, by (4414) and (4407).

**6114** *Lambert's Theorem.*—The area of a focal sector of an ellipse, as  $PSP'$  (Fig. 28), in terms of  $\phi, \phi'$ , the eccentric angles of  $P, P'$ , is

$$\frac{ab}{2} \{\phi - \phi' - e(\sin \phi - \sin \phi')\} = \frac{ab}{2} \{\chi - \chi' - (\sin \chi - \sin \chi')\}.$$

In the second value,  $\sin \frac{\chi}{2}$  and  $\sin \frac{\chi'}{2}$  are  $= \frac{1}{2} \sqrt{\frac{r+r'+c}{a}}$  respectively, where  $r = SP, r' = SP'$ , and  $c = PP'$ ,\* a result of use in Astronomy.

#### THE HYPERBOLA.

**6115** The length of an arc of the hyperbola  $b^2x^2 - a^2y^2 = a^2b^2$  and the abscissa of its centroid may be approximated to, as in (6084) for the arc of an ellipse, by the substitutions from (4278),

$$\int ds = a \int \sec \phi \sqrt{(e^2 \sec^2 \phi - 1)} d\phi$$

and 
$$\int x ds = a^2 \int \sec^2 \phi \sqrt{(e^2 \sec^2 \phi - 1)} d\phi.$$

**6117** *Landen's Theorem.*—This theorem gives any arc of an hyperbola in terms of the arcs of two ellipses, as follows:

$$\int \sqrt{(a^2 + b^2 + 2ab \cos C)} dC = \int \sqrt{(a^2 - b^2 \sin^2 A)} dA + \int \sqrt{(b^2 - a^2 \sin^2 B)} dB + 2a \sin B + \text{const.},$$

that is—Arc of ellipse whose semi-axes are  $a+b$  and  $a-b$  = Arc of ellipse whose major axis is  $2a$  and eccentricity  $b : a$  + difference between a right line and the arc of an hyperbola whose major axis is  $b$  and eccentricity  $a : b$ .†

\* Williamson's Integ. Calc., Art. 137. † Ibid., Art. 167.



**6118** Area  $ANP$  (Fig. of 1183) bounded by  $x, y$ , and the curve

$$= \frac{b}{2a} \left\{ x \sqrt{(x^2 - a^2)} - a^2 \log \frac{x + \sqrt{(x^2 - a^2)}}{a} \right\}. \quad (1931)$$

$$\mathbf{6119} \quad = \frac{1}{2} \left\{ xy - ab \log \left( \frac{x}{a} + \frac{y}{b} \right) \right\}. \quad (4271)$$

**6120** Area of sector between  $CA, CP$  and the curve

$$= \frac{ab}{2} \log \left( \frac{x}{a} + \frac{y}{b} \right).$$

**6121** Area between two ordinates  $y_1, y_2$ , when the asymptotes are the coordinate axes

$$= \frac{ab}{2} \log \frac{x_2}{x_1}.$$

**PROOF:**  $\sin 2ACO \int y dx = \frac{2ab}{a^2 + b^2} \int \frac{a^2 + b^2}{4} \frac{dx}{x}. \quad (4387)$

**6122** The centroid of  $ANP$ ,  $A$  being the area (6118), is given by

$$A\bar{x} = \frac{b}{3a} (x^2 - a^2)^{\frac{3}{2}}; \quad A\bar{y} = \frac{b^2}{2a^2} \left( \frac{x^3}{3} - a^2 x + \frac{2a^3}{3} \right).$$

**6124** The  $M. I.$  of  $ANP$  about the  $x$  and  $y$  axes are

$$\frac{b^3}{24a^3} (2x^3 - 5a^2 x) \sqrt{(x^2 - a^2)} + \frac{ab^3}{8} \log \frac{x + \sqrt{(x^2 - a^2)}}{a}.$$

$$\mathbf{6125} \quad \frac{b}{8a} (2x^3 - a^2 x) \sqrt{(x^2 - a^2)} - \frac{a^3 b}{8} \log \frac{x + \sqrt{(x^2 - a^2)}}{a}.$$

#### THE ELLIPTIC PARABOLOID.

**6126** Equation,  $\frac{x^2}{a} + \frac{y^2}{b} = 2z.$

**6127** Vol. of segment  $= \pi \sqrt{(ab)} \bar{z}^2, \quad \bar{z} = \frac{2}{3}z.$

**6129**  $M. I.$  about the axes of  $x, y$ , and  $z$  respectively,

$$m \left( \frac{az}{3} + \frac{z^3}{2} \right), \quad m \left( \frac{bz}{3} + \frac{z^3}{2} \right), \quad m \frac{a+b}{3} z.$$

**6132** The *surface*  $S$  of the same segment may be found from

$$S = 4 \int_0^{\sqrt{2az}} \int_0^{\sqrt{(2bz - \frac{b}{a}x^2)}} \sqrt{\left(1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}\right)} dx dy. \quad (5874)$$

**6133** If the surface of the paraboloid be bounded by a curve of constant gradient  $\gamma$  (5881), the area becomes

$$S = \frac{2}{3}\pi ab (\sec^3 \gamma - 1). \quad (5883)$$

#### THE PARABOLOID OF REVOLUTION.

**6134** Equation,  $x^2 + y^2 = 2az$  or  $r^2 = 2az$ .

**6136** *Surface of segment*,  $S = \frac{2}{3}\pi \sqrt{a} \{(2z+a)^{\frac{3}{2}} - a^{\frac{3}{2}}\}$ . (5880)

**6137** *Volume*  $= \pi az^2 = \frac{1}{2}\pi r^2 z$ ,  $\bar{z} = \frac{2}{3}z$ . (5887, '99)

**6140** *M. I.* about axis of figure  $= \frac{mr^2}{3}$ . (6131)

**6141** For *M. I.* about  $OX$  and  $OY$  put  $a = b$  in (6129–30).

#### THE ELLIPSOID.

**6142** Equation,  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ , semi-axes  $a, b, c$  (5600).

**6143** The *surface of the segment* cut off by the plane whose abscissa is  $x$ , will be found from

$$S = 4 \int_x^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \left( \frac{a^4b^4 + b^4(c^2-a^2)x^2 + a^4(c^2-b^2)y^2}{a^4b^4 - a^2b^4x^2 - a^4b^2y^2} \right)^{\frac{1}{2}} dx dy.$$

PROOF.—By (5874) and (5629, '7), eliminating  $z$  by means of the equation of the surface.

**6144** The *volume* of the solid segment and the *centroid* are given by

$$V = \frac{\pi bc}{3a^2} (2a+x)(a-x)^2, \quad \bar{x} = \frac{3(a+x)^2}{4(2a+x)}.$$

PROOFS.—Let (Fig. 177) represent one octant of the ellipsoid;  $OA, OB, OC$  being the principal semi-axes. The elemental section

$$4PNQ = \pi NP \cdot NQ dx = \pi \frac{b}{a} \sqrt{a^2-x^2} \frac{c}{a} \sqrt{a^2-x^2} dx.$$

$$\text{Therefore Vol.} = \frac{\pi bc}{a^3} \int_x^a (a^2-x^2) dx = \frac{\pi bc}{3a^2} (2a^3-3a^2x+x^3) = \&c.$$

The moment with respect to the plane of  $yz$

$$= \frac{\pi bc}{a^3} \int_x^a (a^2 x - x^3) dx = \frac{\pi bc}{4a^3} (a^2 - x)^2,$$

and division by the volume gives  $\bar{x}$  as above.

**6146** The  $M. I.$  of the solid segment about the axis  $a$

$$= \frac{\pi bc (b^2 + c^2)}{60a^4} (a - x)^3 (8a^2 + 9ax + 3x^2).$$

PROOF.—(Fig. 177.)

$$M. I. = \int_x^a \pi NP \cdot NQ \cdot \frac{NP^2 + NQ^2}{4} dx \quad (6112) = \frac{\pi bc (b^2 + c^2)}{4a^4} \int_x^a (a^2 - x^2)^2 dx = \&c.$$

**6147** The  $M. I.$  about the axis  $b$

$$= \frac{\pi bc}{15a^2} \left\{ \frac{c^2}{a^2} (a - x)^3 (8a^2 + 9ax + 3x^2) + 2a^5 - 5a^2 x^3 + 3x^5 \right\}.$$

$$\begin{aligned} \text{PROOF: } M. I. &= \int_x^a \pi NP \cdot NQ \left( \frac{NQ^2}{4} + ON^2 \right) dx & (5921) \\ &= \frac{\pi bc}{4a^4} \int_x^a (a^2 - x^2)^2 dx + \frac{\pi bc}{a^3} \int_x^a (a^2 - x^2) x^2 dx = \&c. \end{aligned}$$

**6148** The *volume* of the whole ellipsoid  $= \frac{4}{3}\pi abc$ .

PROOF.—By making  $x = 0$  in (6144).

*Otherwise:* Let  $\xi\eta\zeta$  be the point on the auxiliary sphere of radius  $r$  corresponding to  $xyz$  on the ellipsoid. By (5638-9),  $rx = a\xi$ ,  $ry = b\eta$ ,  $rz = c\zeta$ .

$$\text{Therefore} \quad \int dx dy dz = \frac{abc}{r^3} \int d\xi d\eta d\zeta = \frac{abc}{r^3} \frac{4}{3}\pi r^3. \quad (6061)$$

**6149** For the *centroid* of the semi-ellipsoid  $\bar{x} = \frac{3a}{8}$ . (6145)

**6150** The  $M. I.$  about the axis  $a = m \frac{(b^2 + c^2)}{5}$ . (6146)

**6151** The *volume* of a segment cut off by any plane  $PNQ$  (Fig. 177), where  $OA = d$  is the semi-conjugate diameter, and  $AN = h$ , is

$$V = \pi abc \frac{h^2 (3d - h)}{3d^3}.$$

PROOF.—Taking the area of the section from (5655), the volume of the segment will be

$$\frac{\pi abc \sin \theta}{p} \int_x^d \left( 1 - \frac{x^2}{d^2} \right) dx, \quad \text{where} \quad \sin \theta = \frac{p}{d},$$

$\theta$  being the inclination of  $d$  to the cutting plane. Integrate, and put  $x = d - h$ .

## PROLATE SPHEROID.

Put  $c = b$  in equation (6142) of the ellipsoid; then  $a$  will be the semi-axis of revolution.

**6152** The *surface of the zone* between the plane of  $yz$  and a parallel plane at a distance  $x$  is

$$S = \pi b \left\{ \frac{a}{e} \sin^{-1} \frac{ex}{a} + \frac{x}{a} \sqrt{(a^2 - e^2 x^2)} \right\}.$$

PROOF.—By (5878).  $S = \frac{2\pi be}{a} \int_0^x \sqrt{\left(\frac{a^2}{e^2} - x^2\right)} dx.$

Then by (1933). Otherwise, make  $b = c$  in (6143), and reduce.

**6153** COR.—The *whole surface*  $= 2\pi b \left(b + \frac{a}{e} \sin^{-1} e\right).$

**6154** The *centroid* of the surface of the zone in (6152) is given by

$$\bar{x} = \frac{2\pi be}{3aS} \left\{ \frac{a^3}{e^3} - \left(\frac{a^2}{e^2} - x^2\right)^{\frac{3}{2}} \right\}.$$

PROOF.—From  $S\bar{x} = \frac{2\pi be}{a} \int_0^x x \sqrt{\left(\frac{a^2}{e^2} - x^2\right)} dx.$

**6155** The *M. I.* of the same zone is

$$= \pi ab^3 \left(\frac{1}{e} - \frac{1}{4e^3}\right) \sin^{-1} \frac{ex}{a} + \frac{\pi b^3}{a} \left(1 + \frac{1}{4e^2} - \frac{x^2}{2a^2}\right) x \sqrt{(a^2 - e^2 x^2)}.$$

**6156** And for the whole surface, by making  $x = a$  and doubling,

$$M. I. = \pi ab^3 \left(\frac{2}{e} - \frac{1}{2e^3}\right) \sin^{-1} e + \pi b^4 \left(1 + \frac{1}{2e^2}\right).$$

PROOF:  $M. I. = 2\pi \int y^3 \sqrt{\left(1 + \frac{b^4 x^2}{a^4 y^3}\right)} dx = \frac{2\pi b^3 e}{a^3} \int (a^2 - x^2) \sqrt{\left(\frac{a^2}{e^2} - x^2\right)} dx$   
 $= \frac{2\pi b^3 e}{a} \int \sqrt{\left(\frac{a^2}{e^2} - x^2\right)} dx - \frac{2\pi b^3 e}{a^3} \int x^2 \sqrt{\left(\frac{a^2}{e^2} - x^2\right)} dx.$

The first integral by (1933). For the second, by Rule VI. 2048, we obtain the formula

**6157**  $\int x^2 \sqrt{(a^2 - x^2)} dx = \frac{a^4}{8} \sin^{-1} \frac{x}{a} + \frac{2x^2 - a^2}{8} x \sqrt{(a^2 - x^2)},$

in which  $\frac{a}{e}$  must now be written for  $a$ .

**6158** For the *volume, moment of inertia, and abscissa of centroid* of the solid prolate spheroid, make  $c = b$  in (6144–51),  $a$  being the axis of revolution.

## OBLATE SPHEROID.

**6159** Put  $b = a$  in the equation (6142) of the ellipsoid; then  $c$  will be the semi-axis of revolution.

The surface of the zone between the plane of  $xy$  and a parallel plane at a distance  $z$ , is

$$S = \frac{\pi a}{c^2} z \sqrt{(c^4 + a^2 e^2 z^2)} + \frac{\pi c^2}{e} \log \frac{aez + \sqrt{(c^4 + a^2 e^2 z^2)}}{c^2}.$$

PROOF.—By (5878).  $S = \frac{2\pi a^2 e}{c^2} \int_0^z \sqrt{\left(\frac{c^4}{a^2 e^2} + z^2\right)} dz$ . Then by (1931).

**6160** COR.—The whole surface  $= 2\pi a^2 + \frac{\pi c^2}{e} \log \frac{1+e}{1-e}$ .

**6161** The centroid of the surface of the zone in (6159) is given by

$$\bar{z} = \frac{2\pi a^2 e}{3c^2 S} \left\{ \left( \frac{c^4}{a^2 e^2} + z^2 \right)^{\frac{3}{2}} - \frac{c^6}{a^3 e^3} \right\}.$$

PROOF.—As in (6154).  $\bar{z}$  for the surface of half the spheroid is obtained in this case by making  $z = c$ , but in (6154) put  $x = a$ .

**6162** The M. I. of the same zone is

$$M. I. = \frac{\pi a^3}{c^2} \left( 1 - \frac{c^2}{4a^2 e^2} - \frac{z^2}{2c^2} \right) z \sqrt{(c^4 + a^2 e^2 z^2)} \\ + \frac{\pi c^2 (4a^2 - 3c^2)}{4e^3} \log \frac{aez + \sqrt{(c^4 + a^2 e^2 z^2)}}{c^2}.$$

**6163** And for the whole surface, by making  $z = c$  and doubling,

$$M. I. = \pi a^4 \left( 1 - \frac{c^2}{2a^2 e^2} \right) + \frac{\pi c^2 (4a^2 - 3c^2)}{2e^3} \log \frac{a(1+e)}{c}.$$

PROOF:  $M. I. = 2\pi \int x^3 \sqrt{\left(1 + \frac{a^4 z^2}{c^4 x^2}\right)} dz = \frac{2\pi a^4 e}{c^4} \int (c^2 - z^2) \sqrt{\left(\frac{c^4}{a^2 e^2} + z^2\right)} dz$ .

The first integral involved is given at (1931), and the second is obtained in the same way as in the Proof of (6155), giving

**6164**  $\int x^3 \sqrt{x^2 + a^2} dx = \frac{2x^3 + a^2 x}{8} \sqrt{(x^2 + a^2)} - \frac{a^4}{8} \log \{x + \sqrt{(x^2 + a^2)}\}.$

**6165** For the *volume, moment of inertia, and abscissa of centroid* of the solid oblate spheroid, make  $b = a$  in (6144–51),  $c$  being the axis of revolution.

JOINT INDEX  
TO THE  
SYNOPSIS  
AND TO THE  
PAPERS ON PURE MATHEMATICS,  
CONTAINED IN THE UNDERMENTIONED  
BRITISH AND FOREIGN JOURNALS  
AND  
TRANSACTIONS OF SOCIETIES.



"There is an immense amount of knowledge lying scattered at the present day,  
and almost useless from the difficulty of finding it when wanted."

—*Professor J. D. Everett.*

## KEY TO THE INDEX.

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Prefixed to each title will be found the symbol by which the work is referred to in the Index. The words "*with Vol.*" or "*with Year,*" signify that any number following the symbol in the Index denotes, respectively, the Volume or Year of the journal. The year is given in all cases in which the work consists of more than one series of volumes. In order to connect the volumes with the years of publication, a Chronological Table is prefixed to the Index; in which table successive series of numbers in any column indicate successive series of volumes of the publication.

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- A. *with Vol.*—Archiv der Mathematik; 1843 to 1884; 70 vols. [B. M. C.: P.P. 1580.] \*
- Ac. *with Vol.*—Acta Mathematica, Zeitschrift Journal, herausgegeben von G. Mittag-Leffler; Stockholm, 1882 to 1885; 7 vols.
- AJ. *with Vol.*—American Journal of Mathematics; Baltimore. Editor: J. J. Sylvester, F.R.S.; 1878 to 1885; 7 vols. [B. M. C.: P.P. 1575.b.]
- An. *with Year.*—Annali di Scienze Matematiche e Fisiche, compilati da Prof. Barnaba Tortolini; Rome, 1850–57; *afterwards*—Annali di Matematiche pura et applica; Rome, 1858–65. Series II., Annali di Matematiche pura et applica, compilati da F. Brioschi e L. Cremona; Milan, 1868–85; 23 vols. in all. [B. M. C.: P.P. 1573 and 952.]
- At. *with Year.*—Atti della Reale Accademia delle Scienze di Napoli; 1819 to 1878; 15 vols. [B. M. C.: for 1819–55, 8 vols., Acad. 2813; for 1863 to 1878, 7 vols., Acad. 96.]
- C. *with Vol.*—Comptes rendus hebdomadaires des séances de l'Académie des Sciences; Paris, 1835 to 1885; 100 vols. [B. M. C.: Acad. 424 and R.R. 2099.c.] †
- CD. *with Vol.*—Cambridge and Dublin Mathematical Journal. Editor, W. Thomson, B.A.; 1846 to 1854; 9 vols. [B. M. C.: P.P. 1565.]
- CM. *with Vol.*—Cambridge Mathematical Journal; 1839 to 1845; 4 vols. [B. M. C.: P.P. 1565.]
- CP. *with Vol.*—Cambridge Philosophical Transactions; 1822 to 1881; 13 vols. [B. M. C.: Acad. 3008.]
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\* i.e., British Museum Catalogue, Press mark P.P. 1580.

† R.R. signifies *Reading-Room volumes within reach.*

- E. with Vol.**—Educational Times Reprint of Mathematical Questions and Solutions, with additional papers; London, half-yearly, 1863 to 1885; [44](#) vols. Editor: W. J. C. Miller, B.A. [B. M. C.: 2242.c.]
- G. with Vol.**—Giornale di Matematichead uso degli studenti delle università Italiane, pubblicato per cura del professore G. Battaglini; Naples, 1863–85; [23](#) vols. [B. M. C.: P.P. 1572.]
- L. with Vol.**—Journal of the Institute of Actuaries, or, The Assurance Magazine; London, 1850–84; [24](#) vols. [B. M. C.: P.P. 1423.g.g. and [126.](#)]
- J. with Vol.**—Journal für die reine und angewandte Mathematik, herausgegeben von A. L. Crelle; 1826–1856; and Journal als Fortsetzung des von A. L. Crelle gegründeten Journals von C. W. Borchardt; Berlin, 1856–1884; [97](#) vols. [B. M. C.: P.P. 1585 and R.R. 2022.g.]
- JP. with Vol.**—Journal de l'Ecole Polytechnique; Paris, 1796 to 1884; [34](#) vols. [B. M. C.: T.C. [1.b.](#)]
- L. with Year.**—Journal de Mathématiques pures et appliquées, ou Recueil mensuel de mémoires sur les diverses parties des Mathématiques, publié par Joseph Liouville; Paris, 1836 to 1884; [49](#) vols. [B. M. C.: P.P. 1575 and R.R. 2022.g.]
- LM. with Vol.**—London Mathematical Society's Proceedings; 1866 to 1885; [16](#) vols. [B. M. C.: Acad. 4265, [2.](#)]
- M. with Vol.**—Mathematische Annalen, in Verbindung mit C. Neumann begründet durch R. F. A. Clebsch unter Mitwirkung der Herren Prof. P. Gordan, Prof. C. Neumann, Vols. [1–9](#); and Prof. K. V. Mühl, gegenwärtig herausgegeben von Prof. F. Klein und Prof. A. Mayer, Vols. [10](#), &c. Leipzig, 1869–1885; [25](#) vols. [B. M. C.: P.P. [1581.b.](#)]
- Man. with Year.**—Manchester Memoirs, or, Memoirs of the Literary and Philosophical Society of Manchester; 1805 to 1884; [23](#) vols. [B. M. C.: [255.d.](#), [9–12](#), and Acad. 1360.]
- Me. with Year.**—The Oxford, Cambridge, and Dublin Messenger of Mathematics; 1862 to 1871; [5](#) vols. Continued as—The Messenger of Mathematics. Editors: W. A. Whitworth, M.A., C. Taylor, D.D., R. Pendlebury, M.A., J. W. L. Glaisher, F.R.S.; Cambridge, 1872 to 1885; [14](#) vols. [B. M. C.: P.P. [1565.b.](#) and [463.](#)]
- Mél. with Vol.**—Mélanges mathématiques de l'Académie des Sciences de Saint Petersburg; 1849 to 1883; [6](#) vols. [B. M. C.: Acad. [1125/9.](#)]
- Mém. with Year.**—Mémoires de l'Académie Impériale des Sciences de



- Saint Petersburg; 1809 to 1883; [51](#) vols. [B. M. C.: 1809-30, *T.C.* [9.a.](#), [14-20](#); 1831-59, *Acad.* [1125/2](#); 1859-83, *Acad.* [1125/3](#).]
- Mo. *with Year*.—Monatsbericht der Königlich-Preussischen Akademie der Wissenschaften zu Berlin; 1856 to 1881; *continued as*—Sitzungsberichte der Königlich, &c.; 1881 to 1885; [30](#) vols. [B. M. C.: *Acad.* [855](#).]
- N. *with Year*.—Nouvelles Annales de Mathématiques. Journal des Candidats aux Ecoles Polytechnique et Normal. Rédigé par MM. Terquem et Gerono; Paris, 1842 to 1882; [41](#) vols. [B. M. C.: *P.P.* 1544.]
- No. *with Year*.—Nova Acta Regiæ Societatis Scientiarum Upsaliensis; 1775 to 1884; [26](#) vols. [B. M. C.: for 1775-1850, [14](#) vols., *T.C.* [6.b.](#), [13-19](#); for 1851-1884, [12](#) vols., *Acad.* 1076.]
- P. *with Year*.—The Philosophical Transactions of the Royal Society of London; 1781 to 1884; [104](#) vols: *i.e.*, vols. [71](#) to [174](#). [B. M. C.: *R.R.* 2021.*g*.]
- Pr. *with Vol*.—Proceedings of the Royal Society of London; 1800 to 1885; [39](#) vols. [B. M. C.: for vols. [1](#) to [23](#), *Acad.* 3025, [21](#); for vols. [24](#), &c., *R.R.* 2101.*d*.]
- Q. *with Vol*.—Quarterly Journal of Mathematics; [20](#) vols. Cambridge, 1857-78; Editors: J. J. Sylvester, F.R.S., N. M. Ferrers, F.R.S., G. G. Stokes, F.R.S., A. Cayley, F.R.S., M. Hermite, F.R.S.; 1878-85, N. M. Ferrers, F.R.S., A. Cayley, F.R.S., J. W. L. Glaisher, F.R.S. [B. M. C.: *P.P.* 1566 and 251.]
- TA. *with Vol*.—Transactions of the American Philosophical Society; Philadelphia, 1818-71; [14](#) vols. [B. M. C.: *Acad.* [1830/3](#) and *T.C.* [18.b](#) [11](#).]
- TE. *with Vol*.—Transactions of the Royal Society of Edinburgh; 1788 to 1880; [29](#) vols. [B. M. C.: *T.C.* [15.b](#) [1](#) and *R.R.* 2099.*g*.]
- TI. *with Vol*.—Transactions of the Royal Irish Academy; 1786 to 1879; [26](#) vols. [B. M. C.: *Acad.* 1540 and *R.R.* 2099.*g*.]
- TN. *with Vol*.—Transactions of the Royal Society of New South Wales; Sydney, 1867-83; [17](#) vols. [B. M. C.: *Acad.* 1971.]
- Z. *with Vol*.—Zeitschrift für Mathematik und Physik, herausgegeben von Dr. O. Schlömilch und Dr. B. Witzschel; Dr. O. Schlömilch, Dr. E. Kahl, und Dr. M. Cantor; Leipzig, 1856-1883. [28](#) vols. [B. M. C.: *P.P.* 1581.]

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## EXPLANATION OF ABBREVIATIONS, &c.

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a. c.	= areal coordinates.	i. eq.	= indeterminate equation.
alg.	= algebraic.	imag.	= imaginary.
ap.	= application.	l. c. m.	= lowest common multiple.
anal.	= analytical.	num.	= numerical.
ar. p.	= arith. progression.	o. c.	= oblique coordinates.
c. c.	= Cartesian coordinates.	p. c.	= polar coordinates.
cn.	= construction.	p. d. e.	= partial difference equations.
cond.	= condition.	p. e.	= polar equation.
curv.	= curvature.	perp.	= perpendicular.
d. c.	= differential calculus.	pl.	= plane.
d. e.	= differential equations.	pr.	= problem.
d. i.	= definite integral.	q. c.	= quadriplanar coordinates.
eq.	= equation.	rad.	= radius.
ex.	= example or exercise.	sd.	= solid or 3-dimensional.
ext.	= extension.	sol.	= solution.
f.	= formula.	sym.	= symmetrically.
f. d. c.	= finite difference calculus.	ta.	= table.
f. d. e.	= finite difference equation.	t. c.	= trilinear coordinates.
geo.	= geometrical.	tg. c.	= tangential coordinates.
g. p.	= geometrical progression.	tg. e.	= tangential equation.
gn.	= general.	th.	= theorem.
gz.	= generalization.	tr.	= treatise (i.e., more than 50 pages).
h. c. f.	= highest common factor.	transf.	= transformation.
i. c.	= integral calculus.		

15<sub>3</sub>.—The suffix means that three articles under the same heading will be found in the volume.

18—21.—Means that one article on the subject will be found in each of the four consecutive volumes.

References to the Synopsis stand first, and are the numbers of Articles, not of Pages. An asterisk (\*) is prefixed where such numbers will be found.

The unclassified references following a principal title commonly refer to papers on the general theory of the subject; but some papers are occasionally included amongst these of which the titles are too long for insertion, and do not admit of abbreviation.

Subjects which might well have been included under the same heading appear sometimes under different ones, for the following reasons:—Exigencies of space have decided the insertion of the number of the volume only of the particular work in question, and a subsequent examination of the Index of that volume is required in order to find the page. It, therefore, became desirable not to change the original title of the paper when there was danger, by so doing, of making it unrecognizable. When, however, the same subject appears in two parts of this Index under different names, cross references from one to the other are given. Some changes, however, have been made when the synonym was perfectly obvious; for instance, when a reference to a journal, published fifty years ago, is found under the heading of "Binary Quantics," the actual title of the article will, in all probability, be "Homogeneous Functions of Two Variables," and so in a few other instances.

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- \* eight circles touching three, cn: 4189: [Me.13](#): [Q.5](#).
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- \* equation of: 4136—48, p.c 4151: [Pr.27](#): [TL26](#)
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- \* two; eq. for angle of intersection: 4180—1.
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$$\iint \frac{\sin \theta}{tt'} dx dy, t=0, t'=0 \text{ being the tangents from } xy: \text{ [LM.2](#) .}$$
  
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 \*  $C(n, r)$  an integer: 366: [L.42](#).  
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 \*  $C(n, r) = C(n-1, r-1) + C(n-1, r): \text{ [102](#) .}$   
 $C(m+m', p) = \sum_0^p C(m, r) \cdot C(m, p-r): \text{ [L.42](#) .}$
- Combinations—(continued):  
 \* problem or theorem: 105—7: [A.21](#): [C.97](#): [CD.2,4,7,8<sub>2</sub>](#): [L.5](#): [J.3,45,56](#): [L.38](#): [Mém.11](#): [N.53,73](#): [Z.15](#).  
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  - \* and circle, intersection th: 1263: [A.59](#): [N.64](#).
  - \* collinear relation to circle: [Z.1](#).
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  - \* conjoint lines of: [L.38](#).
  - \* conjugate diameters: 1193—1213, 4346; ths 1278—85: [CM.1](#): [L.37](#): [N.42](#), ths [44,69](#): [Q.3](#).
  - \* parallelogram on: 1194: 4367.
  - \* relation to ellipse when equal: [A.18](#).
  - \* conjugate points: [Q.8](#).
  - \* construction of: 1245, 4822: [A.28,43](#): [E.29](#): [Q.4](#): [N.59,73](#): with help of circle of curv. [A.24](#).
  - \* from conj. diameters: 1253: [A.52](#).
  - \* contact of: 4527—33: [A.1,60](#): [C.78](#): [Pr.34](#).
  - \* at 2 points: [Q.3](#): [N.74](#).
  - \* do. with each of 2 conics or circles: 4803—6: [CD.5,6](#): [E.31,34](#).
  - \* 4-pointic with a quartic: [M.12](#).
  - \* 5-pointic: 5190—1: [C.78](#): [E.5,23](#): [J.21](#): [P.59](#).
  - \* with surfaces: [C.91](#): [P.70,74](#).
  - \* convexity from focal property: [N.56](#).
  - \* criterion of Mobius: [J.89](#).
  - \* criterion of species: 4464—77; t.c 4689: 5000.
  - \* curvative:—centre and radius of: 4534—49: geo 1254—66: ths [Me.73](#): [N.79,85](#).
  - \* geo cn: 1265: [A.17](#).
  - \* chord of: 1259,—64: [Q.6](#): locus of mid-point [A.70](#).
  - \* as curves in space: tr [A.37](#): [M.64](#).
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  - \* equations of: 4251, 4273; p.c 4336; tg.c 4663, 4870: [J.2](#).
  - \* general: 4400, 4714, 4719; t.c 4755 and 4765; tg.c 4664 and 4872; p.c 4493: t.c [A.51](#): [CP.4,5](#): t.c [G.6,7](#): [Mém.52](#): [N.43,45,65](#). (See also "Conics, general equation.")

- Conics:—equations of—(continued):
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  - \* cond. for a hyperbola: 4468; t.c 4689: [A.39](#).
  - \* cond. for a rectangular hyperbola: 4737; t.c 4690; tg.c 5000.
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  - \* cond. for two right lines: 4469, 4475, t.c 4662.
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  - \* normals: 1171, sd 5629—32: [A.16,24,32](#), cn [43,47](#): [An. cn 64,78](#): [C.72,84](#): [J. cn 48,56,62](#): [Me.66](#): [Mél.2](#): [N.70,81](#): [Q.8](#): [Z.11,18,26](#).
  - \* circle through feet of: [N.80](#).
  - \* cutting off the min. or max. arc or area: [N.44](#).
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  - \* equations of: 4286, 4483, 4512.
  - \* intercepts: 4294: segments; 4309, 4486.
  - \* least distance between two: [A.21,38](#).
  - \* number of real: [J.59](#): [N.70,72](#).
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  - 5 points: cn 4531, eq 5024: [A.27](#): [A.9,24,64](#): [An.50](#): [N.57](#).
  - 4 foci of a conic: [Q.5](#).
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- of curves having the same polar surface: [An.76](#).
- \* depression of order by unity: 3262—9. with different. total integrals: [L.84](#). of dynamics: [C.5,26,40](#); [CD.2](#): [G.1,4](#): [L.37,49,52,55,72,74](#): [M.2,17,25](#): [M.61,4](#): [Pr.12](#): [P.54,55,63](#). ap. to engineering: [JP.4](#). and elliptic functions: [L.49](#). elliptic: [G.19](#): [M.21](#). elliptic multiplier: [M.21](#).
- \* exact: 3187, 3270—5: [G.12](#): [C.1,10,11](#). of families of surfaces: [Me.77](#). with fractional indices: [JP.15](#). of functions of elliptic cylinders: [M.22](#). general methods: [L.81](#).
- \* generation of: 3150. geometrical meaning of: [Q.14](#).
- \* homogeneous: 3186, 3234, 3262—8: [C.13](#): [CM.4](#): [J.86](#). hyperelliptic: [J.32,55](#): [Me.62](#). of hypergeometrical series: [J.56,57,73](#). integrability of: [Z.12](#): immediate, [C.82](#). whose integrals satisfy relations of the form  $F[\phi x] = \psi x Fx$ : [C.93](#). whose integrals are satisfied by the periodicity modulus of elliptic integrals of the first kind: [J.83](#). integrating factors: pp468—471, 3394: [C.68,97](#). of  $Pdx + Qdy + Rdz$ : [Q.2](#). integration: by Bessel's function: [Me.80](#). by Gamma function: [TE.20](#).
- \* by definite integrals: 3617—28: [C.17](#): [J.74](#). by differentials of any index: [C.17](#): [L.44](#). by elimination: [CP.9](#). by elliptic functions: [An.79,82](#): [C.41](#): [JP.21](#). by separation of operative symbols: [Z.15](#).
- \* by series: 3604—16: [C.10,94](#): [LM.6](#): [Me.79](#): [Q.19](#): [TI.7](#). by theta-functions: [C.90](#). irreducibility of: [J.92](#). isoperimeters, pr: [Mém.50](#). of Lamé: [J.89](#). of light: [M.1](#). in linear geometry: [M.5](#). of motion: [C.55](#): of elastic solids, [Q.13](#): of fluids, [CP.7](#): of a point, [C.26](#). with integrals "monochrome and monogène": [C.40](#).

## D. E.—(continued):

- \* Parseval's theorem: 3628. and p.d.e of first order: [J.23](#). particular integrals of: [CM.2](#): algebraic, [C.86](#). relations of the constants: [C.93](#): [J.10](#): [JP.6](#). in problem of  $n$  bodies: [An.83](#). of perturbation theory: [Mém.83](#). with quadratic integrals: [J.99](#). for roots of algebraic equations: [P.64](#): [Pr.13](#).
- \* rule for equivalence of two solutions 3167.
- \* singular solution of: 3169—78, 3301—6, 3401—3: [C.19,94](#): [CM.2](#): [JP.18](#): [M.22](#): [Man.83,84](#): [Q.12,14](#). of sources: [AJ.75](#). of a surface: [G.2](#). satisfied by the series  $1 \pm 2q + 2q^4 \pm 2q^9 + \&c. . 2\sqrt{q} + 2\sqrt{q^3} + 2\sqrt{q^5} + \&c. .$ : [L.49](#): [J.36](#). satisfying Gauss's function  $F(a, \beta, \gamma, x)$ : [L.82](#). synectic integrals of: [C.40](#). and tortuous curves: [L.53](#). transformation of: [An.52](#): [CD.9](#): in curvilinear coords: [J.85](#).
- D. E. linear: [A.28,35,40,41,43,45,46,53,59,65,69](#): [Ac.3](#): [AJ.7](#): [An.50,85](#): [At.75](#): [C.7,29,58,73,84,88,90,91,92,94](#): [CD.3,4,9](#): [CP.9,10](#): [G.15](#): [J.23,24,25,40,42,55,63,70,76,79,80,81,83,87,88,91,98](#): [L.38,64](#): [M.5,11,12](#): [Me.75](#): [P.48,50,51](#): [Pr.5,18,19,20](#): [Q.8](#): [Z.3,7,9](#). without absolute term, condition of solutions in common: [C.95](#). with algebraic integrals: [C.96,97](#): [J.80,90](#): [M.21](#). determination of arbitrary constant: [At.65](#): [Q.19](#). argument & parameter interchanged in the integral: [J.78](#). bibliography of: [AJ.7](#). with coefficients that are algebraic functions of an independent variable: [C.92,94](#).
- \* with constant coefficients: 3238—50: [An.64](#): [CM.1,2](#): [E.34](#): [JP.33](#): [L.42](#): [N.47,84](#). with periodic coefficients: [C.91,92](#): doubly periodic: [C.90,92,98](#): [J.90](#). with rational coefficients, algebraic integrals of: [C.96](#): [JP.32,34](#). with rational coefficients, upon whose solution depends the quadrature of an irrational algebraic product: [C.91, 92](#).



## D. E. linear—(continued):

- with variable coefficients: [C.92](#):  
[J.66,68,76](#): [L.80,81](#).
- which connect a complete function of  
the 1st kind with the modulus:  
[C.86](#).
- homogeneous: [Ac.1](#): [J.90](#): [Mo.82](#).
- integrating factors of: [C.97,98](#).
- integration by Abelian functions, [C.](#)  
[92](#): [J.73](#); by finite differences,  
[Q.1](#); by series, [J.76](#).
- which admit of integrals whose loga-  
rithmic differentials are doubly  
periodic functions: [L.78](#).
- whose particular integrals are the  
products of those of two given  
linear d.e: [A.41](#).
- irreducibility of: [J.76](#).
- Landen's substitution, geo: [J.91](#).
- Malmsten's theorem: [J.40](#).
- singular solution: [J.73,83,84](#).
- transformation of: [C.91,96](#).
- \*  $n$  variables, 1st order: 3320—32: [C.](#)  
[14,15](#): [G.13](#): [J.20,80](#): [L.38](#).
- $n$  variables, 2nd order: [L.37](#): 2 varia-  
bles, [C.70](#).
- $n$  variables, any order: [Mém.13](#).
- \*  $Pdx + Qdy + Rdz = 0$ :  $P, Q, R$  involving  
 $x, y, z$ , 3320; [geoM.16](#); [Z.20](#):  $P,$   
 $Q, R$ , integral functions of  $x$  only,  
[Q.19](#);  $P = (ax^n + bx^{n-1} + \&c.)^n$ ,  
 $Q, R$  similarly with  $y$  and  $z$ , [Q.20](#).
- \*  $Xdx + Ydy + Zdz + Tdt = 0$ : condition  
of being an exact differential,  
3330.  
 $xdt + ydx + zdy + tdz = 0$ : [A.30](#).
- \* D. E. of first order, linear: p467: [C.86](#):  
[G.13](#): [algG.18](#); [M.23](#).
- \* exact: 3187.
- \* homogeneous: 3186.
- integration by a particular integral:  
[C.86](#).
- reduction to a continued fraction of a  
fraction which satisfies a: [C.98](#).
- \* separation of variables: 3185: [CM.1](#).
- \*  $Mdx + Ndy = 0$ : 3184: [N.74,77](#).
- \*  $(ax + by + c)dx + (a'x + b'y + c')dy = 0$ :  
3205, p471: [L.59](#).
- $(ax + by + c)^n dx + (a'x + b'y + c')^n dy$ : [A.](#)  
[64](#).
- $\frac{dx}{\sqrt{P}} + \frac{dy}{\sqrt{Q}} = 0$ ;  $P, Q$  being quartics in  
 $x, y$ : [C.66](#): [LM.8](#): [ME.79](#).
- $\frac{f(x)dx}{F(x)} + \frac{f(y)dy}{F(y)} = 0$ ,  $f(x)$  of 1st deg.;  
 $F(x)$  of 5th deg.:  
[C.92](#).

## D.E. of first order, linear—(continued):

- \*  $P_1 dx + P_2 dy + Q(xdy - ydx) = 0$ ;  $P_1,$   
 $P_2$  being homogeneous and of the  
 $p$ th deg. in  $x, y$ ;  $Q$  homogeneous  
and of the  $q$ th deg.: 3212.  
 $P_1, P_2, Q$  different linear functions  
of  $x, y$ : [C.78,83](#): [L.45](#): [J.24](#).
- \*  $y_x + Py = Q$ , where  $P, Q$  involve  $x$   
only: 3210.
- \*  $y_x + Py = Qy^n$ : 3211.  
 $yy + Py + Q$ : [Mém.11](#).
- $y_x + y^2 = \frac{A}{(P + 2Qx + Rx^2)^2}$ , where  $P,$   
 $Q, R$  are functions of  $x$ : [Mém.11](#).
- $y_x + a + by + y^2 = 0$ : [J.25](#).
- $y_x \sqrt{m+x} = \lambda y \sqrt{m-x}$ : [A.42](#).
- $y_x = f(y)$ : [J.9](#);  $y_x = f(x, y)$ : [An.73](#):  
[L.55](#).
- $y_x = \frac{3y(y+1) - 4x}{x(8y-1)}$ : [C.88](#).
- $y_x + f(x) \sin y + F(x) \cos y + \phi(x) = 0$ :  
[L.46](#).
- \*  $u_x + bu^2 = cx^m$  (Riccati's eq.): 3214:  
[A.40](#): [C.11,85](#): ( $m = -6$ ) [E.7](#):  
ext28: [JP.14](#): [L.41](#): [P.81](#): [Q.7](#):  
[11,16](#).
- allied eqs: [L.51](#): [Me.78](#): [Q.12](#).
- sol. by continued fractions: [Mém.18](#).
- by definite integrals: [J.12](#).
- transformation of: [Me.83](#).
- \* D. E. of first order: 3221—36: [A.29](#):  
[C.40,45,66](#): [M.3](#).
- two variables: [An.76](#): [J.40](#): [Mém.](#)  
[62](#): [N.50](#): singular solution, [J.38](#).
- \* Clairaut's equation,  $y = px + f(p)$ :  
3230: [CM.3](#): [Me.77](#).
- integration by second order d.e: [A.46](#).
- homogeneous in  $x$  and  $y$ : 3234.
- reduction to a linear form with respect  
to the derivatives of an unknown  
function: [C.87](#).
- related transcendents: [Ac.3](#).
- separation of variables: [CD.9](#).
- \* singular solution: 3230: [A.56,58](#):  
[CP.9](#): [J.48](#): [Me.73,77](#).
- \* solution by differentiation: 3236.
- \* solution by factors: 3222.
- transf. by elliptic coords: [J.65](#).
- verified by a reciprocal relation be-  
tween two systems of values of  
variables: [C.15](#).
- $dx^2 + dy^2 = ds^2$  and analogous eqs:  
[L.73](#).
- \*  $adx + bdy = ds$ : 3287.
- $dx^2 + dy^2 + dz^2 = ds^2$ : [L.48](#).



- D. E. of first order—(continued):  
 $dx^2 + dy^2 + dz^2 = \lambda (da^2 + d\beta^2 + d\gamma^2)$ : L.  
50.  
 $F(u, u_x) = 0$ : C.93.
- \*  $x\phi(p) + y\psi(p) = \chi(p)$ : 3226.
- D. E. of second order, linear: A.29, 32.55, 64: An.63, 79, 82, 83: C.82, 90, 91, 93, 97: J.51, 74: L.36: Me.1, 4: M.11: Mo.64: Z.5.  
 with algebraic integrals: C.90: J.81, 85: L.76.  
 with doubly periodic coefficients: Ac.2.  
 homogeneous: M.22.  
 integration by Gauss's series: Z.19.  
 transformation of: An.52.
- \*  $y'' = a$ : 3288.  
 $y'' = Py$ : C.9.
- \*  $y'' + a^2y = Q$ : 3522, '5: geoMe.66:  $Q = \cos nx$ , 3526:  $Q = 0$ , 3523—4.  
 $y'' = Ay(a + 2bx + cx^2)^{-2}$ : L.44.
- \*  $y'' = ax + by$ : 3281.  
 $xy'' = y$ : Z.2.  
 $(1 \mp x^2)y'' \pm my = 0$ , &c.: CM.3.  
 $x^2(y'' + q^2) = p(p-1)y$ : CM.2.  
 $y'' = \{h + n(n+1)k^2 \operatorname{sn}^2 x\}y$  (Lamé's eq.): C.85.  
 $y'' = \phi_0 + y\phi_1 + y^2\phi_2 + \&c.$ , when  $\phi_0$  &c. are trigonometrical series: C.98.  
 $y'' = y(e^x + e^{-x})^{-2}$ : L.46.  
 $y'' + ax^my = f(m)$ : E.6.  
 $y'' = ay + \psi(x)$ : A.45.
- \*  $y'' = f(y)$ : 3257.
- \*  $y'' = f(x, y)$  (Jacobi): 3285.  
 $y'' + py' + ry = 0$ : C.85, 90: Q.19.  
 $y'' = x^2y' - nxy$ : A.53.  
 $xy'' + my' + nxy = 0$ : L.45, 78.  
 $xy'' + y' + Ax^my = 0$ : C.39.  
 $x^2y'' + rxy' = (bx^m + s)y$ : An.51: CD. 5.  
 $x^4y'' + 2x^3y' + f(y) = 0$ : A.28, 30.  
 $y'' + f(x)y' + F(y)y'^2 = 0$ : 3284: L.42.
- \*  $(a + bx)y'' + (c + dx)y' + (e + fx)y = 0$ : A.58.  
 $(a + bx^n)x^2y'' + (c + ex^n)xy' + (f + gx^n)y = Q$  (Pfaff): 3598: J.2, 54: and like eqs., Z.2, 3: with  $b = 0$ , A.38.  
 $x^2(a - bx)y'' - 2x(2a - bx)y' + 2(3a - bx)y = 6a^2$ : A.28, 30.  
 $xy'' + y' + y(x + A) = 0$ : Me.81, 84.  
 $xy'' + y' + y(x - A) = d_x \frac{\cos(x+l)}{x+l}$ : Me. 82.  
 $x^2y'' - 2xy' + 2y = x^2yf^{-2}$ : A.28, 30.
- D. E. of 2nd order, linear—(continued):  
 $sy'' + (r + qx)y' + (p + nx + mx^2)y = 0$ : A.23: Z.8, 9.
- \*  $(1 - x^2)y'' - xy' + q^2y = 0$ : 3282.
- \*  $(1 + ax^2)y'' + axy' \pm q^2y = 0$ : 3283, 3594.  
 $2x(1 - x^2)y'' - y' + n(n+1)y = 0$ : Q.18.  
 $x(1 - x)y'' + \frac{x}{2} - \frac{1}{2}x)y' + \frac{1}{2}y = 0$ : Me. 82: Q.17.  
 $(m+x)(n+x)y'' + (m-n)y' - \lambda^2(m+x)^2y = 0$ : A.42.  
 $(mx^2 + nx + p)y'' + (qx + r)y' + sy = 0$ : JP.13: Z.4.  
 $\lambda\mu y'' + A\lambda y' + B\mu y = 0$ ,  $\mu y'' + A\lambda y' + B\lambda\mu y = 0$ , and  $\mu y'' + A\lambda y' + B\mu = 0$ ; with  $\lambda \equiv a + bx + cx^2$  and  $\mu \equiv b + 2cx$ : A.42, 43.  
 $d_x \{(x - x^3)y_x\} - xy = 0$ : L.54.
- \*  $y'' + Py^2 + Qy + R = 0$ ,  $P, Q, R$  being functions of  $x$ : 3280.  
 $Py'' + Qy' + Ry = 0$ : Ac.1.  
 $zy''_{xx} + az_y y'^2_x + f(y) = 0$ : Me.71.
- D. E. of second order: Ac.1: An.79: JP.29: C.67, 69, 80, 91: L.90: L.39: LM.11, 12, 13, 16: Z.15.  
 with algebraic integrals: C.82.  
 derived from linear eq: Me.73.  
 with elliptic function coefficients: Ac.3.  
 in the neighbourhood of critical points: C.87.  
 polynomials which verify: Ac.6.  
 solution by definite integrals: A.27.  
 by factors: C.68.  
 by Challis's method, and application to Calc. of Variations: A.65, 66.  
 $yy'' = \frac{1}{2}y'^2 + 2py^2$ : L.73.  
 $Myy'' + Ny'^2 = f(x)$ : N.79.
- \*  $y'' + Py' + Qy^3 = 0$ ,  $P, Q$  functions of  $x$ : 3276.
- \*  $y'' + Py' + Qy^n = 0$ : 3278.
- \*  $y'' + Py'^2 + Qy'^n = 0$ : 3279.
- \*  $y'' + Qy'^2 + R = 0$ : 3277.  
 $I_{2x} + \frac{1}{x}I_x + I = 0$ , where  $I$  is Bessel's function: J.56.
- D. E. of third order, linear: C.88, 92: Q. 7, 8, 14: M.24.  
 $y' = y'''$ : JP.15:  $u_1 = u_{3x}$ : C.3.  
 $x^2y''' - y = 0$ : Z.8.  
 $y''' = \frac{3mx^2y'' + 6m(\mu + 2)xy' + 3m(\mu + 2)(\mu + 1)y}{A.42}$ .  
 $y''' = x^m(Ax^2y'' + Bxy' + Cy)$ : A.58.
- D. E. third order: An.83: C.98: M.23.
- \* D. E. of higher order, linear: 3237—50: A.65: C.97: J.16: M.4: Q.18.



- D. E. of higher order, linear—(cont.):  
of orders  $p$  and  $m+p$ , th: [C.43](#).
- $y_{4x} = xy_x - y$ : [A.1](#).
  - \*  $y_{nx} = f(x)$ : 3256.
  - $y_{nx} = x^m y$ : [L.39](#).
  - $y_{nx} = (a + \beta x) y$ : [J.10](#).
  - $x^{2m} y_{mx} = a^m y$ : [A.32](#).
  - $x^{2m} y_{2mx} = y$ : [A.42](#).
  - $xy_{nx} = y$ : [A.26](#).
  - $x^m y_{nx} = \pm y$ : by definite integrals: [C.48, 49](#): [J.57](#).
  - $y_{nx} = x^m y$ : by definite integrals: [J.19](#).
  - $x^{m+1} y_{2m+1} = \pm y$ : by Bessel's function: [M.2](#).
  - $y_{nx} = x^m y + A + Bx + Cx^2 + \dots + Nx^n$ : [Z.10](#).
  - $y_{nx} = Ax^m y_x + Bx^{m-1} y$ : [A.28, 38](#).
  - $x^{2n} y_{nx} = Ax y + B y$ : [A.33](#).
  - $y_{nx} - xy = y_{2x} + abx^n y$ : by definite integrals: [J.17](#).
  - $y_{nx} = Ax^2 y_{2x} + Bxy_x + Cy$ : [A.53](#): [M.3](#).
  - $y_{nx} = Ax^m y_{2x} + Bx^{m-1} y_x + Cx^{m-2} y$ : [A.29, 30, 33, 38](#).
  - $xy_{nx} + ay_{(n-1)x} = bxy$ : [J.2](#): [Z.10](#).
  - $xy_{nx} - y_{(n-1)x} + mx^2 y = 0$ : [A.40](#).
  - $xy_{nx} + \lambda y_{(n-1)x} = x(xy_x + \mu y)$ : [A.86](#).
  - $Axy_{nx} + By_{(n-1)x} = x^m (Axy_x + By)$ : [Z.8](#).
  - $x^2 y_{mx} + q^m x^2 y = p(p-1)y_{(m-2)x}$ : [J.2](#).
  - $Ax^2 y_{(n+2)x} + Bxy_{(n+1)x} + Cy_{nx} = x^m (ax^2 y_{2x} + bxy_x + Cy)$ : [A.38](#).
  - \*  $y_{nx} + a_1 y_{(n-1)x} + \dots + a_n y = 0$ : 3239: [A.40](#).
  - \* ditto  $= f(x)$ : 3243, 3516:  $a_1 \dots a_n$  functions of  $x$ , 3237: [J.39](#).
  - [n](#) fractional and all lower orders integral: [L.36](#).
  - $(d_x + a)^n y = f(x)$ : [CM.4](#).
  - \* ditto  $= e^{ax}$ : 3528: ditto  $= \sin mx$ : 3529.
  - \*  $(p+qx)^n y_{nx} + a_1 (p+qx)^{n-1} y_{(n-1)x} + \dots + a_n y = f(x)$ : 3250: with  $p = 0$ , [C.96](#).
  - $a_{m+n} y_{(m+n)x} + \dots + (a_m + x) y_{mx} + \dots + a_0 y = 0$ : [A.47](#).
  - $x^{n-1} (a + bx) y_{nx} + x^{n-2} (c + d) y_{(n-1)x} + \dots + ty = 0$ : [J.39](#).
  - $y_{2x} + my_x + ny_{\frac{x}{2}} + py = q$ : [L.44](#).
- \*D. E. of higher order: 3251—69.
- \*  $y_{nx} = F(y_{(n-1)x})$ : 3258.
  - \*  $y_{nx} = F(y_{(n-2)x})$ : 3260.
  - $Py_{nx} + Q = 0$ , where  $P, Q$  are functions of  $x, y$ , and the first  $n-1$  derivatives of  $y$ : [J.31](#).
- D.E., simultaneous system of: [An.69, 82, 84](#): [C.10, 43, 47, 92](#): [CM.1](#): [LM.14](#): [Me.13, 80](#): [Pr.12](#).
- Hamiltonian: [Q.14](#).
  - integration and inversion of the integrals: [C.23](#).
  - Jacobi's: [CD.3](#).
  - \* method of multipliers: 3353.
  - number of arbitrary constants: [Me.77](#).
  - \* reduction of order: 3350.
  - reduction to a P.D. eq: [C.44](#).
  - theorem of Abel: [C.24](#).
  - theorem analogous to Lagrange's in the Perturbation theory: [L.51](#).
  - theorem of a new multiplier: [J.27](#).
  - transformation and integration of: [L.45](#).
  - \*  $x_{2t} = ax + by$  and  $y_{2t} = cx + dy$ : 3354 and a similar example.
  - \*D.E., simultaneous linear: 3340—59: [A.14](#): [C.9, 92](#): [E.5](#): [N.66, 84](#).
  - Pfaff's method: [C.14](#): [J.2](#).
  - transformation of: [J.98](#).
  - \*  $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ : 3346: [Q.14](#).
  - \*  $\frac{dx}{P_1 - xP} = \frac{dy}{P_2 - yP} = \frac{dz}{P_3 - zP}$ : 3347.
  - \*  $x_t + P(ax + by) = Q$  and  $y_t + P(cx + dy) = R$ : 3348.
  - \*  $tx_1 + 2(x - y) = t$  and  $ty_1 + (x + 5y) = t^2$ : 3349.
  - \* equations in  $x, x_{2t}, x_{4t}, \&c. \dots y_t, y_{2t}, y_{4t}, \&c.$ : 3357.
  - \* homogeneous in  $x, y, z \dots$  and their second derivatives only: 3358.
  - \*D.E., simultaneous first order: 3340—49: [C.43](#): [J.48](#): [Pr.62](#).
  - \*D.E., symbolic methods: 3470—3636: [CD.1](#): [P.61](#): [Q.3, 17](#).
  - \*  $F(d_x)u = Q$ : 3515.
  - \*  $u_{2x} + a^2 u = Q$  and similar: 3522.
  - \* exceptional case of the inverse process: 3526.
  - \* reduction of an integral of the  $n$ th order: 3530.
  - \*  $ax^m y_{mx} + bx^n y_{nx} + \&c. = Q$ : 3531.
  - \*  $ay_m + by_n + \&c. = f(e^\theta, \sin \theta, \cos \theta)$ : 3535.
  - \*  $a\pi_1^m z + b\pi_1^n z + \&c. = u_1 + u_2 + \&c.$ : 3540.
  - \*  $F(\pi)u = Q$ : 3541.
  - \* Reduction to the form  $(\Pi^n + A_n \Pi^{n-1} + \dots + A_0)u = Q$ , where  $\Pi = Mdx + Ndy + \&c.$ : 3546.



- D.E., symbolic methods—(continued):
- \*  $F(xd_x, yd_y, \dots)u = \Sigma Ax^m y^n \dots$ : 3558.
  - \* to transform  $(a+bx+\dots)u_{nx}+(a'+b'x+\dots)u_{(n-1)x}+\dots \&c. = Q$  into the symbolic form: and the converse: 3571, 3573.
  - \*  $u+a_1\phi(D)eu+\&c. = U$ : 3575.
  - \* to transform  $u+\phi(D)e^ru = U$ : 3579—80.
  - \* to reduce a homog. eq. to the form  $y_{nx}+qy = X$ : 3585.
  - \* Differential expressions: 1407: prAn. 85.  
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- in quotients of roots: [N.45](#).
- in sums of the  $C(n,2)$  roots of another eq.: [N.43](#).
- insolubility of quintics, &c.: [J.1](#).
- irrational: [Man.51](#).
- \* with integral coefficients: 503: [C.24](#): [J.53](#): complex, [J.53](#).
- irreducible: [An.51](#): [Mo.80](#): of prime degrees, [AJ.7](#).
- \* linear: see "Linear equations."
- \* miscellaneous: 214.
- numerical: [C.10,12,32,78,81](#): [G.13](#): [J.10](#): [L.36,38,41,83](#).
- and commensurable quadratic factors: [L.45](#).
- of  $n$ th degree with two real roots: [C.98](#).
- from observations: [A.21](#).
- \* with only one positive root: [411](#): [C.98](#).
- of payments: [A.34,36](#): [CD.1](#): [CM.2](#).



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- \* reciprocal: [466](#): [A.44](#): [C.16](#)<sub>2</sub>: of a quartic, [N.66](#)  
reduction of: [C.97](#): [CD.6](#): to reciprocal eqs., [A.35](#)  
relation to linear d.e and f.d.e: [L.36](#)  
roots of: see "Roots of equations."
- \* simultaneous (see also "in two or three variables"): [59](#), [211](#), [582](#): [C.25](#): [LM.6](#): [thN.48](#) and [81](#): quadratics, [N.60](#)  
deducible the one from the other: [C.22](#)  
of the form  $x^m + y^n + z^m = a$ : [N.46](#)
- \* solution of: [45](#), [54](#), [59](#), [211](#), [466](#)—[533](#), [582](#): [A.64](#): [trAn.52](#): [C.3](#), [5](#), [62](#), [64](#): [J.4](#), [27](#), [87](#): [Mo.56](#), [61](#)
- \* by approximation: [506](#)—[533](#): [A.30](#): [Ac.4](#): [C.11](#), [17](#), [45](#), [60](#), [79](#), [82](#): [E.4](#): [G.8](#): [J.14](#), [22](#): [Me.68](#): [N.51](#), [62](#), [78](#), [80](#), [84](#): [No.58](#): [P.5](#): [Q.3](#): [TL.7](#): [Z.23](#)
- \* Horner's method: [533](#): [P.19](#)
- \* Lagrange's method: [525](#): [C.91](#)
- \* Newton-Fourier method: [527](#)—[8](#): [AJ.4](#): [G.2](#): [Me.66](#): [N.46](#), [56](#), [60](#), [69](#), [79](#)  
Weddle's method: [Z.7](#), [8](#)  
by continued fractions: [J.33](#)  
by definite integrals: [Me.81](#): [P.64](#): [Z.3](#)  
by diminishing the powers of the roots: [C.41](#)  
by elimination of integers: [N.70](#)  
by geometry: [C.87](#)  
by imaginary values: [J.20](#)  
by infinite series: [J.33](#)  
by interpolation: [C.5](#)  
by logarithms: [C.95](#)  
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by radicals: [C.58](#): [Q.15](#)  
by series: [An.57](#): [C.49](#), [52](#): [J.6](#): [Mém.33](#)  
by transcendents: [An.63](#): [Q.5](#)
- \* by trigonometry: [480](#): [A.1](#)  
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symbolic, non-linear: [C.22](#)  
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transformation of: [C.64](#)
- \* in one variable: [45](#)—[58](#), [214](#)—[16](#), [400](#)—[550](#): approx [A.20](#)  
graphic solution: [C.65](#)  
 $x^{2n+1} - x - k = 0$ : [An.59](#)

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- $x^m - px + q = 0$ : number of real roots: [C.98](#)
- $x^{2n} + qx^n + p^n = 0$  and derivatives: [N.65](#)<sub>2</sub>
- $x^{n-1} + x^{n-2} + \dots + 1 = 0$  irreducible if  $n$  be a prime: [L.56](#)
- $ax^{2m+n} + bx^{m+n} + cx^n + d = 0$ : [G.14](#)
- $(x-1)! + 1 = x^n$ : [L.56](#)
- $(x^p - a^p) \psi(x) = 0$ : [N.82](#)
- $(1+x)^m(1+bx) = 0$  when  $x$  is small: [A.2](#)
- \* in two variables: [59](#)—[67](#), [211](#), [217](#)—[8](#): [A.20](#), [25](#): [CM.2](#): [J.14](#): [N.47](#), [48](#), [63](#): [Pr.8](#): [Q.18](#)  
of any degree with a variable parameter: [L.59](#)  
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- \*  $(y-c)(z-b) = a^2$ , sym in  $x, y, z$ : [219](#)
- \*  $y^2 + z^2 + yz = a^2$ , &c., sym: [220](#)
- \*  $x^2 - yz = a^2$ , &c., sym, and  $x = cy + bz$  &c., sym: [221](#)—[2](#)  
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- \* Equiangular spiral: [5288](#): [Me.62](#): [N.69](#), [70](#)  
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- Equipotential curves: [Me.82](#): [Pr.24](#)  
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- Errors of observation: [A.18](#), [19](#): [An.58](#): [C.93](#): [JP.13](#): [N.56](#): [P.70](#): [TE.24](#)
- Errors of constants: [Mo.83](#)
- \* Euclid, enunciations: p. xxi.  
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- Euler's algorithms: [A.67](#).
- \* Euler's constant: 2744: [Pr.15,16,18,19, 20](#), Table 27.  
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- Euler's equation: [N.72](#): integration of it by the lines of curvature of a ruled hyperboloid, [N.75](#).
- Euler's equations of motion solved by elliptic integrals: [Q.14](#).
- Euler's formula for  $(1+x)^n$ : [L.44](#).
- \* Euler's integrals: 2280—2323: [A.41](#): [Ac.1,2](#): [An.54](#): [C.9,17,94,95,th96](#): [J.15,21,45](#): [fL.43](#): [Me.83](#): [Z.9](#).
- \*  $B(l, m)$ : 2280: [An.69](#): [G.9](#).  
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ap. to series and functions of large numbers: [JP.16](#).  
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- Euler's numbers: [AJ.5](#): [An.77](#): [C.66, 83](#): [J.79,89](#): [prsL.44](#): [Me.78,80](#).
- Evectant of Hessian of a curve: [E.32](#).
- \* Even and odd functions: 1401.
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- \* of a catenary: 5159.  
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- \* of an ellipse: 4547,4958: [C.84](#): [N.52, 63](#).  
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- \* of a parabola: 4549,4959: [Q.5](#): [N.65](#).  
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- \* of a tortuous curve: 5731: [A.25](#).
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- \* Ex-circle of a triangle: [711,953](#): 4749: [A.54](#): [thN.60](#).  
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- \* Expansions of a function in a series (see also "Series" and "Summation"): [A.31](#): [An.7](#): [thsAJ.3](#) and [4](#): [C.7,13,17,20](#): [CM.4](#): [J.90](#): [L.38,46,76](#): [M.16](#): [Mél.3](#): [Mém.33](#): [N.82,83](#): num, [Q.3](#): [Z.2](#).
- Expansions of a function in a series—
- \* of circular functions: 2955: [A.11](#): [CM.3](#): [J.43](#): [L.36](#): [Q.12](#): of imag. arcs, [J.6](#).  
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- \* of explicit functions: 1500—47.  
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- \* by binomial theorem: [125](#).
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 $n$ , alg. functions from  $n$  eqs: [G.11](#).  
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 $(1+ax+bx^2+\dots+lx^{n-1})^{-1}$ : [AJ.6](#).  
 $\{(x-a_1)^2+\dots+(x-a_n)^2\}^{-1}$ : [C.95](#).  
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- $(1-ax-bx^2)^{-n}$ : [J.43](#).  
 $\left\{\left(1-\frac{t}{2^n}\right)\left(1-\frac{t}{3^n}\right)\dots\right\}^{-1}$ : [J.40](#).  
 $(x-z)^n$  in powers of  $z^2-1$ : [C.86](#).  
 $(x+y)^{(n)}$ : [CM.3](#).  
 $n$ th derivative of  $\sqrt{(a^2-b^2x^2)}$ : [A.4](#).  
 $\left(\frac{1-\sqrt{(1-4t)}}{2}\right)^n$  in powers of  $t$ , when  
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- \* Bernoulli's numbers: 1545.
  - circular functions: [J.24](#): [Q.5](#).
  - an arc in linear functions of sines or tangents of fractions of the arc in g.p.: [L.43](#).
  - powers of arc in powers of sines: [J.11](#).
  - \*  $\pi$ : 2931—2, 2945, 2960—2: [Me.78](#).
  - \* powers of  $\pi$ : [Me.78](#):  $\pi^2$ , [858](#):  $\pi^{-1}$ , [Me.83](#).
  - \*  $\sin \theta$  and  $\cos \theta$  in powers of  $\theta$ : [764](#), 1531: [A.5,29](#): [C.16](#).
  - \*  $\sin^m \theta$  and  $\cos^m \theta$  in sines or cosines of multiple arcs: 772—4: [A.24,55](#): [C.12](#): [CD.3](#): [J.1,5,14](#): [N.71](#): [TL.7](#).
  - \*  $\sin n\theta$  and  $\cos n\theta$  in powers of sine or cosine: [758](#), [775—79](#), 1533: [C.82](#): [CM.2](#): [Me.76](#): [Mém.13,15,18](#): [N.73,83](#): [Q.4](#): convergency of the series, [J.4](#).
  - \*  $\cos n\theta$  in powers of  $\cos \theta$ : [780](#): [Q.12](#).
  - \*  $\sin^{-1}x$ : 1528,—64: [J.25](#): [N.74](#): remainder, [Z.1](#).
  - $\cos^m ax$ : [A.11](#).
  - \*  $\tan x$ : 1525, 2913,—17: [A.16](#): [C.88](#): [N.57](#).
  - \*  $\cot x$ : 2911,—16: [C.88](#): [Q.17](#).
  - \*  $\sec x$ : 1526: [A.16](#): [C.88](#): [J.26](#): [N.57](#): [Q.17](#).
  - \*  $\operatorname{cosec} x$ : 2914,—8.
  - \*  $\tan^{-1}x$ : [791](#).
  - \*  $\tan n\theta$ : [760](#).
  - \*  $a^x \sin x$  in differences of  $\sin x$ : 3749.
  - $\frac{\sin nx}{\cos nx} / \cos x^n$ : [A.4](#).
  - $(1-\mu \cos \phi)^{-1} = \sum a_n \cos 2n\phi$ : [A.21](#).
  - $(a^3+b^3-2ab \cos \phi) \exp.-(m+\frac{1}{2})$ : [TE.5](#).
  - $(a+b \cos \phi + c \cos \phi')^{-n}$  in cosines of multiples of  $\phi$  and  $\phi'$ : [J.15](#).
  - $\cos k \cos^{-1}(\cos \omega + a)$ : [C.15](#).
  - $\frac{\sin(\theta + \theta_1 + \dots + \theta_{n-1})}{\cos}$ : [A.34](#).
  - \*  $y$  in powers of  $x$  when  $x = \frac{\sin y}{\sin(y+a)}$ : [796,1558](#).

## Expansion of—(continued):

- \* do. when  $x = \frac{\log y}{y}$ : 1570.
- $\theta \cot \theta$  in powers of  $\sin^2 \theta$ : [Q.6](#).
- $\cot^{-1}(m-1) - \cot^{-1}(m+1)$ : [A.47](#).
- differential coefficients by f.d.c and the converse: [J.16](#).
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- and of their powers: [C.83](#):  $\cos am x$ , [L.64](#).
- equations: [L.50](#).
- exponential functions: [N.82](#).
- \*  $e = \text{limit of } (1+x)^{\frac{1}{x}}$ : 1590: [A.3,23](#): [Q.7](#).
- \*  $e^x$ : [C.30](#): [N.48](#).
- \*  $\frac{x}{e^x \pm 1}, \frac{e^x - 1}{e^x + 1}$ : 1539, 1543—4.
- \*  $e^{ay}$  in powers of  $ye^{by}$ : 1571.
- $(ae^x - 1)^{-1}$ : [At.57](#).
- \*  $\frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$ : 2962.
- $e \exp. \left(\lambda x + \frac{\mu}{x}\right)$ : [Z.3](#).
- \*  $e^{ax} \cos bx$ : [798](#).
- $\int_0^\infty e^{-ax} \frac{\sin bx}{\cos bx} dx$ , and summation of the series: [J.41](#).
- \*  $e \exp. \sin x$ : 1529.
- \*  $e \exp. a \sin^{-1}x$ : 1535.
- \*  $e \exp. \log(z+x \sin y)$ : 1557.
- $e \exp. -\phi(x,y,z\dots)$ : [C.58](#).
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- $Al(x), Al_1(x), Al_2(x)$  (Weierstrass's functions) in powers of the modulus: [C.82,85,86](#): [L.79](#).
- $\frac{f^n(x)}{n!}$ , by Taylor's th: [CM.4](#).
- \*  $f(x+h)$  (see Taylor's th): 1500—9, 1520: Abel's th. 1572; Stirling, 1546; Boole, 1547: [AJ.3](#).
- \*  $f(x)$  (Maclaurin): 1507: 3759.
- \*  $f(x+h, y+k)$ : 1512, 1521.
- \*  $f(x,y)$ : 1516, 1523: [Me.3](#).
- \*  $f(0)$  (Bernoulli): 1510.
- \*  $\phi(a+bx+cx^2+\dots)$  (Arbogast): 1536: [CD.1,6](#).
- \*  $\frac{1}{\phi(z)}$  (Cayley): 1555.
- \*  $f\{z+x\phi(y)\}$  in powers of  $x$  (Lagrange): 1552: Laplace's th. 1556.

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- \*  $f(y)$  in powers of  $\psi(y)$  (Burmam): 1559.
- \*  $f\{\psi^{-1}(x)\}$  and  $\psi^{-1}(x)$ : 1561—3.
- \*  $\phi(e')$  (Herschel): 3757.
- \*  $u_{x+n}$ : 3740;  $\Delta^n u$ : 3761.
- \*  $\Delta^n u_x$  and  $\Delta^n u_y$ : 3741—2.
- \* in differential coefficients of  $u$ : 3751.
- \*  $\Delta^n x^m$  and  $\Delta^n 0^m$ : 3743—4.
- \*  $u_{nx}$  in differences of  $u$ : 3752.
- \*  $\int_0^n u_x dx$  in terms of  $u_0, u_1, u_2$ , &c.: 3778.
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- \* logarithms: 152—9: [N.82](#).
- \*  $\log(1 \pm x)$ ,  $\log \frac{1+x}{1-x}$ , &c.: [155—9](#).
- \*  $\log y$  and  $\log y^n$  in powers of  $a^{-1}$  when  $y^3 - ay + b = 0$ : 1553—4.
- \*  $\log(a + bx + cx^2 + \dots)$ : 1537.
- \*  $\log(1 + 2a \cos x + a^2)$ : 2922.
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- \*  $\log 2 \frac{\cos(\frac{x}{2})}{\sin(\frac{x}{2})}$ : 2927.
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- \* notation: [94](#), 2451: [Q.2](#).

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- \*  $n! = \Gamma(1+n)$ : 2290.
- approx. to when  $n$  is large: [C.9, 50](#): [J.25, 27](#): [L.39](#).
- $n! = n^n e^{-n} \sqrt{(2n\pi)}$  (Stirling): [Q.15](#).
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- \* Fagnani's theorem: 6088: [A.26](#): LM. 5, 13, 23: [Z.1](#).
- curves having Fagnanian arcs: LM.11.
- stereometric analogy: [Z.17](#).



- Faisceaux:—of binary forms having the same Jacobian: [C.93](#).  
 of circles: [C.76](#).  
 of conics: [Z.20](#).  
 curvature relations: [Z.15](#).  
 formation of: [C.45](#): [CM.3](#).  
 intersections of: [N.72](#): degree of the resulting curve, [J.71](#).  
 of lines and surfaces: [N.53,83](#).  
 plane: [N.53](#): defined by a first order d.e, [C.86](#).  
 of tortuous cubics in connection with ray-complexes: [Z.19](#).
- Fan of Sylvester: [E.33](#).
- Faure's theorems: [G.1.19](#): and Painvin's, [N.61](#).
- Fermat's theorems:—of  $(N^{p-1}-1) \div p$ : [369](#): [A.32](#): [AJ.3](#): [J.8](#).  
 of  $x^n + y^n = z^n$  being insoluble when  $n$  is an odd prime, &c: [An.57](#): [C.gz84](#) and [96,91](#): [J.40,87](#): [TE.21](#).  
 analogous theorem: [J.3](#).  
 case of  $n = 14$ : [J.9](#).  
 and periodic functions: [Me.76](#).  
 $x + y = \square$ ,  $x^2 + y^2 = \square^2$ : [Mém.26](#).  
 of the semicircle: [A.27,30,31](#): [gzA.31](#).  
 method of maxima: [C.50<sub>2</sub>](#).
- Feuerbach's: th of the triangle, [Me.84](#): circle, [A.59](#).
- Fifteen girl problem: [E.34,35](#): [Q.8,9](#).  
 "Fifteen" puzzle: [AJ.2<sub>2</sub>](#).
- \*Figurate numbers: [289](#): [A.5,69](#).
- \*Finite differences, calculus of: 3701—3830: [A.13,18,24,63](#): [C.70](#): [J.11<sub>2</sub>](#), [12,13<sub>2</sub>,14<sub>2</sub>,15<sub>2</sub>,16](#): [Me.82](#): [Mél.5](#): [Mém.13](#): [N.69](#): [thsP.16,17](#).  
 ap. to complex variability: [An.82](#): ap. to i.eq, [An.50](#).
- \* first and  $n$ th differences: 3706.  
 $\Delta^2 u = 0$ : [An.73](#).
- \*  $\Delta^n 0^m$ : 3744: [Q.5,8,9](#): Herschel's table, [N.54](#).  
 $\Delta^n 0^m \div \Pi(m)$ , table of: [CP.13](#).  
 $\delta u_0$ ,  $\delta^2 u_0$ , &c., in a function of  $\Delta u_0$ ,  $\Delta^2 u_0$ , &c.: [N.61](#).
- \*  $\Delta^n u$  in successive derivatives of  $u$ : 3761: [N.73](#).  
 $\Delta 1^p$  and Bernoulli's numbers: [An.59](#).  
 $hu'_x = \Delta u_x - \frac{h}{2} \Delta u'_x + \frac{B_1 h^2}{12} \Delta u''_x - \&c.$ : [Ac.5](#).  
 $\Delta \sin x$  and  $\Delta \cos x$ : [CM.1](#).
- Finite difference equations: [AJ.4](#): [An.59](#): [CD.2](#): [CM.1,3,4](#): [CP.6](#): [JP.6](#): [L.83](#): [P.60](#): [Pr.10](#).  
 of integrable form: [C.54](#).  
 of mixed differences: [Q.10](#).
- Finite difference eqs.—(continued):  
 of the kind  $u_{x,y} = u_{x-y,z+y}$ : [CM.4](#).  
 linear: [AJ.7](#): [An.50](#): [At.65](#): [Q.1](#).  
 first order, constant coefficients: [C.8](#).  
 determination of arbitrary constants: [A.27](#): [At.65](#): [G.7](#).  
 integration to differences of any order: [J.12](#).  
 with variable coefficients: [C.17](#).  
 partial:—  
 constant coefficients: [C.8](#).  
 linear of 2nd order: [C.98](#).  
 of physics: [C.73](#).
- Finite differences:—exercises: No. [44,47](#).  
 formulæ: [CD.9](#): [Q.2](#).  
 sum and difference: [J.58](#).  
 of functions of zero: [TL.17](#).  
 $H(n, r)$  value of: [Q.9](#).  
 integrals: [C.39,57](#): [JP.4](#): [L.44](#).  
 expressed by definite integrals: [An.53](#).  
 $\Sigma e^x y$ : [A.6](#): No. [44](#).  
 inverse method: [C.74](#): [P.7](#).  
 involving  $\sqrt{1}$ : [Me.78](#).  
 of powers converted into d.i: [JP.17](#).  
 Fleflectnodal planes of a surface: [Q.15](#).
- Flexure: [AJ.2<sub>2</sub>](#): [Me.2](#).  
 of ruled surfaces: [An.65](#).  
 of spaces: [LM.9](#).  
 of spherical surfaces: [Me.77](#).
- \*Fluctuating functions: 2955a: [LM.5](#): [M.20](#): [TL.19](#).
- Fluents: [P.1786](#).  
 of irrational functions: [P.16](#).
- \*Focal: chords of conics: 1226, 4235, 4339.  
 circle of conics: [Mél.2](#).  
 distances: 4298: [N.64](#).  
 pedal of a conic: [N.66](#).
- \*Focal properties:—of conics: 1163, 1167—9, 1181, 1286—8, 4298—4306, 4336—45, 4378, 4382, 4516, 4550—58, 4719—[21](#), 5008—16: [CD.7](#).  
 of curves: [CD.7](#).  
 of homographic figures: [N.71](#).
- \* of a parabola: 1220, 1223—6, [1230—4](#), 4231, 4235—8: [G.22](#).  
 of a quadric surface: [An.59](#): [N.58](#).  
 Focal quadrics of a cyclide: [Me.85](#).  
 Foci: [J.64](#): [N.42,44,53,85](#): [Q.2,a,c,9](#).
- \* of conics: 1160: [trA.25,63,64,cn69](#): [gzC.22](#) and [L.39](#): [CP.3](#): [N.69,74,78,81,82](#): [t.cQ.8,13,12](#) and 45: gen. eq, [N.48](#).  
 analogous points in higher plane curves: [J.10](#).



- Foci:**—of conics—(*continued*):
- \* coordinates of: 4516.
  - eq. of: LM.11: [a.c.E.40](#)
  - exterior: [N.79](#)
  - \* to find them: [Q.25](#): from gn.eq.5008.
  - through four points: [N.83<sub>a</sub>](#)
  - \* of four tangents: 5029: [N.83](#)
  - locus, a cubic: [M.5](#)
  - negative: A.64<sub>2</sub>
  - under three conditions: [Q.8](#)
  - of curves: [C.82](#); *n*th class, [86](#): [N.59,79](#)
  - of cones: [N.79](#)
  - of differential curve of a parabola: [A.58](#)
  - of in-conic of an *n*-tic, locus of: [E.21](#)
  - of lines of curvature of an ellipsoid: [Z.26](#)
  - of quadrics: [N.42,66,74,75,78](#)
  - of quartics: [J.56](#)
  - of the section of a quadric by a plane: [N.64,70](#)
  - by another quadric: [N.47](#)
  - of surfaces: [C.74](#): of revolution, N.74
  - \*Folium of Descartes: 5360: [N.44](#)
  - Forms, theory of: [M.18](#): of higher degree, [Mo.83](#): [Pr.38](#)
  - reciprocity principle: [An.56](#)
  - Formulae: [G.15,19](#)
  - for log 2, &c.: [Me.79](#)
  - in the Fund. Nova: [Me.76](#)
  - \* of reduction in i.c.: 1965: [Me.3](#)
  - Four colors problem: [AJ.2<sub>a</sub>](#)
  - Four-point problem: [E.5,6,8<sub>a</sub>](#)
  - Four right lines not 2 and 2 in same plane: [J.5](#)
  - Fourier-Bessel function: [J.69](#): [M.3](#)
  - \*Fourier's formula in i.c.: 2726—[42](#): [CM.3](#): [J.36,69](#): [L.36](#): [M.4](#): [Me.73](#): [Q.8](#): [gzZ.9](#)
  - ap. to calculation of differentials: [J.13](#)
  - \*Fourier's theorem: [518](#): [528](#): [An.50](#), [75](#): [J.13](#): [M.19<sub>2</sub>](#): [Me.77,82,83](#)
  - ap. to a function of a complex variable: [M.21](#)
  - Fractions: [AJ.3<sub>a</sub>](#): [G.9,pr16](#): [J.88](#): [L.10](#)
  - continued, decimal, partial, vanishing, &c.: see each title.
  - number expressible by digits  $\geq n$ : [C.96](#)
  - reduction to decimals: [A.1,25](#)
  - transformation into decimals: A.11.
  - \*Frullani's formula: 2700: [LM.9](#)
  - Fuchsian functions: [C.92,93,94<sub>3</sub>,95,96](#)
  - Fuchs's theorem on  $F(z, y, y_2) = 0$ : [C.99](#)
  - Functional equations: [CM.3](#): [J.90](#): [TE.14](#)
  - $f. \phi x = fx$ : [C.88](#)
  - $f. \phi x = 1 + fx$ : [C.99](#)
  - $\phi. fx = F. \phi x$ , to find  $\phi$ : [Mém.31](#)

Functional equations—(*continued*):

- $\phi x + \phi y = \phi (xfy + yfx)$ : [J.2](#)
- $f_1y. \phi_1x + f_2y. \phi_2x + \&c.$
- $= f_1x. \phi_1y + f_2x. \phi_2y + \&c.$ : [J.5](#)
- $\phi x + \phi y = \psi \frac{fy. Fx + fx. Fy}{\chi(xy)}$ : [J.46](#)
- $\int_a^b f(x, \theta) \phi(x + \theta) d\theta = F(x)$ , to find  $\phi$ : [Pr.8](#)
- $\phi x - \phi \frac{ax+b}{cx+d} = \&c.$ : [Q.15](#)
- $f(x) = f\left(\sin \frac{\pi x}{2}\right)$ : [C.88](#)
- Functional images in ellipses: [Q.17](#)
- in Cartesian ovals: [Q.18](#)
- Functional powers: [Mém.38](#)
- symbols: [Q.4](#)
- Functions: [A.28](#): [AJ.6](#): [An.79](#): [C.43](#), [91](#): [CP.1](#): [J.16,prs71,74,84,87](#), [91](#): [L.45](#): [Me.7](#): [Mo.80,81](#): [P.15](#), [16,17,62](#): [Pr.11<sub>2</sub>](#): [prsZ.26](#)
- algebraic, alternating, analytical, circular, circulating, conjugate, continuous, curvital, cyclotomic, derived, discontinuous, elliptic, even and odd, exponential, Fuchsian, gamma, generating, hyperbolic, implicit, infinite, imaginary, integral, irrational, irreducible, isotropic, iterative, monodrome, monogenous, monotypical, non-uniform, periodic, polyhedral, quantitative, rational, representative, transcendental, trigonometrical: see the respective headings.
- analogous:—to algebraic functions: [C.89](#)
- to circular functions: [C.84](#)
- to Euler's: [C.89](#): [M.19](#)
- to functional determinants: [J.75](#)
- sine and cosine: [Q.16](#)
- to modular functions: [Ac.2](#): [C.93](#)
- connected by a linear eq.: [C.17](#)
- condition of  $f(x, y)$  being a function of  $\phi(x, y)$ : [A.21](#)
- development of: see "Expansion."
- defined by d.e: [JP.21,28](#)
- differing very little from zero: [L.74](#)
- errors of geometricians: [J.16](#)
- expressed by other functions, remainder: [C.98](#)
- fractional: [J.8](#): the variable being the root of an equation, [N.56](#)
- from functional equations: [M.24](#)
- from Gauss's equation: [C.92](#)
- with lacunæ: [C.96](#)
- Lagrange, tr: [JP.5,7](#)



## Functions—(continued):

linear: [C.90](#)  
 with linear transformations *inter se*:  
[M.19,20](#).  
 whose logarithms are the sums of  
 Abel's integrals of the 1st and  
 3rd kind: [C.92](#).  
 with non-interchangeable periods:  
[M.20,21,25](#).  
 number of values of: [C.48](#).  
 do. through permuting the variables:  
[JP.10,18](#): [L.50,60](#).  
 of two variables: [Ac.3](#): [C.90,96](#).  
   made constant by the substitution  
   of a discontinuous group: [C.97](#).  
   which arise from the inversion of  
   the integrals of two functions:  
   [C.92](#).  
   whose ratio has a fixed limit: [G.5](#).  
    $f(x, y)$  such that  $f\{zf(x, y)\}$  is sym-  
   metrical: [J.1](#).  
 of three variables satisfying the d. e.  
 $\Delta F = 0$ : [Ac.4](#).  
 of three angles, th. re 1st derivatives:  
[J.48](#).  
 of 4 and 5 letters: [L.56](#).  
 of 4, 5, and 6 letters: [L.50](#).  
 of 7 letters: [C.57,95](#).  
 of 6 variables which take only 6 diffe-  
 rent values through their permu-  
 tation, not including 5 symmetri-  
 cal permutations: [A.68](#).  
 of  $n$  variables: [C.21](#): [Mo.83](#): with  
 $2n$  systems of periods, [C.97](#).  
 analogous to sine and cosine: [Q.16](#).  
 number of values: [J.85](#): do. by  
 permutation of the variables: [C.21](#).  
 obtained from the inversion of the  
 integrals of linear d. e with  
 rational coefficients: [C.90](#): [J.89](#).  
 of an analytical point, ths: [C.95](#).  
 of a circular area from a given inte-  
 gral condition: [Z.26](#).  
 of imaginary variables: [C.32,48](#): [JP](#).  
[21](#): [L.58,59,60,61,62](#): [LM.geo](#)  
[8](#).  
 of large numbers, approx.: [C.20](#).  
 of a real variable, connexion with  
 their derivation: [M.23,24](#).  
 of real arguments, classification ac-  
 cording to their infinitesimal  
 variation: [J.79](#).  
 of the species zero and unity: [C.95](#).  
 of a variable analogous to the poly-  
 nomials of Legendre: [C.95](#).  
 allied to Pfaffians: [Q.16](#).  
 rationally connected: [L.59](#).  
 with recurring derivatives: [LM.4](#):  
[TE.24](#).

## Functions—(continued):

which relate to the roots of the equa-  
 tion of division of a circle or of  
 $n^p - 1 = 0$ : [J.17](#).  
 representation of: [C.92](#): [M.17](#): one-  
 valued, [Z.25](#).  
 approximate: [Z.3](#).  
 by an arbitrary curve: [M.22](#).  
 by Bessel's functions: [M.6](#).  
 by definite integrals: [Ac.2](#).  
 by elliptic functions: [An.82](#).  
 by Euler's sum-formula: [J.56](#).  
 by Fourier's series: [Mo.85](#).  
 by graphic methods: [A.2](#): imag.,  
[J.55](#).  
 by infinite products: [Z.24](#).  
 $y = e^{x^r} \lambda$ , constant and  $r$  a positive  
 integer: [A.42](#).  
 $y = x^n e^{x^r}$ : [A.52](#).  
 reproduced by substitution: [C.19](#).  
 resolution into factors: [Ac.65](#): [C.19](#),  
[30](#): [CP.11](#): [J.18](#).  
 satisfying the eq.  $\Delta F = 0$ : [C.96](#).  
 singularities of: [M.19](#).  
 whose successive derivatives form an  
 arith. prog.: [An.71](#).  
 systems of: [Mo.78](#): of two inter-  
 connected, [C.98](#).  
 of two systems of quantities, cor-  
 relative and numerically equal:  
[C.98](#).  
 which are neither rational nor reduci-  
 ble to irrational algebraic ex-  
 pressions: [C.18](#).  
 which are of use in elliptic functions  
 and logarithms: [No.58](#).  
 which take a given value in a given  
 position: [An.82](#).  
 which have no derivative throughout a  
 certain interval: [An.77](#).  
 which vanish with their variables:  
[TI.16](#).  
 $x^x$ : [An.63](#).  
 $\{(x^x)^x\}^x$  and so on, and the corres-  
 ponding inverse function: [J.42](#).  
 arising from  $\sqrt{(4-2xz+z^2)}$ : [J.2](#).  
 $\phi(x) = \frac{ax+b}{cx+d}$ : [LM.9](#).  
 $f(u, z)$ ,  $u$  being an implicit function  
 of an imaginary variable  $z$ : [Pr.42](#).  
 $f(x)$ , formula of analysis: [J.53](#).  
 $f(x) = 0$ ,  $y = f'(x)$ , th. re  $\phi(y)$ : [E.36](#).  
 $\psi(x) \equiv d_x \log \Gamma(x)$ : 2743.  
 $\psi(a)$  of Jacobi: [J.93](#).  
 $Q(x)$ : [Ac.2](#).

## Functions—(continued):

$$\text{Bessel's, } I(x) \equiv \frac{1}{\pi} \int_0^\pi \cos(x \cos \theta) d\theta \\ = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} +$$

Cauchy's numbers;  $N_{-k,p}$ 

$$\equiv \frac{1}{2\pi} \int_0^{2\pi} e^{-kui} (e^{iu} + e^{-iu})^p \times \\ (e^{iu} - e^{-iu})^p du.$$

cosine integral;  $Ci q \equiv \int_\infty^q \frac{\cos x}{x} dx$ :  
[taP.70.](#)

Dirichlet's function,

$$F(x) = \sum \left( \frac{D}{n} \right) \frac{1}{n^x}: \text{Z.27.}$$

elliptic;

$$\int x^{r-1} \mathfrak{F}(x^p) \{R(x^p)\}^{\pm \frac{1}{rp}} dx: \text{J.23.}$$

\* Euler's;  $B(l, m)$ : 2280.expon-integral;  $Ei q \equiv \int_{-q}^\infty \frac{e^{-x}}{x} dx$ : A.  
[10](#): [taP.70.](#)

$$E_n \equiv \frac{E^n}{1_2^{(n)} \pi} \int_0^\pi \sin^{2n} \omega \cos(e \cos \omega) d\omega: \text{An.} \\ \text{70.}$$

 $E(x)$ : [M.61.6.](#)

$$G \equiv \int_0^1 \frac{\tan^{-1} x}{x} dx = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \&c. \\ = .915965, 594177 \dots: \text{Mém.83.}$$

 $\Gamma(x)$ : see "Gamma function."

Jacobi's

$$\left( \frac{b}{a} \right) \equiv (-1) \exp \sum_{i=1}^{a-1} \frac{a-1}{2} e \left( \frac{bi}{a} \right):$$

[C.59.60](#); [L.47.50.](#)Laplace's  $Y^{(n)}$ : [M.14.](#)log-integral;  $Li q \equiv \int_0^q \frac{dx}{\log x}$ 

$$L^n(1+x) \equiv x - \frac{x^2}{2^n} + \frac{x^3}{3^n} - \&c.$$

Legendre's  $X_n$ : see "Legendre."

$$P, \text{ where } \int_0^P e \exp \left( -\frac{x^2}{h^2} \right) dx = \frac{h\sqrt{\pi}}{4}: \\ \text{Q.10.}$$

 $P^n(\cos \gamma), n = 0$ : [G.22.](#)

$$P \left( \frac{\alpha \beta \gamma}{\alpha' \beta' \gamma'}, x \right): \text{Z.14.}$$

$$\Pi(z) \equiv \int_0^\infty x^z e^{-x} dx; \psi(z) \equiv d_z \log \Pi(z): \\ \text{Q.1.}$$

sine-integral;  $Si q \equiv \int_0^q \frac{\sin x}{x} dx$ : [taP.70.](#)

$$\sum \int e \exp(-x^r) F(z) dz = 0: \text{C.93.}$$

## Functions—(continued):

$$\int \sqrt[3]{1+ax^3} dx: \text{J.9.}$$

$$Y(p, \phi) \equiv \int_0^\pi \frac{E(p, \phi) d\phi}{\sqrt{1-p^2 \sin^2 \phi}}.$$

X, Y, &c., such that  $SXYd\sigma = 0$  and  
and that any function can be ex-  
panded in the form  $\alpha X + \beta Y + \&c.$ :  
[L.M.10.](#)\*Gamma function,  $\Gamma(n)$ : 2284: [A.4,6,61](#):  
[An.69](#): [C.35,92,96](#): [J.35,82,90](#),  
ap 57: [L.42,46,52,55](#): [Q.9](#): [Z.1,](#)  
[25.](#)application of this and other trans-  
cendents: [C.86.](#)of a complex: [Me.84.](#)\* curve  $y = \Gamma(x)$ : 2323.\* deductions: 2286—2316: [A.10.](#)derivatives of: [Q.6.](#)of equidifferent products: [J.36.](#)of an infinite product: [J.39.](#)\* = limit. of  $\frac{\mu! \mu_n}{n^{(\mu+1)}}$ : 2293: [A.30.](#)\* logarithm of: 2294, 2768: [C.9.](#)

\* numerical calculation: 2771.

\* as a definite integral: 2768.

[n](#), negative: [CD.3.](#)

\* numerical calculation of: 2317.

\* the function  $\psi(x) \equiv d_x \log \Gamma(x)$ : 2743  
—70.reciprocal of: [Z.25.](#)reduction of: [J.40.](#)\* transformation of: 2284, 2318: [J.57.](#)

$$\Gamma(n+1) = \sqrt{(2\pi)} e^{-n} n^{n+\frac{1}{2}} (1+e) \text{ (Stir-} \\ \text{ling): } \text{C.50.}$$

$$*\Gamma(m) \Gamma(1-m) = \frac{\pi}{\sin m\pi}: 2313: \text{Ac.3.}$$

$$*\Gamma(x) \Gamma\left(x + \frac{1}{n}\right) \dots \Gamma\left(x + \frac{n-1}{n}\right): 2316: \\ \text{L.55,56.}$$

$$\Gamma(x) = P(x) + Q(x): \text{Ac.2.}$$

$$\sum \frac{1}{\Gamma(x)}: \text{G.6.}$$

Gauche cubics: [C.82](#): [J.60](#): [N.62.](#)3rd class, theorems: [J.58.](#)number of common chords of two:  
[An.70.](#)through five points: [N.83.](#)through six points, cn: [N.66.](#)Gauche curves: [C.70,77,90.](#)[Mo.82](#): thsN.53.classification of: [JP.32.](#)on a cubic surface: [C.62.](#)of a developable surface, singulari-  
ties: [An.70.](#)differential invariants of: [JP.28.](#)



## Gauche curves—(continued):

intersection of two surfaces having common multiple points, singularities of: [C.80](#)  
 on a one-fold hyperboloid: [An.1](#): [C.52,53](#)  
 metric properties of, in linear space of  $n$  dimensions: [M.19](#)  
 representative curve of the surface of principal normals of: [C.86](#)  
 of the zero species: [C.80](#)

Gauche helicoids: rad. of curv.: [N.45](#)  
 in perspective: [JP.20](#)

Gauche:—in-polygons of a quadric: [C.82](#).

perspective of algebraic curves: [C.80](#)  
 projection: [N.65](#)

quadric: [N.67](#): and orthogonal trajectory of generatrices: [thN.48](#)

quartic: [A.62](#): [C.82](#): [L.70](#)

2 points of, 7 points of a gauche cubic and 8 associated points: [C.98](#)

unicursal, a class of: [C.83](#)

surface: [N.61](#)

sextic curve: [C.76](#)

surface: [JP.17](#): [L.37,72](#)

deformation of: [C.57](#)

which can be represented by a p.f.d.e of the 2nd order: [C.61](#)

Gaussian periods of congruent roots corresponding to circle division: [J.53](#)

\*Gauss's function: see "Hypergeometric function."

Gauss's theorems: [J.3](#)

\*General methods in anal. geometry: 4114.

General numerical solution of any problem: [LM.2](#)

\*Generating functions: 3732: [J.81](#): [N.81](#): [Pr.5](#)

and ground-forms of binary quantics of first ten orders: [AJ.2,3](#)

do. of binary 12-ic and of irreducible syzygies of certain quantities: [AJ.4](#)

of some transcendental series: [At.55](#)  
 for ternary systems of binary forms, ta: [AJ.5](#)

\*Geodesics: 5775, 5837—55: [A.39](#): [C.40,41,96](#).p.c97: [CD.5](#): [G.19](#): [J.50,91](#): [M.20](#): [Me.71](#): [N.45,65](#): [Q.1,5](#): [Z.18,26](#)

of cubic surfaces, loci: [CD.6](#)

curvature: 5846: [C.42,80](#): [L.12](#)

on an ellipsoid: [M.20](#)

\* radius of: 5776, 5846.

duals of: [Q.12](#)

\* equations of: 5837.

## Geodesics—(continued):

flexure of: [C.66](#)

forms from cn of their polar systems: [Z.24](#)

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- $y^2 = z^2 + t(z+\beta)^2$ : [N.78<sub>2</sub>](#).
- quadratics in five unknown integers:  
 $x^2+by^2+cz^2+dt^2 = u$ , with the following values of  $b, c, d$ : [1, 1, 1](#); [2, 3, 6](#); [L.45](#): [1, 1, 2](#); [1, 1, 4](#); [1, 1, 8](#); [2, 2, 2](#); [2, 4, 8](#); [4, 4, 4](#); [3, 4, 12](#); [L.61](#): [1, 2, 4](#); [1, 4, 8](#); [2, 2, 4](#); [2, 4, 4](#); [2, 8, 8](#); [4, 4, 8](#); [4, 4, 16](#); [4, 16, 16](#); [8, 8, 8](#); [8, 8, 16](#); [8, 16, 16](#); [16, 16, 16](#); [L.62](#): [2, 3, 3](#); [3, 3, 3](#); [3, 3, 9](#); [L.66](#): [1, 3, 3](#); [L.60](#), [63](#): [1, 1, 3](#); [1, 2, 6](#); [2, 2, 3](#); [2, 4, 6](#); [4, 4, 12](#); [1, 1, 12](#); [2, 2, 12](#); [1, 4, 12](#); [1, 3, 4](#); [3, 4, 4](#); [4, 12, 16](#); [3, 6, 6](#); [3, 3, 3](#); [3, 3, 12](#); [3, 12, 12](#); [12, 12, 12](#); [L.63](#):



## Indeterminate equations—(continued):

1,1,5; 2,3,6; 5,5,5; L.64: 1,5,5;  
1,6,6; 1,9,9; 1,n,n; 2,n,2n; L.65,  
59: with  $c = ab$ , C.42 and L.56.

$ax^2 + by^2 + cz^2 + dt^2 = u$ , with the follow-  
ing values of  $a, b, c, d$ : 2,2,3,4; 2,3,  
3,6; 3,3,3,4; 1,2,6,6; 2,3,4,4;  
L.66: 2,2,3,3; L.65: 3,4,4,4;  
3,4,12,48; L.63.

$x^2 + y^2 + z^2 + t^2 = 4u$ : L.56.

$ax^2 + by^2 + cz^2 + dt^2 + exy + fzt = u$ , with  
the following values of  $a, b, c, d$ ,  
 $e, f$ : 1,2,-2,2,1,2; L.63: 1,2,3,3,  
1,3; 2,2,3,3,2,3; 1,1,6,6,1,6; L.64:  
2,3,2,3,2,2; 2,5,2,5,2,2; L.64: 1,1,  
2,2,0,2; 1,1,1,1,0,1; 1,1,1,1,1,1;  
2,2,3,3,2,0; 1,1,3,3,1,0; L.63: 3,5,  
10,10,0,10; 2,3,15,15,2,0; 2,3,3,3,  
2,0; L.66.

$x^2 + 2y^2 + 2z^2 + 3t^2 + 2yz = u$ : L.64.

$2x^2 + 3y^2 + 3z^2 + 3t^2 + 2yz = u$ : L.66.

$x^2 + y^2 + z^2 + 2u^2 + 2uv + 2v^2 + t^2 = w$ :  
L.64.

$x^2 + y^2 + 2z^2 + 2zt + 2t^2 + 3u^2 + 3v^2 = w$ :  
L.64.

$2(x^2 + xy + y^2) + 3(z^2 + t^2 + u^2 + v^2) = w$ :  
L.64.

$xy + yz + zt + tu = v$ : C.62; L.67.

$y^2 = x_1^2 + x_2^2 + \dots + x_n^2$ : G.7.

quadratics in seven unknown inte-  
gers:

$x^2 + ay^2 + bz^2 + ct^2 + du^2 + ev^2 = w$ , with  
the following values of  $a, b, c, d, e$ :  
4,4,4,4,4; 1,4,4,4,4; 2,2,4,4,4; 1,1,  
4,4,4; 1,2,2,4,4; 1,1,1,4,4; 1,1,2,  
2,4; 1,1,1,1,4; 4,4,4,4,16; L.65:  
1,1,1,1,1; 1,1,1,1,2; 1,1,1,2,2;  
1,2,2,2,2; 2,2,2,2,2; 2,2,2,2,4; 3,3,  
3,3,3; L.64.

higher degrees:

cubic: AJ.2.

$x^3 = y^2 + a$ : N.78,83: with  $a = 17$ ,  
N.77.

$x^3 + y^3 = az^3$ : N.78,80.

$x^3 + y^3 + z^3 + u^3 = 0$ : A.49.

$ax^4 + by^4 = z^2$ : C.87,91,94: N.79:  
 $a = 7$ ,  $b = -5$ , L.79.

$x^4 \pm 2^m y^4 = z^2$  and similar eqs: L.53.

$x^4 + ax^2 y^2 + y^4 = z^2$ : Mém.20.

$ax^4 + bx^2 y^2 + cy^4 + dx^3 y + exy^3 = fz^2$ :  
C.88.

$x^5 + y^5 = az^5$ : L.43.

$x^7 + y^7 = z^2$ , impossible: C.82; L.40.

## Indeterminate equations—(continued):

$n$ -tic solution by alg. identities:  
C.87.

$x^n + y^n = z^n$  impossible if  $n > 2$  (Fer-  
mat's last th.): A.26,58: An.64:  
C.24,89,90,98: J.17.

$x^{2n} - y^{2n} = 2x^n$ : L.40.

$ax^m + by^m = cz^n$ : N.79.

$x^m = y^n + 1$ , impossible: N.50,70,71.

$x^2 - ay^2 = z^n$ : C.99.

simultaneous:

$x = u^2$ ;  $x + 1 = 2v^2$ ;  $2x + 1 = 3w^2$ :  
N.78.

$x^2 + a = y^2$ ;  $y^2 - a = z^2$ : An.55; C.78.

$x^2 + x + 2 = y^2$ ;  $x^2 - x - 2 = z^2$ : N.76.

$ax + by + cz = 0$ ;  $Ayz + Bzx + Cxy$   
 $= 0$ : A.28.

$x^2 + y^2 - z^2 = \square$ ;  $x^2 - y^2 + z^2 = \square$ ;  
 $-x^2 + y^2 + z^2 = \square$ : E.20.

$x^2 + axy + y^2 = \square$ ;  $y^2 + ayz + z^2 = \square$ ;  
 $z^2 + azx + x^2 = \square$ : E.20,21.

$x + y + z = \square$ ;  $x^2 + y^2 + z^2 = \square$ ;  
 $x^3 + y^3 + z^3 = \square$ : E.17.

six eqs. in nine unknowns: N.50.

exponential,  $x^y = y^x$ : A.6: Z.23.

$a^x - b^y = 1$ : N.57.

\*Indeterminate forms: 1580—93: A.26:  
AJ.1 exponential: J.1: Mé.75:

N.48,77:  $\frac{0}{0}$ , A.21;  $Z.1$ :  $\frac{\infty}{\infty}$ ; L.41,

\* 42: N.46:  $\frac{f(x)}{x}$  when  $x = \infty$ , 1592:  
 $0^0$ , J.11,12:  $0 \exp 0^x$ , J.6.

\* with two variables: 1592a.

\*Indeterminate multipliers: 213, 1862,  
3346: N.47.

\*Index law: 1490.

Indian arithmetic, th: L.57: calculation  
of sines, N.54.

\*Indicatrix: 5795: C.92: Mé.72: N.74.

\* a rect. hyperbola, condition: 5824.

\* two coinciding lines: 5825.

\* an umbilicus: 5819.

to determine its axes: L.78,82.

determination of a surface from the  
indicatrix: A.59.

\*Indices: 29: N.76,77,78.

in relation to conics: N.72,  
of functions, calculus of: JP.15.

\*Induction: 233: C.39: G.15: L.48.

\*Inequalities: 330—41: A.1,2,4.  
in integrals: Mé.3: f.d.c. Mém.59.  
 $(|n|)^2 > n^n$ : N.60.



## Inequalities—(continued):

- $a^x > x$ : [A.14](#)  
 if  $x^2 + y^2 = z^2$ ,  $x^m + y^m > z^m$ : [A.20](#)  
 geo. mean of  $n$  numbers < arith. mean: [332](#): [N.42](#)  
 Infinite:—equalities: [M.10](#): [prG.22](#)  
 functions: [An.71](#): [J.54](#)  
 from gnomonic projection: [Me.66](#)  
 linear point-manifoldness: [M.15,17,20,21,23](#)  
 point-mass: [M.23](#)  
 products: [J.27](#): [N.69](#)  
 value expressed by  $\Gamma$  functions: [Ac.3](#)  
 exhibiting circular arcs, logarithms and elliptic functions of the 1st kind: [J.73](#): [Ac.4](#)  
 use of in mathematics: [C.73](#)  
 \*Infinitesimal calculus: 1407: [M.11,18](#)  
 Infinitesimal geometry: [An.59](#): [C.82](#)  
 of a surface, formulæ: [G.13](#)  
 Infinity, points at on alg. surfaces: [C.59](#)  
 \*Inflexional tangents: 5789: [A.35](#)  
 of a cubic curve: [E.30](#): [J.38,58](#)  
 Inflexion curves: [Z.10](#)  
 \*Inflexion points: 5175: [CM.4](#): [J.41](#)  
 of cubic curves: [J.28](#): axis, [E.31](#)  
 Herse's equation: [N.81](#)  
 Inscribed figures:  
 In-circle:—of a quadrilateral, locus of centre: [A.52](#)  
 \* of a triangle: [709, 953](#), 4747—50: tg.e 4889: [CM.1](#)  
 \* in-centre: [709](#), t.c.4629, tg.e.4882.  
 In-conic:—of a circle: [thJ.31](#)  
 four of a conic: [prJ.39](#)  
 of a developable quartic: [An.59](#)  
 of a polygon: [M.25](#)  
 of a quadric: [J.41](#)  
 \* of a quadrilateral: tg.e.4907: [N.63](#): four, [N.56](#)  
 \* of a triangle: 4739—46, tg.e.4887: [A.2](#): [N.50](#): max, [A.8](#): [Q.2](#): lat. rect., [E.34](#)  
 In-cubic of a pencil of six lines: [Q.9](#)  
 In-hexagon:—of a circle: [A.22](#)  
 \* of a conic: 4781: [N.57,82](#)  
 In-parabola of a triangle: [CD.7](#)  
 In-pentahedron of a cubic: [Ac.5](#): [M.5](#)  
 \*In-polygons:—of a circle: [746](#): [CM.1](#): [J.35](#): [N.50](#)  
 regular of [15,30,60,120](#), &c., sides: [A.62](#)  
 do. 9 and 11 sides: [LM.10](#)  
 do. 17 sides: [TL.13](#)  
 do. four of 30 sides: [N.78](#)  
 do. 5, 6 and 10 sides, relation: [A.40, 43,45,48](#)

## In-polygons—(continued):

- two stars, one double the other: [A.61](#)  
 of a circle and conic (Poncelet): [G.1](#)  
 \* of a conic: 4822: [thsN.47](#): [cnTN.69](#)  
 \* with sides through given points, cn: 4823: [An.51](#)<sub>2</sub>  
 semi-regular: [N.63](#)  
 of a cubic (Steiner): [M.24](#)  
 of a curve: [Q.7](#)  
 of a polygon, th: [CD.5](#)  
 of a quadric with sides through given points: [LM.2](#)  
 In-quadrics:—of a developable: [Q.10](#): quartic, [An.59](#)  
 6 of a quadric, 2 touching 4: [An.69](#)  
 \*In-quadrilateral:—of a circle: [733](#): [A.5](#): [cnE.21](#): area, [N.44](#): [P.14](#)  
 \* of a conic: 4709.  
 of a cubic: [N.84](#)  
 In-sphere of a tetrahedron: [A.61](#)  
 In-spherical quadrilateral: [N.49](#)  
 In-square: of a circle: [J.32](#)  
 of a quadrilateral: [A.6](#)  
 In-triangles:—of a circle: [P.71](#)  
 with sides through given points: [J.45](#): [N.44](#)  
 of a conic: [J.7](#): Maccullagh's th, [N.65](#)  
 with given centroid: [G.23](#)  
 similar: [A.9](#)  
 of a triangle: [thsQ.21](#)  
 two (Steiner's "Gegenpunkte"): [J.62](#)  
 In- and circum-circles:—of a polygon: [N.45](#)  
 distance of centres: [A.32](#)  
 of a quadrilateral: Fuss's [prMe.13](#)  
 \* of a triangle: [935](#): [A.38](#)  
 \* distance of centres: [936](#), 4972: eq, 4644.  
 In- and circum-conics: of a pentagon: [N.78](#)  
 of a polygon: [J.64,70](#): regular, [Z.14](#)  
 of a quadrilateral: 60 theorems, [N.45](#): [N.76](#)  
 of a self-conjugate triangle: [Me.81](#)  
 \* of a triangle: 4724, 4739: [An.52](#): [G.22](#)  
 In- and circum-heptagons of a conic: [A.3](#)  
 In- and circum-pentagon: of a circle: [A.22,43](#)  
 In- and circum-polygons (see also "Regular polygons"):—of a circle: [L.16](#): [N.80](#): [P.11](#): [Q.11](#)  
 \* sum of squares of perps., &c.: [ths 1099](#)  
 difference of perimeters, [ths](#): [N.43](#)  
 of two circles, respectively: [C.53](#): [G.21](#): [L.78](#)



- In- and circum-polygons—(*continued*):  
 of a conic: [A.4](#): ellipse [An.52](#): [An.57](#): [L.64](#): [N.57,84](#).  
 of two conics: [C.90](#).  
 of a curve: [C.78](#).  
 of a homonymous polygon: [A.50](#).
- In- and circum-quadratics of a tetrahedron: [eqsN.65](#).
- In- and circum-quadrilaterals:—of a circle: [A.48](#).  
 of a conic: 4709: and pentagon ths, [N.48](#).
- In- and circum-spheres:—of a tetrahedron: [N.73](#).  
 of a regular polygon: [A.32](#).  
 \* of a regular polyhedron: [910](#).
- In- and circum-triangles:—of a circle (Castillon's pr): [Q.3](#).  
 equilateral, of another triangle: [Me.74](#).  
 and square of an ellipse: [A.30](#).  
 \* of two conics: 4970: [N.80](#).  
 \* envelope of base: 4997.  
 respectively of two conics having a common pole and axis: [CD.4](#).
- \*Instantaneous centre: 5243.
- \*Interest: 296—301: and insurance, [A.26](#).
- Integrability of functions: [An.50,73](#): [C.28](#): [J.59,79](#): [JP.17](#): [L.49](#).  
 criterion for max. and min. values of a primitive: [An.52](#).
- \*Integral calculus: 1900—1997: [A.ext.18,26](#): Euler's, [A.20](#): [C.14,42](#): Newton, [CD.8](#): [G.19](#): [L.47](#): [Me.72,74,75](#): [Mém.18,36](#).  
 paradoxes: [C.44](#).
- \* theorems, &c.: 2700—42: [A.45](#): [C.13](#): [L.geo.ap50,56](#): [Me.77](#): [Mém.prs15,30](#).
- Integral functions: [C.88,89,98](#): [G.4,22](#): [h.c.f. of G.2](#).  
 with binomial divisors: [J.70](#).  
 and continued fractions: [An.77](#).  
 reciprocal relation of: [A.67](#).
- \*Integrals or Integration: 1908: [A.1,2,4,5,6,10,23](#): [Ac.1,32,44](#): [C.90](#): [CD.9](#): [CP.3](#): [J.2,4,8,17,25,36,39,61,92](#): [JP.9,10,11](#): [L.39](#): [M.6,16,73,75](#): [Mém.31](#): [P.14,36,37](#): [Pr.7,39](#): [Q.11,13](#): [Z.7,11,15,18,22,23](#).
- \* approximation to: 2127, 2262, 2991: [A.9,14](#): [C.97](#): [CM.2](#): [G.3](#): [J.1,16,18,37,48](#): [L.80](#).
- \* Gauss's: 2997: [C.84](#): [M.25](#).  
 by the principle of Abel's derivative: [J.23](#).
- \* by differentiating under the sign of integration: 2258.  
 by elliptic functions: [G.11](#): [L.46](#).
- Integrals or Integration—(*continued*):  
 from orthogonal surfaces' theory: [L.38](#).  
 by Pfaff's method: [A.47](#).  
 by series: [Me.71](#).  
 by substitution: [A.18](#).  
 by Tchebychef's method: [L.74](#): [M.5](#).  
 comparison of transcendents: [Me.79](#): [Pr.8](#).  
 complex, representing products and powers of a definite integral: [J.48](#).  
 connected with trinomial integrals: [L.55](#).  
 convergency of: [M.13](#).  
 definite:—applied to Euler's, &c.: [J.16](#).  
 with finite differences: [J.12](#).  
 from indefinite: [J.41,51,52](#).  
 whose derivatives involve explicit functions of the same variable: [C.12](#).  
 determination of functions under the sign  $\int$ : [JP.15](#).
- \* difference between a sum and an integral: 2230: [G.9](#).  
 division into others of smaller intervals: [A.4](#).
- \* eight rules for definite integration: 2245.  
 equations for obtaining functions as integrals: [J.79](#).  
 expressible only by logarithms: [An.76](#).  
 extended independently of the conception of differentials: [A.61](#).  
 formulæ of: [A.1](#): [J.18,19](#): [M.4](#): [Me.76](#): [Mo.85](#): [N.85](#): failure of  $f$ , [CM.2](#).  
 and gamma-function: [LM.12](#): [Z.9,12](#).  
 higher, of composite functions: [A.20](#).  
 with imaginary limits: [C.23](#): [J.37](#).  
 use of imaginaries in: [M.14](#).  
 inverse method: [CM.4](#): [CP.4,5](#): [L.78](#).  
 involving elliptic functions: [Pr.29](#): [Q.19](#).
- \* limits of: 2233—44: [L.74](#).  
 multiplication of: [Pr.23](#).  
 number of linear independent of 1st kind: [An.82](#).  
 of alg. differentials by means of logarithms: [Mo.57](#): [An.75](#): [C.90](#): [J.12,24,78,79](#): [Mo.84](#): [N.81](#) (see "Integrals").  
 of algebraic surfaces: [C.99](#): [J.26](#): octic, [An.52](#): cubature, [C.80](#).
- \* of circular functions: 1938—97: 2453—2522: No.1799: [LM.4](#): [M.6](#): [Mém.9](#).



## Integrals or Integration—(continued):

- sine and cosine: [G.6](#): [M.11](#).
- \* of exponential and logarithmic functions: 2391—2431: [E.17,18](#).
- \* of circular, logarithmic and exponential functions: 2571—2643 (see "Integrals").
- of a complex function: [A.66](#): th of Cauchy, [Ac.84](#).
- \* of a closed curve: 5204: [C.23](#): [E.28](#): [Z.17](#).
- of differentials containing the square root of a cubic or biquadratic: [Me.57](#).
- \* of discontinuous functions: 2252: [C.23](#): [LM.6](#).
- of dynamics: [L.52,55,58](#).
- of explicit functions, determination of algebraic part of result: [An.61](#).
- \* of functions which become infinite between the limits: 2240: [J.20](#): [JP.11](#): [Q.6](#).
- of infinite relations: [M.14](#).
- of irrational alg. curves by logarithms: [An.61](#).
- \* of irrational functions: 2110—20: [A.12](#): [An.56](#): [C.32,89](#): [L.53,64](#): [Mém.30](#).
- \* limits of: 1903,—6, 2233, 2775.
- \* for quadrature of curves: 5205—11: [C.68,70](#): circle, [J.21,23](#): [JP.27](#).
- triple integrals: [J.59](#).
- \* of rational functions: 2021—[32](#), 2071—2103: [L.27](#): [N.73](#).
- \* of rational fractions: 1915: [CD.3](#): [Mém.33](#): [N.72](#).
- of total differentials: [C.99](#).
- of transcendental functions: [JP.26](#).
- of two-membered complete differentials: [J.54](#).
- periods of: [C.36,38,75](#): [G.75](#): [JP.27](#).
- \* principal values of infinite definite: 2240: [A.68](#).
- properties by elliptic coordinates: [L.51](#).
- quotient of two [di](#) of the form  $\int dx dy \dots dz$ : [J.67](#).
- reduction to elliptic functions: [An.60](#): [LM.12](#): [Me.77,78](#).
- residues of: [JP.27](#).
- Riemann's of first kind: [An.79](#).
- singular values of: [A.11](#).
- \* successive: 2148: [L.62](#): 2nd order, [M.20](#) and [Z.11](#).
- \* summation of: 2250: [J.47](#): [JP.12,21](#).
- tables of definite, by B. de Haan, note on: [C.47](#).
- and Taylor's theorem: [Me.84](#).
- theorems: [L.48](#): [P.55](#): [Q.10,12](#).

## Integrals or Integration—(continued):

- \* transformation of: 2245—52: [A.10](#): [CM.4](#): [J.f15,22,36](#): [L.36](#): [Mél.3](#): [Q.1](#).
- \* variation of arbitrary constant: 2247: [J.33](#).
- whose values are algebraic: [J.10](#): [JP.14](#): [L.38](#).
- ALGEBRAIC FUNCTIONS. *Indefinite*:  
unclassified: [An.75](#): [C.90](#): [J.12,24](#), [78,79](#): [Mo.84](#): [N.81](#).
- \* simple functions of  $x^2 \pm a^2$ : 1926—37:  $x^m$ , [A.4](#):  $\sqrt{(a^2 - x^2)}$ , [A.38](#):  $\frac{1}{x^2 + a^2}$ , [N.82](#):  $\frac{x^m}{\sqrt{(1 - x^2)}}$ , [An.68](#).
- \* fractions involving a binomial surd: 2008—19.  
 $\frac{1}{(1+x)\sqrt[3]{(2x^2-1)}}$ : [Mém.13](#).  
 $\frac{x^m}{(x^n - a)\sqrt{(x^n - b)}}$ : [Q.18](#).  
 $\frac{\sqrt{(1+x^4)}}{1-x^4}$ : 2015: [L.80](#): [Z.8](#).  
 $\frac{(1+x^2)^2}{(1+x^2)(1+bx^2+x^4)^{\frac{1}{2}}}$ : [Mém.10](#).  
 $\frac{x}{(x^3+8)\sqrt{(x^3-1)}}$ : [A.3](#).
- \*  $\frac{x^m}{x^n \pm 1}$  and deductions: 2021—8.
- \*  $x^m(a+bx^n)^p$ : 2035—60: [A.36](#): [Mém.11](#).
- \*  $\frac{1}{x^2+a^2}$ : 2061—5: [J.36](#).
- \*  $\frac{1}{(x-a)^p(x-b)^q}$ : [A.40](#);  $\frac{1}{x^n(x-1)^n}$ : 2007.
- \* functions of  $(a+bx+cx^2)$ : 2071—80: 2103—9.  
 $\frac{x^m}{\sqrt[3]{(a+bx+cx^2)^n}}$ : [A.55](#).
- \* functions of  $(a+bx^2+cx^4)$ : 2081—5.
- \* functions of  $(a+bx^n+cx^{2n})$ : 2086—2102.
- rational algebraic functions of irrational ones:
- \* integrated by rationalizing: 2110—[20](#).
- reducible to elliptic integrals, viz.:  
rational functions of  $\sqrt{X_4}$ ,  $\sqrt{X_3}$ , and  $\sqrt{X_6}$ :  
where  $X_4$  is a quartic in  $x$ : 2121—41.
- \*  $F(x, \sqrt{X_4})$ : 2121: [LM.8](#): [L.57](#).
- \*  $\frac{F(x)}{\sqrt{X_4}}$ : 2133—6: [J.36](#): [LM.14](#).

## Integrals:—ALGEBRAIC—(continued):

$$\frac{x+a}{\sqrt{X_1}}: \text{L.64: M61.3.}$$

$$\frac{1}{\sqrt{X_1}}: \text{C.59: CD.1: E.36: J.10.}$$

$$\frac{f(x)}{F(x)\sqrt{X_1}}: \text{C.51.}$$

$$\frac{1}{\sqrt{X_3}}: 2141: \text{J.17: Man.79.}$$

$$F(\sqrt{X_3}): \text{L.57.}$$

$$\frac{1}{(1-x^3)^{\frac{1}{3}}}: \text{Me.82.}$$

$$\frac{1}{(x^3-1)\sqrt{(x^3-b^3)}}, \text{ \&c., reduced to}$$

Jacobi's functions: [Q.18.](#)

$$\frac{f(x)}{\sqrt{\pm(1-x^3)}}: \text{J.32.}$$

sundry: [CM.2: L.47.](#)

Limits 0 to 1:

Euler's: see "Euler's integrals."

$$\frac{x^{l-1}(1-x)^{m-1}}{J.11,17_2}: 2280: \text{A.40: C.16:}$$

$$\frac{x^{l-1}+x^{m-1}}{(1+x)^{l+m}}: 2341: \text{C.55.}$$

$$\text{similar forms: } 2342-4, 2352, 2356-67.$$

$$\frac{x^{a-1}-nx^{na-1}}{1-x-x^n}: 2367: \text{A.10.}$$

$$\frac{x^m(1-x)^n}{(1+ax)^p}: \text{L.59.}$$

$$\frac{x^p(1-x)^q}{\{1+\sqrt{(1+ax)}\}^{2p+2q+2}}: \text{L.57.}$$

$$\frac{x^{m+1}(1-x)^{m-1}}{(a+bx+cx^2)^{m+1}}: \text{L.56-7.}$$

$$\frac{1}{(a-bx)^m(1-x)^{1-n}x^n}: \text{J.42.}$$

Limits 0 to  $\infty$ :

$$\frac{x^m}{1 \pm x}, \text{ \&c.: } 2309-12, 2345-55: \text{A.38.}$$

$$\text{J.24: L.41: Q.12: Z.19.}$$

$$\frac{1}{(x^2+a^2)^n}: 2364.$$

$$X^n x^{-r}: \text{Me.83.}$$

$$\frac{x^{a-1}}{1+x^a+x^{2a}}, 2n > a > 0: \text{N.48.}$$

$$\frac{x^{a-1}}{1+x^2+\dots+x^{2(a-1)}}, \text{ \&c.: A.16.}$$

$$\frac{x^{a-1}(1+x^2)}{1+x^{4n+2}}, \text{ and 12 similar: A.35.}$$

## Integrals:—ALGEBRAIC—(continued):

$$\frac{x^{2n}F\left\{\left(ax-\frac{b}{x}\right)^2\right\}}{\text{A.9.}} \text{ (Cauchy): } 2712:$$

$$\frac{x^n}{1+2x\cos\phi+x^2}: \text{A.12: E.41.}$$

deductions from this involving integrals of the forms

$$\int_0^1 \frac{1-x^n}{1-x} dx \text{ and } \int_0^{\frac{\pi}{2}} \frac{\sin \frac{m\pi}{n}}{\sin a\pi}: \text{A.35.}$$

Other limits:

$$\int_{-1}^1 \frac{F(x)dx}{\sqrt{(1-x^2)}}, \int \frac{F(x)dx}{(x-c)^n}: 2368-9.$$

$$\int_0^{\sqrt{\frac{a}{b}}} (a-bx)^{\frac{p}{q}} x^{m-1} dx: \text{A.35.}$$

$$\int_{-1}^1 \frac{\sin a}{1-2x\cos a+x^2} dx=f(a): \text{geoN.85.}$$

## CIRCULAR FUNCTIONS. Indefinite:

$$\sin x, \sin^{-1}x, \text{ \&c.: } 1938-49: \sin^n x, \text{ \&c., } 1954-7: \cos^n x, \text{N.74.}$$

$$\frac{1}{a+b\cos x}: 1951: \text{J.9: } \frac{1}{(a+b\cos x)^n}, 1958: \text{Me.80.}$$

$$\text{products and quotients of sine and cosine and their powers: } 1959-80, 2066-70: \text{A.49.}$$

$$\text{binom. funcs. of sin and cos: } 1982-92.$$

$$\text{ditto of tangent: } 1983, 1991.$$

$$\frac{F(\sin x, \cos x)}{a\cos x+b\sin x+c}: 1994-7: \text{A.12: J.19,32.}$$

$$(a+2b\cos x+c\cos 2x)^{-1}: 1993;$$

$$\frac{x^m}{x^{2n}-2x^n\cos n\theta+1}, 2029.$$

$$\frac{F(\cos x)}{(a_1+b_1\cos x)(a_2+b_2\cos x)\dots\&c.}: 1996.$$

a function of sine or cosine in a rapidly converging series, and successive integration of it: [J.4,15.](#)

$$\frac{\sin^n\phi}{(1-k\sin\phi)\sqrt{(1-k^2\cos^2\phi)}}: \text{A.11.}$$

$$(\sin amx)^{2n}: \text{J.81.}$$

$$\sqrt{\{1-k^2\sin^2\frac{1}{2}(a+x)\sin^2\frac{1}{2}(a-x)\}}: \text{J.39.}$$

$$\frac{\sin^n x}{x^m}: \text{A.17: with } m=1, \text{G.7.}$$



Integrals:—CIRCULAR. *Indef.*—(cont.):

- \*  $\frac{\sin kx}{x \text{ or } \sin x} f(x)$  (Fourier): 2726—42: [A.38.](#)

Limits 0 to  $\frac{\pi}{2}$ :

- \*  $\sin^n x$ : 2453—[5](#), 2458, 2472: [E.29](#).  
 $n = \frac{1}{2}$ , [E.28](#).
- \*  $\tan^{2m-1} x$ : 2457.
- \*  $\sin^n x \cos^p x$ : 2459—65.
- \*  $\cos^p x \frac{\sin nx}{\cos nx}$ : 2481, 2484—[92](#): L.43.
- \*  $\frac{\cos^{n-1} x \sin nx}{\sin x} = \frac{\pi}{2}$  and similar: [A.59](#).
- \*  $x \cos^{n-1} x \sin nx$ : 2494.
- \*  $\frac{1}{a^2 \cos^2 x + b^2 \sin^2 x}$  and similar: 2496—2501, 2344.
- \*  $\cos^{p-n-1} x \sin^{n-1} x \frac{\sin px}{\cos px}$ ; and  
 $\sin^{p-2} x \sin px$ : 2585—8.
- \*  $x \cot ax$ : [A.34](#): [No.19](#);
- \*  $x \tan \frac{1}{2}\pi (1-bx)$ : 5340.

Limits 0 to  $\pi$ :

- \*  $\sin^n x \cos^p x$ : 2459.
- \*  $\sin nx \frac{\sin px}{\cos px}$ : 2467—9.
- \*  $\left(\frac{\sin}{\cos}\right)^p x \frac{\sin nx}{\cos nx}$ : 2474—82, 2493.
- \*  $\sin^{2n} x d_{nx} f(\cos x)$ : 2495.
- \*  $\cos n(x-a \sin x)$ : [L.41](#): 0 to  $2\pi$ , [C.39](#).
- \*  $\cos(a+nx) \cos(b+px) \cos(c+qx)$ : C.  
[54](#).
- \*  $\frac{x \sin x}{1+a^2 \cos^2 x}$ : [Pr.25](#): with  $a = 1$ , 2506  
and [Q.11](#).
- \*  $\frac{x \sin x \text{ or } \sin x \sin rx \text{ or } \cos rx}{1-2a \cos x + a^2}$ : 2623—9.
- \*  $\frac{\sin^{2n} x \text{ or } \cos px}{(1-2a \cos x + a^2)^n}$ : [Q.11](#): [LM.11](#):  $n=1$ ,  
[L.74](#).
- \*  $x F(\sin x \cos x)$ : [JP.27](#).
- about [250](#) integrals with limits chiefly  
from 0 to  $\pi$ , some from 0 to  $\infty$ :  
[Pr.25, 26, 27, 29, 30, 31, 32, 33](#).

Limits 0 to  $\infty$ :

- \*  $\frac{\cos x}{\sqrt{x}}$ ,  $\cos x^2$ ,  $\cos(ax)^2$ : 2507—9: Q.  
[12](#): 2602.
- \*  $\left(\frac{\sin}{\cos}\right)^2 x$ : [Q.13](#);  $\frac{\sin x^n}{\cos x^n}$ : [Me.72](#).

Integrals:—CIRCULAR. 0 to  $\infty$ —(cont.):

- \*  $x^{\pm m} \frac{\sin bx}{\cos bx}$ : 2579—81.
- \*  $\frac{\cos bx}{x+a}$ , also  $\int_x^{\infty} \frac{\cos x}{x} dx$ : [A.10](#).
- \*  $\frac{\sin^n x}{x^p}$ : 2510: [A.30](#): [E.26](#): [Z.5](#):  $n = 2$   
and [3](#), 2511—2.
- \*  $\frac{\cos qx - \cos px}{x \text{ or } x^2}$ : [2513—5](#).
- \*  $\frac{\sin ax \sin bx}{x}$  and similar: 2514—22.
- \*  $\frac{\cos 2ax \sin^p x}{x^m}$ : 2722—[5](#).
- \*  $\phi(\sin ax, \cos bx)$ , reduction of: [J.15](#).
- \*  $\phi(\sin ax, \cos bx)$ : J.23, [25](#).
- \*  $\frac{\sin rx}{x}$ : 2572: [L.49](#): [Z.7,8](#).
- \*  $\frac{\cos rx}{a^2+x^2}$ : 2573: [A.10, 11, 59](#): [J.33](#): [Me.](#)  
[72, 76](#): [Z.7,8](#).
- \*  $\frac{x \sin rx}{a^2+x^2}$ : 2575: [A.10](#): [J.33](#).
- \*  $\frac{\cos rx}{(a^2+x^2)^n}$ : [L.40](#): [Q.18](#).
- \*  $\phi(n) \frac{\sin rx}{\cos rx}$ : A.11: with  $\phi(x) = x^m$ ,  
[L.46](#):  $\phi(x) = \tan^{-1} cx$ , A.11.
- \*  $\frac{\sin(ax)^2}{\cos(a^2+x^2)}$  and similar: [Z.13](#).
- \*  $\frac{\sin ax}{(1+x^2) \sin bx}$  and related integrals:  
[Ac.7](#).
- \*  $\frac{1 - \cos^m x}{x^2}$ : Z.7,8.
- \*  $\frac{1 \text{ or } x \sin cx}{(1+x^2)(1-2a \cos cx + a^2)}$ : 2630—2.
- \*  $\frac{\tan^{-1} ax}{x(1+x^2)}$ : 2503;  $\frac{1}{x^2} \tan^{-1} \frac{x}{a} \tan^{-1} \frac{x}{b}$ ,  
2504.
- \*  $\frac{\tan^{-1} ax - \tan^{-1} bx}{x}$ : 2505.

Other limits:

- \*  $\int_0^{2\pi} \frac{dx}{a+b \cos x + c \sin x}$ , lim. 0 to  $2\pi$ :  
[A.55](#).
- \*  $\int_0^1 \frac{\tan^{-1} x}{1+x} dx$ , deduced from (2416):  
[L.69](#).

Integrals :—CIRCULAR—(continued) :

- \*  $\int_0^1 \frac{\tan^{-1} ax \, dx}{x\sqrt{1-x^2}}$  : 2502.
- \*  $\int_{-\infty}^{\infty} \frac{F(x)}{x^n} dx$ , where  $F(x)$  is a rational integral circular function : [CM.3](#) sine-integral, &c. : see "Functions."

EXPONENTIAL FUNCTIONS. *Indefinite* :

- \*  $e^x, a^x$  : 1924;  $x^m + a^x$ , 2004.
- \*  $e^x \{\phi(x) + \phi'(x)\}$  : 1998.
- \*  $e \exp(\frac{1}{2}x^2)$  : [E.34](#).
- X being a rational integ. funct. of  $x$  :  $e^x X$  : [J.13](#);  $e^x$ , [Mém.83](#);  $e \exp(-x^2)$ , [Q.1](#).
- \*  $e \exp(-x^2)$ , or the same  $\times x^2$  or  $\sqrt{(a+bx^2)}$  or  $(c+dx^2)$  : [L.52](#).

Limits 0 to  $\infty$  :

- \*  $e^{-kx} x^n$  : 2284—91: see "Gamma function."
- \*  $e \exp(-kx^2)$  : 2425: evaluation by a continued fraction, [J.12](#).
- \* other forms : 2426—31, 2595, ['8](#), 2601.
- \*  $e \exp\left(-x^2 - \frac{a^2}{x^2}\right)$  : 2604—5: [L.56](#).
- \*  $\frac{e^{-nx}}{1+x^2}$  : [Z.6](#); ditto  $\times x^m$ , [L.46](#).
- \*  $\frac{e^{-nx}}{x^2-a^2}$  and the same  $\times x$  : [A.10](#).
- \*  $e^{-ax} e^{i(x^2+bx)}$  : [M61.2](#).
- \*  $e \exp(ax^n) \cdot Ei(\pm ax^n)$  : [Q.18](#).

Other limits :

expon-integral : see "Functions."

$$\int_1^{\infty} e \exp(-x^2) [\equiv \operatorname{Erfc} \xi \text{ (Glaisher)}] : \text{Me. } 76.$$

$$\int_0^1 e^{-kx} F(x) \, dx : \text{C.77}_4.$$

$$\int_0^1 x^{2n} e \exp\left(-x^2 - \frac{a^2}{x^2}\right) : \text{C.12.}$$

$$\int_{0a}^{2\pi} F(re^{i\theta}) e^{-n\theta i} : \text{A.15.}$$

LOGARITHMIC FUNCTIONS. *Indefinite* :

- \*  $\log x$  : 1950;  $x^m (\log x)^{\pm n}$ , 2003—6.
- \*  $x^m \log(1+x)$ , &c. : [A.39](#).
- \*  $F(x) \log x$  : [G.12](#).
- \*  $\frac{x^i (\log x)^m}{x \pm 1}$  and a similar form : 2030—2.

Integrals :—LOGARITHMIC. *Indef.*—(continued) :

$$\frac{x^m \log x}{(1-x^n)^p}, \text{ type of several: } \text{Mém.18.}$$

Limits 0 to 1 :

- \*  $\log \frac{1}{x}$  : 2284;  $x^i \left(\log \frac{1}{x}\right)^n$ , 2291.
- \* involving  $\log x$  or  $\log(1 \pm x)$  : 2391—2403, 2416—22.
- \*  $\frac{\log(1+x)}{1+x^2}$  : 2416: [C.59](#); [J.6](#); [L.43,44](#).
- \*  $\frac{\log x}{\sqrt{(1-x^2)}}$  : [2636](#); [Q.12](#);  $\frac{x^{m-1} \log x}{1+x^m}$ , and many cases, [J.34](#).

$$\frac{1}{x} \log \frac{1+x}{1-x} : 2403 : \text{L.73.}$$

$$\frac{x^m (x-1)^n}{\log x} : \text{Z.3}; \frac{x^m-1}{\log x}, \text{A.37.}$$

about [540](#) expressions chiefly formed from  $\log(1 \pm x)$  or  $\log(1 \pm x_n)$  or  $\log(1+x+x^2)$  or  $\log(1+x^2+x^4)$ , joined to a single factor of the form  $x^m$  or  $(1 \pm x^m)$  or  $\frac{1 \pm x^m}{1 \mp x^m}$  with integral values of  $m$  and  $n$  : [A.39,40](#).

about [280](#) expressions, nearly all comprised in the form

$$\frac{x^m (1-x^n)^{\frac{p}{q}}}{(\log x)^t},$$

with integral values of  $m$ ,  $n$ ,  $p$ ,  $q$ , and  $t$  : [A.40](#).

about [130](#) expressions of the forms,

$$x^m (1-x^n)^p (\log x)^2 \text{ or } 3,$$

$$x^m (1-x^n)^{\frac{p}{q}} \log x, \text{ and}$$

$$\frac{x^m (x^{nr}-1)}{x^r-1} \log x, \text{ with integral}$$

values of  $m$ ,  $n$ ,  $p$ ,  $q$ , and  $r$  : [A.40](#).

- \* Limits 0 to  $\infty$  : 2423—4.

Other limits :

$$\log \frac{(1-x)}{x} : 2408—12 : \text{L.73}; \text{Z1.}$$

$$\frac{1}{x} \log \frac{1+x}{1-x}, \text{ limits 0 to } \sqrt{2}-1 : 2415.$$

log-integral : see "Functions."

CIRCULAR-EXPONENTIAL FUNCTIONS. *Indefinite* :

- \*  $e^{ax} \left(\frac{\sin}{\cos}\right)^n bx$  : 1999;  $e^x \sin^n x \cos^n x$ , 2000.



Integrals :—CIRCULAR-EXPONENTIAL—  
(continued) :

$$(e^{ix} \sin x)^{n-1} : \text{L.74.}$$

Limits, 0 to  $\infty$  :

- \*  $\frac{e^{-ax} \sin rx}{x} : 2571, 2591.$
- \*  $e^{-ax} \sin bx : 2583 : 0 \text{ to } 1, \text{ Mém.30.}$
- \*  $e^{-ax} x^m \frac{\sin}{\cos} bx : 2577, 2589 : \text{L.33} : \text{Z.7.}$
- \*  $e^{-ax} \left( \frac{\sin}{\cos} \right)^m x : 2608-11 : \text{A.7} :$
- \* with limits  $-\frac{\pi}{2}$  to  $\frac{\pi}{2}$ , 2612.
- \*  $\frac{e^{ax} + e^{-ax}}{e^{rx} + e^{-rx}} \sin mx$ , &c. : 2593—2600.
- \*  $e^{-kx} \cos mx \sin^n x : 2717-20.$
- \*  $e \exp \left( -\frac{a^2 x^2}{\cosh} \frac{\cos}{2bx} \right)$ , and similar :  
2614—8 : Z.1,10.
- \*  $e \exp \left\{ -\left( x^2 + \frac{a^2}{x^2} \right) \cos \theta \right\} \times \&c. :$   
2606 : Q.12.
- \*  $\frac{\cos x - e^{-ax}}{x} : 2619.$

CIRCULAR-LOGARITHMIC FUNCTIONS :

- \*  $\frac{x^p (\log x)^m}{x^{2n} - 2x^n \frac{\cos n\theta + 1}{2}} : 2033.$
- Limits 0 to 1 :
- \*  $\frac{\sin}{\cos} (m \log x) / \log x$ , and similar : 2641  
—3.
- $\log(1 - 2x \cos \phi + x^2)$ , &c., &c. : A.34.
- $\log \frac{1 - 2x \cos \phi + x^2}{x(1 - x^2)}$ , & 13 similar : L.73.
- $\log \sin x \log \cos x$ , &c. (limits 0 to  $\frac{\pi}{4}$ ) :  
E.22.
- Limits 0 to  $\frac{\pi}{2}$  :
- \*  $\log \sin x : 2635 : \text{CM.2} : \text{E.23} : \text{Q.12} : 0 \text{ to } n\pi \text{ and similar integrals, A.16.}$
- \*  $\frac{\log(1 + c \cos x)}{\cos x} : 2633 : 0 \text{ to } \pi, 2634.$
- $\tan x \log \operatorname{cosec} x : \text{E.27.}$
- $\log(1 + n^2 \sin^2 x)$  : L.46.  
 $\sqrt{(1 - k^2 \sin^2 x)}$

Integrals :—CIRCULAR LOGARITHMIC—  
(continued) :

$$\log \frac{\sin}{\cos} x \text{ and } \log \sqrt{(1 - k^2 \sin^2 x)}, \text{ each with the above denominator : } \text{L.92.}$$

Limits 0 to  $\pi$  :

- \*  $\log(1 - 2a \cos x + a^2) : 2620-2 : \text{L.38} : \text{Q.11.}$
- \*  $\cos rx \log(1 - 2a \cos x + a^2) : 2625 : \text{A.13.}$
- \*  $x \log \sin x : 2637 ; x \log \sin^2 x, 2638.$

Limits 0 to  $\infty$  :

- $\frac{\cos ax}{b^2 + x^2} \log 2cx : \text{A.11.}$
- \*  $\frac{\log(1 - 2a \cos cx + a^2)}{1 + x^2} : 2631.$

Other limits :

- $\int_a^b \frac{a + b \cos x}{a - b \cos x} dx : \text{A.53.}$
- $\int_0^{2\pi} F(\cos nx) \log \sin \frac{x}{2} dx : \text{Z.10.}$
- $\int_0^{\frac{\pi}{2}} \cos^{2(n-1)} x \log \tan x dx : \text{A.16.}$
- $\int_0^{n\pi} x^m \log(1 - 2a \cos x + x^2) dx$ , and similar : J.40.

EXPONENTIAL-LOGARITHMIC FUNCTIONS :

- $\int_0^x \frac{e^{ax} - e^{-ax}}{e^{rx} - e^{-rx}} \log(x^2 + u^2) dx$ , &c. : L.38.
- $\int_0^{2n\pi} e^{cx} \log(1 - 2a \cos x + a^2) dx : \text{J.40.}$

MISCELLANEOUS THEOREMS :

- \* formulæ of Frullani, Poisson, Abel, Kummer, Cauchy, &c. : 2700—13.
- \*  $\int_a^b f'(x) = f(b) - f(a) : 1901-3 : \text{A.16.}$
- \*  $\int_a^b \frac{\phi(ax) - \phi(bx)}{x^n} dx : \text{Me.81} : n=0, 2700.$
- \*  $\int \phi(x) \{\psi(x)\}^{\pm n} dx : 2001-2.$
- $\int_0^x \frac{f(\theta x)}{x^r} dx$  ( $\theta$  of Taylor's th) : LM.13.
- $\int_{-\infty}^{\infty} F(x) dx$ , approx. to : C.97.
- $\int f(x-R) dx = \int f(y) dy : \text{CD.4.}$
- $\int_{-\infty}^{\infty} \phi\left(x + \frac{1}{x}\right) \frac{f(x)}{x} dx$ , and similar :  
Me.75,76,77.
- $\int f(x) \phi(x) dx : \text{L.49.}$
- $\int_0^1 uv dx \geq \int_0^1 u dx \int_0^1 v dx : \text{AJ.7.}$

Integrals : — MISCELLANEOUS TH. —  
(continued) :

$$\int_0^1 (u)_n du : \text{Z.25.}$$

$$\int_0^1 (u+k)_{k+2} du = (-1)^k \int_1^2 (u)_{k+2} du : \text{A.38.}$$

$$\int_0^1 P_m(x) P_n(x) dx \quad (P_m, x \equiv \text{Legendre's coeff.}) : \text{Pr.23.}$$

$$\int \frac{du}{uU\psi(u)} = \log \psi(u) + C : \text{J.19.}$$

$$\int \frac{(x+p)(x+q)^n d\theta}{x^{n+1}(x+p+q)^{n+1}}, \quad \theta \equiv \text{pos integer} : \text{C.78.}$$

$$\int_0^{\frac{\pi}{2}} \phi(\sin 2x) \cos x dx \\ = \int_0^{\frac{\pi}{2}} \phi(\cos^2 x) \cos x dx : \text{A.21 : L.53.}$$

$$\int f(x) \sin \omega \pi x d(\log x) : \text{Mo.85.}$$

$$\int_0^\infty \frac{r}{r^2+x^2} \Phi \frac{\cos x \Psi \sin^2 x}{\sin x \cos^2 x}, \text{ trans. of : J.36.}$$

integrals deduced from

$$z'' + Pz' + Qz = 0, \text{ and } y'' + (P+2R)y' + \{Q+R(P+R)+R'\}y = 0 : \text{J.18.}$$

$$\int \Pi(x, y) dx : \text{J.61 : Mo.61 : Q.7.}$$

\* if  $f(x+iy) = P+iQ$ , th : 2710.

[Upwards of 8,000 definite integrals have been collected and arranged in a 4to volume by D. Bierens de Haan; Leyden, 1867 (B.M.C. : 8532. ff.)]

\*Integrating factors of d.e : pp.468—471, 3394.

\*Integrator mechanical : 5450.

Intercalation : C.M.3.

\*Intercepts, to find : 4115.

\*Interest : 296. N.48,61,64.

Interpolating functions : C.11.

\*Interpolation : 3762 : A.32,61,62,70 : A.J.2 : C.19,48,68,92 : J.5 : L.37,46 : Me.78 : N.59,76 : Q.7,8.

of algebraic functions, Abel's th : J.28.

Cauchy's method : A.2 : C.37, L.53.

by circular functions : N.85.

by cubic and quintic equations : C.25.

formulæ : J.2 : C.99, Mo.65.

\* Lagrange's : 3768 : J.1,84 : N.57,61.

Newton's : N.57,61,71.

for odd and even functions : C.99.

\* and mechanical quadrature : 3772 : A.20.

Interpolation—(continued) :

by method of least squares : C.37, Mé.m.59.

\* by a parabolic curve : 2992 : C.37.

Stirling's series : Me.68.

and summation : L.11,12,14.

tables : L.11.

of values from observation : Me.12 : tr, Mé.m.59.

Intersection : — of circles and spheres : L.38.

\* of 2 conics : 4916 : CD.5,6 : N.66.

\* of 2 curves : 4116, 4133 : CM.3 : J.15 : L.54 : by rt. lines, Me.80.

\* of 2 planes : 5528.

of 2 quadrics : C.62 : N.68.

of right line and conic : see "Right line."

of successive loci, ths : N.42.

of surfaces : J.15 : L.54 : by rt. lines, Me.80.

\*Invariable line, plane, conic, and quadric : 5856—66.

\*Invariants : 1628 : An.54, C.85, E.42 : G.1,2,15 : J.62,68,69 : L.55,61,76 : M.3,5,17,19 : Me.81 : N.58,59,69,70 : P.82 : Pr.7 : Q.12 : Z.22.

of binary cubics : An.65.

of binary forms : of 8th deg., C.84 :

G.2 : M.5 : simultaneous, M.1.

of higher transformations : J.71.

superior limit to number of irreducible : C.86.

of a binary quadric : M.3.

\* of a binary quantic : 1648 : E.40 : Me.79 : of two, 1650.

of a binary quartic : M.3 : Q.10.

of a binary quintic, table of irreducible : A.J.1.

of a bi-ternary quadric : J.57.

\* of a conic : 4417 : 4936—5030.

\* of two conics : 4936 : N.75.

of three conics : Q.10.

\* of particular conics : 4945.

of a correspondence : G.20.

differential : M.24 : of given order and degree belonging to a binary 10-ic, C.89.

of d.e linear : C.88 : of 4th order, Ac.3.

and covariants of  $f(x^2, y^2)$  relative to linear transformation : G.17.

of linear transformations : M.20.

mutual relation of derived invariants : J.85.

of an orthogonal transformation : J.65 : LM.13.

of a pair of homog. functions : Q.1.

partial : LM.2.

of points, lines, and surfaces : Q.4.



- Invariants—(continued):  
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 involution: AJ.4  
 \* quadrature: 3772: A.58,59: J.6,63  
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 \* Meunier's theorem: 5809: gzC.74  
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 Minimum:—theory of: L.56: prM.20  
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 area: J.67  
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 of a hexagonal “alvéole,” pr: N.43  
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 circum-tetrahedron of an ellipsoid: Z.25  
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 curves on surfaces: J.5: see “Geodesics.”  
 distance of 2 right lines: G.4; of a point, ths: A.8  
 ellipse through 3 points and ellipsoid through 4: L.42  
 ellipsoid, th: Mo.72  
 N. G. F. of a binary septic: AJ.2



- Minimum :—theory of—(*continued*) :  
 numerical value of a linear function  
 with integral coefficients of an  
 irrational quantity : [C.53,54](#).  
 perimeter enclosing a given area on a  
 curved surface : [J.86](#).  
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[M.61,2](#) : [M.6m.59](#).  
 \* sum of distances from two points :  
[920—1](#).  
 sum of squares of distances of a point  
 from three right lines : [Z.12](#).  
 sum of squares of functions : [N.79](#).  
 Minimum surfaces :—[eq A.38](#) : [G.14](#),  
[22](#) : [J.81,85,87,ext78](#) : [Mo.67,72](#) :  
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 algebraic : [M.3](#) : lowest class-number,  
[An.79](#).  
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 sion of algebraic curves : [C.87](#).  
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 catenoid, [M.2](#) : determined by  
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 : [N.73](#).  
 of  $\int \sqrt{(dx^2+dy^2+\dots)}$  when the varia-  
 bles are connected by a quadric  
 equation : [J.43](#).  
 Models : [LM.39](#) : of ruled surfaces,  
[Me.74](#).  
 Modular :—equations : [An.79](#) : of 8th  
 degree, [59](#) : [C.48,49,66](#) : [M.1,2](#) :  
[Mo.65](#) : see also under “ Elliptic  
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 functions and integrals : [An.51](#) : [J.](#)  
[18,19,20,21,23,25](#) : [M.20](#).  
 indices of polynomials which furnish  
 the powers and products of a  
 binomial eq : [C.25](#).  
 relations : [At.65](#).  
 Modulus :—of functions, principal :  
[C.20](#).  
 of series : [C.17](#).  
 \* of transformations : 1604 : [A.17](#).  
 \* Momental ellipsoid : 5953.  
 \* Momental ellipsoid : 5925, 5934 ; for a  
 plane, 5936.  
 \* Moment of inertia : 5903 : [An.63](#) : [At.](#)  
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 \* of ellipsoid : 6150 : [CD.8](#) : [J.16](#).  
 by geometry of 4 dimensions : [Q.16](#).  
 \* principal axes : 5926, 5967, 5972 : [At.](#)  
[43](#).  
 \* of a quadrilateral : 5951 : [Q.11](#).  
 of solid rings of revolution : [Q.16](#).  
 \* of a tetrahedron : 5957.  
 \* of a triangle : 5944 : [Me.4](#) : [Q.6](#) :  
 polar, [N.83](#).  
 \* of various laminæ and solids : 6015—  
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 Remblais ” : [LM.14](#).  
 Monocyclic systems and related ones :  
[J.98](#).  
 Monodrome functions : [C.43](#) : [G.18](#).  
 Monogenous functions (Laurent's th) :  
[Ac.4](#) : [C.32,43](#).  
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 Movements : [JP.15](#).  
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 groups of : [An.69](#).  
 of a plane figure : [thAn.68](#) : [JP.20](#),  
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 of an invariable system : [C.43](#).  
 of a point on an ellipsoid : [A.11](#) : [J.54](#).  
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 of a right line : [C.89](#) : [N.63](#).  
 of a solid : [JP.21](#).  
 transmission of and the curves result-  
 ing : [JP.3](#).  
 of “ ähnlich - veränderlicher ” and  
 “ affin-veränderlicher ” systems :  
[Z.24](#) and [19](#).  
 \* Multinomial theorem : 137 : [Me.62](#).  
 Multiple-centres, geo. theory : [L.45](#).  
 Multiple curves of alg. surfaces : [An.73](#).  
 Multiple Gauss sums : [J.74](#).  
 \* Multiple integrals : 1905, 2825 : [A.64](#) :  
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[LM.8](#) : [Me.76](#) : [Z.1,3](#).  
 \* double : 2710, 2734—42 : [A.13](#) : [Ac.5](#) :  
[An.70](#) : [J.27](#) : [G.10](#) : [L.58](#) : [Mém.](#)  
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 approximation to : [J.6](#).  
 Cauchy's theory, ext. of : [C.75](#).  
 \* change of order of integration :  
 2775 : [A.19](#).  
 expressing an arbitrary function :  
[J.43](#).  
 reduction of : [J.45](#) : [Z.9](#).  
 residues of : [C.75](#).



Multiple integrals, double—(continued):

$$\int_a^b \int_c^d \frac{(x^2 - y^2) dx dy}{\sqrt{\{(x^2 - b^2)(c^2 - x^2)(b^2 - y^2)(c^2 - y^2)\}}} \\ = \frac{\pi}{2} : \text{L.38.}$$

same with log of numerator: L.50.

$$\iint \frac{(x' - x)(dy dz' - dz dy') + \text{sym}}{\sqrt{\{(x' - x)^2 + \dots\}^3}} \\ = 4m\pi : \text{C.66.}$$

$$\text{transf. of } \iint \frac{d\phi d\psi}{\sqrt{(\sin^2 \nu - \sin^2 \phi \cos^2 \psi)}} : \\ \text{J.20.}$$

$$\frac{1}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{\cos ix \cos jx dx dy}{\sqrt{\{1 + a^2 + 2a(\mu \cos a + \nu \cos y)\}}} : \\ \text{C.96.}$$

$$\iint F(ax + by + cy) dx dy : \text{A.37.}$$

$$\iint F(x + iy) dx dy : \text{J.42.}$$

$$\iint \frac{F(u, t, z)}{G(u, t, z)} dt du : \text{C.96.}$$

$$\int_0^\infty \int_0^\infty \phi(ax^m \pm by^n) x^{p-1} y^{q-1} dx dy : \text{J.} \\ 37.$$

evaluation: A.5: by Fourier's th, CM.4

expansion of: Q.8.

Frullanian: LM.15.

limits of: LM.16.

reduction of: An.57: L.41,39.

by transf. of coordinates: C.13.

$$\int^* F(x^2 + y^2 + \dots) \phi(ax + by + \dots) dx dy \\ \dots : \text{L.57.}$$

of theory of attraction: CD.7.

- \* transformation of: 2774: A.10: An. 53: No.47: CM.4: M.61.2: Mém. 38: Q.4,12.

an indef. double: J.8,10.

- \* a triple integral: 2774—9: J.45.

$y_1 dx_1 + \dots + y_n dx_n$ : LM.11.

- \* triple: 2774: A.30: J.45.

which are unaltered in form by transformation of the variables: J.15, 91.

$$\iiint \dots dx dy dz \dots : \text{Q.23.}$$

- \*  $\iiint \dots x^{l-1} y^{m-1} z^{n-1} \dots dx dy dz \dots$  with different limiting equations: 2825: CM.2: L.51.

- \* some other integrals evaluated by  $\Gamma$  functions: 2826—34.

$$\iiint F(ax + by + cz, a'x + b'y + c'z, \\ a''x + b''y + c''z) dx dy dz, \text{ limits} \\ \pm \infty : \text{A.30.}$$

Multiple integrals—(continued):  
volume integral of

$$\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r : \text{Z.14.}$$

$$\iiint e \exp(-x^2 - y^2 - z^2), x^p y^q z \, dx dy dz : \\ \text{N.54.}$$

$$\iiint \dots \phi(ax^2 + by^2 + \&c.) x^p y^q \dots dx dy \dots \\ \text{with limits 0 to } \infty \text{ in each case} \\ (\text{Pfaff}) : \text{J.28.}$$

$$\iint \dots \frac{dx dy \dots}{\{(a-x)^2 + (b-y)^2 + \dots\}^n} \text{ by dis-} \\ \text{continuous functions: TL21.}$$

do. with  $n = \frac{m}{2}$  and with a numerator

$$(a-x) F\left(\frac{x^2}{h^2} + \frac{y^2}{k^2} + \dots\right) : \text{CM.3.}$$

$$\iint \dots F(x, y, z \dots) PQ \, dx dy dz \dots \text{ where} \\ P = (1-x)^{a-1} (1-y)^{b-1} \dots \\ Q y^2 z^{c-1} t^{a+b+c} \dots : \text{L.59.}$$

arising from (2604), viz:

$$\int_0^\infty e \exp\left(-x^2 - \frac{a^2}{x^2}\right) dx : \text{Pr.42.}$$

- \* Multiple points: 5178: CM.2: thG.15: Me.2: Q.2,6.

on algebraic curves: An.52: L.42: N.51,59,81, at  $\infty$  64.

on two curves having branches in contact: C.77.

on a surface: J.28.

- \* Multiplication: 28: J.49; abridged, N. 79.

by fractions: Me.68.

Multiplicator equations: M.15.

Multiplicity or manifoldness: J.84,86: thAc.5: Z.20.

Music: E.27,28: Pr.37.

Nasik squares and cubes: Q.8,15.

Navigation, geo. prs. of use in: A.38.

Negative in geometry: No.1792.

Negative quantities: At.55: N.44,67: TE.1788.

Nephroid: LM.10.

Net surfaces: J.1,2: M.1: any order, An.64.

quadric: J.70,82: M.11.

quartic: M.7.

and series: C.62.

trigonometrical: Z.14.

having a 3-point contact with the intersection of two algebraic surfaces: G.9.

Newton, autograph m.s.s of: TE.12.

- \* Nine-point circle: 954, 4754: A.41: E. 7,30,th35,pr39: G.1,th34: Me.64, 68: Q.5—8.

- Nine-point circle—(continued):  
 an analogous circle: [A.51](#).  
 \* contact with in- and ex-circles: [959](#):  
[Me.82](#): [Q.13](#).  
 and 12-point sphere, analogy: [N.63](#).  
 Nine-point conic of a tetrahedron:  
[Me.71](#).  
 Nonions (analogous to Quaternions):  
[C.97,98](#).  
 Non-uniform functions: [C.88](#).  
 Nodal cones of quadrinodal cubics: [Q.10](#).  
 Node cusps: [Q.6](#).  
 Nodes, two-plane and one-plane: [M.22](#).  
 \*Normals: 1160: 4122—3, 5122: [A.13](#),  
[53](#): [LM.9](#): [p.eMe.66](#): [Z.cn2](#) and 3.  
 of envelopes: [Me.80](#).  
 \* plane of a surface: 5772.  
 \* principal: 5722: condition for being  
 normals of a second curve, [C.85](#).  
 of rational space curves: [J.74](#).  
 section of ellipsoid (geodesy): [A.40](#).  
 \* of a surface: 5771, 5785: [C.52](#): [CD.3](#):  
[CM.2](#): [L.39,47,72](#): [M.7](#).  
 coincident: [L.48](#).  
 transformation of a pencil of: [C.88](#).  
 \*Notation (see also "Functions"):  
[A,B,C,F,G,H](#): 1642.  
 \* [A.P.](#), [G.P.](#), [H.P.](#): [79,83,87](#).  
 \*  $(a_1b_2c_3)$ : [554](#).  
 $(n) \left(\frac{b}{a}\right)$ , Jacobi's function (see  
 "Functions").  
 \*  $a^b$  or  $a^{n/b}$ : 2451.  
 \*  $a + \frac{1}{b} \frac{1}{c} +$ : [160](#).  
 \*  $B_{2n}$ , Bernoulli's nos.: 1539.  
 \*  $C(n,r)$  or  $C_{n,r}$ : [96](#). Otherwise  
 $C(n,3) \equiv$  number of triads of  $n$  things,  
 &c.  
 $\left(\frac{n}{r}\right) \equiv$  rth coeff. of  $n$ th power of  $(1+x)$ :  
 also Jacobi's function (see "Functions").  
 \*  $D$ : 3489:  $d_x, d_{ux}$ , &c.: 1405.  
 \*  $d^2y, dx^3, \frac{dy}{dx}$ , &c.: 1407.  
 \*  $\frac{d(uvw)}{d(xyz)}$ : 1600.  
 \*  $\Delta$ : [582](#), 1641, 3701;  $\Delta'$ , 1645.  
 \*  $E$ : [902](#), 3735.  
 \*  $e$ : [150](#), 1151.  
 $e \exp x + \frac{1}{x}$  or  $e \left[ x + \frac{1}{x} \right] \equiv e^{x + \frac{1}{x}}$   
 \*  $f(x)$ : [400](#), 1400;  $f^{-1}(x)$ , [430](#).  
 \*  $f(x) f^n(x)$ : [424](#), 1405.

## Notation—(continued):

- \*  $\phi(a\beta\gamma) \equiv u$ : 4656;  $\Phi(\lambda\mu\nu)$  or  $U$ , 4665.  
 \*  $Gu_x$ : 3732.  
 \*  $H(n,r)$ : [98](#).  
 \*  $J$ : 1600.  
 $N.G.F \equiv$  numerical generating function.  
 $N \equiv b \pmod{r}$  signifies that  $N-b$  is  
 divisible by  $r$ .  
 \*  $|n \equiv n^{(n)} \equiv n!$ : [94](#), [3713](#).  
 \*  $\pi$  as operator: 3500.  
 \*  $P(n,r)$  or  $P_{n,r} \equiv n^{(r)}$ : [95](#). Also,  
 $P(n,r) \equiv$  number of triplets of  $n$   
 things, &c.  
 \*  $\psi(x)$  or  $Z'(x) \equiv d_x \log \Gamma(x)$ : 2743.  
 \*  $R, r, r_a$ : 909—13.  
 \*  $S_m, S_{m,p}$ : [534](#);  $S_n$ , 2940.  
 \*  $\sin^{-1}$ , &c.: [626](#);  $\sinh$ , &c., 2180.  
 \*  $\Sigma$ : 3781—3.  
 \*  $u_n$ : 3499.  
 $\zeta_1(n) \equiv$  sum of divisors of  $n$ .  
 $\left\lfloor \frac{n}{d} \right\rfloor$  or  $\left\{ \frac{n}{d} \right\}$  or  $E\left(\frac{n}{d}\right) \equiv$  integer next  
 $< \frac{n}{d}$ .  
 $\left\lceil \frac{n}{d} \right\rceil \equiv$  integer next  $> \frac{n}{d}$ .  
 $\left(\frac{n}{r}\right) \equiv$  rth coeff. of  $(1+x)^n$ .  
 $\nless$   $\equiv$  not less than;  $\ngtr$   $\equiv$  not greater  
 than.  
 $(+)$   $\equiv$  denominator to be stated after-  
 wards.  
 $(\text{X})$  and  $(\text{Y})$ : 1620.  
 algebraic: [CP.3](#).  
 for some developments: [C.98](#).  
 continuant  $\equiv$  contd. fraction determi-  
 nant.  
 median  $\equiv$  bisector of side of a triangle  
 drawn from the opposite vertex.  
 subfactorial: [Me.78](#).  
 suggestions: [Me.73](#).  
 \*Numbers (see also "Partition of," and  
 "Indeterminate equations"): [349](#):  
[A.2,16,26,58,59](#): [Ac.2](#): [AJ.4,6](#):  
[C.f12,43,44,45,60](#): [CM.1](#): [G.16](#),  
[32](#): [J.9,39,40,48,77](#): [tr.27,28](#) and  
[29](#): [JP.9](#): [L.37—39,41,45,58,59](#),  
[60](#): [LM.4](#): [Mém.22,24](#): [tr\(Euler\)](#),  
[30](#): [N.44,62,79](#): [Q.4](#): [TE.23](#).  
 ap. of algebra, [JP.11](#); of  $\Gamma$  function,  
[No.81](#); of infinitesimal analysis,  
[J.19,21](#).  
 formulæ: [L.64,65](#).  
 relation of the theory to i.e.: [C.82](#).  
 5 z



## Numbers—(continued):

approximation:—to  $\sqrt{N}$ , [E.17](#); to functions of large numbers, [C.82](#).  
 binomial eqs. with a prime mod: [C.62](#).  
 cube: [Q.4](#).  
 cubic binomials:  $x^3 \pm y^3$ : [C.61](#).  
 determined by continued fractions: [LM.29](#).  
 digits, calculus of, th: [J.30](#).  
 digits terminating a power: [A.58](#): [N.46](#).

Dirichlet's th.  $\sum \left( \frac{m}{D^2} \right) = \zeta(m)$ : [L.57](#).

Dirichlet's f. for class numbers as positive determinants: [L.57](#).

division of: [A.26](#): [J.13](#): [Mém.3](#): [Pr. 7.10](#); by 7 and 13, [A.25.26](#); by  $mx^2 + ny^2$ , [Mém.15](#): [P.17.88](#): [Q. 19.20](#).

divisors of  $y^2 + Az^2$  when  $A = 4n + 3$  a prime: [J.9](#).

divisors arising from the division of the circle: [L.60](#).

$4m + 1$  and  $4m + 3$  divisors of a number: [LM.15](#).

factors of: [Mém.41](#).

Gauss's form: [L.56](#).

integral quotients and remainders: [An.52](#).

large, analysis of: [A.2](#): [C.2.29](#).

method with continuous variables: [J.41](#).

multiples of: [C.2](#).

non-pentagonal th: [J.31](#).

number of integers prime to  $n$  in  $n! \equiv \phi(n)$ : [L.57](#).

odd: [A.1](#); and prime to all squares, [C.67](#).

Pellian equation: [prA.49](#): [LM.15](#): sol. by ell. functions, [Mo.63](#).

perfect: [C.81](#): [N.79](#).

polygonal, Fermat's th. of: [P.61](#).

polynomials having determinate prime divisors: [C.98](#).

powers of, 12 theorems: [N.46](#).

prime to and  $< N$ : [A.3.29](#): [E.28](#): [J.31](#): [N.45](#).

prime to and  $<$  the product of the first  $n$  primes: [A.66](#).

prime with respect to a given ar.p: [C.54](#).

prime to the radix having multiples made up of repeating digits: [Me.76](#).

products of divisors of: [Q.20](#).

quadratic forms of: [Mém.53](#).

rational linear functions taken with respect to a prime modulus, and connected substitutions: [C.48](#).

## Numbers—(continued):

representation of by forms: [C.92](#); by infinite products, [A.1](#).

square having prime factors of the form  $4n + 1$ : [N.78](#).

squares of: [J.84](#): [M.13](#): [Pr.6.7](#).

three in ar.p: [N.62](#).

sums depending upon  $E(x)$ : [L.60](#).

sums of digits: [Me.66](#): [TE.16](#).

sum and difference of two squares: [thN.63](#).

\* sums of divisors: [377](#): [Ac.6](#): [G.7](#): [L.63](#): [Mém.2](#): [Mém.50](#).

\* sums of powers of (see also "Series"): [276](#), 2939: [An.61.65](#): [thCD.5](#): [Me.75](#): [N.42.56.70](#): [Q.8](#).

of cubes: [An.65](#): [L.66](#); of the odd nos., [A.64](#).

of  $n$  primes: [N.79](#); 4th powers, [A.54](#).

of squares: [A.67](#).

of uneven orders: [Mo.57](#).

symmetrical functions of: [Q.7](#).

systems: [Z.14](#); history of, by Humboldt, [J.4](#).

theorems: [A.7.10.20.49](#): [An.70](#): [C. 25.43.83](#): [CM.2](#): [G.8](#): [L.48.52](#):

[N.75](#); Cauchy's, [gzC.53](#); Eisenstein's, [J.27.50.83](#): [LM.7](#): [Q.5.6](#).

Gauss's on  $X = \frac{x^p - y^p}{x - y}$ : [C.98](#).

Lagrange's arithmetical: [A.47](#).

$p^n + q^n$  in terms of  $pq$ : [N.75](#).

on  $2^n \pm 1$ : [C.85.86](#): [Me.78](#).

2, biquadratic character of: [C.57](#), [66](#): [L.59](#).

on  $(n+1)^m - n^m$ : [N.44](#):  $p_2(n)$ , [L.69](#).

\* on  $n^r - n(n-1)^r +$ , &c.: [285](#): [A.30](#).

on  $2m$  positive numbers: [N.43](#).

on  $P(m) + E\left(\frac{m}{m-1}\right)P(1) +$ : [Q.3](#).

on the greatest product in whole numbers of given sums: [J.57](#).

on an odd sum of 12 squares: [L.60](#).

on products of sums of squares: [G.2](#).

on 4 squares: [N.57](#).

on 2, 4, 8, and 16 squares: [Q.17](#).

on  $\phi(a) + \phi(a') +$ , &c. =  $n$ , where  $a$ ,  $a'$ , &c., are the divisors of  $n$ : [Cm.3](#).

Numeration, ancient decimal: [C.6.8](#).

Numerical approximations: [N.42.53](#).

Numerical functions: [L.57](#): [Me.62](#).

simply periodic: [AJ.1](#).

sums of, approximately: [C.96](#).

which express for a negative determinant the number of classes of a quadratic form, one at least of whose extreme coefficients is odd: [C.62](#).



- Obelisks : [A.9.11](#).
- Oblique :—bevelled wheels, en : [J.2](#).
- \* coordinates : 4050, 5511—9 : [fN.54](#).
- cyclic surface : [TL.9](#).
- and osculating circle of a conic : [G.22](#).
- Octahedron function : [Q.16](#).
- Octahedron, centroid of : [LM.9](#).
- Octic equations : [G.7.10](#); and curves, [M.15<sub>2</sub>](#).
- Octic surface : [G.13](#) : [M.4](#) : [Q.14](#).
- \* Operative or symbolic calculus : 1483, 3470—3628 : [A.14](#) : [C.17](#) : [G.20<sub>2</sub>](#) : [J.5.59](#) : [LM.12<sub>3</sub>](#) : [Me.82.85](#) : [P.37](#) : [44.60—63](#) : [Pr.10<sub>2</sub>](#) : [11<sub>2</sub>](#) : [12<sub>2</sub>](#) : [13<sub>4</sub>](#) : [Q.4.5.8](#).
- applications : [G.19](#) : [Me.82](#).
- algebraic : [TE.14](#); ap. to geometry, [CM.1.2](#);  $\pm$ , [CM.3](#).
- expansions : [Pr.14<sub>2</sub>](#).
- formulae : [C.39<sub>3</sub>](#).
- \* index symbol : 1485 : [CD.6](#).
- integration : [CD.3](#) : [exMe.76](#).
- representation of functions : [C.43](#) : [CD.2](#).
- seminvariant operators : [Q.20](#).
- on the symbols  $x^y$ ,  $\log x$ ,  $\sin x$ ,  $\cos x$ ,  $\sin^{-1}x$ ,  $\cos^{-1}x$  : [A.9.11](#).
- theorems : [A.57](#) : [CD.8<sub>2</sub>](#) : [LM.11](#) : [Q.3.15.16<sub>3</sub>](#); from Lagrange's series, [Q.16](#); from  $\pi pu - \rho wu = pu$ , [CD.5](#).
- \* Operators :  $d_x$ , 1405;  $e^{hd_x}$ , 1520—1, [Q.9<sub>2</sub>](#).
- \*  $(d_x - m)^{-1}$ , &c. : 3470—85; gz of 3474, [C.43](#).
- \*  $D(D-1) \dots (D-n+1)$  : 3489.
- \*  $\pi = xd_x + yd_y + \&c.$  : 3500.
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- \*  $F(xd_x)U$ , where  $U = f(x) = a + bx + cx^2 + \dots$  : 3486.
- \*  $f(D)uv$ , 3494 :  $uf(D)v$ , 3495, with  $f$  as above.
- $D^n f(xD)U$  : [E.36](#).
- \*  $\{\phi(D)e^{ax}\}^n Q$  : 3491.
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- $e^{D \cdot \frac{1}{x}} F(x) = \frac{x+1}{x} F(x+1)$  : [E.36](#).
- $\Omega F(a, x)$  : [Q.13](#).
- $2^{m+1} \sqrt{a \sqrt{v} \pm b \sqrt{wi}}$ , &c. : [C.96](#).
- $f(x+hD) \cdot 1$  : [E.39](#).
- \*  $\psi(d_x + y)\phi x = \phi(d_y + x)\psi y$  : 3498.
- \* reduction of  $F(\pi_1)$  : 3503.
- \*  $F(\pi)U$  and  $F^{-1}(\pi)U$  : 3509—10.
- \*  $C(u, m)u_n/m!$  : 3514.
- transformation of  $Vd_x, Ud_x, \&c$  : [G.21](#).
- Orthocycle : [Q.17](#).
- Orthogonals, algebraic system of : [C.69](#).
- \* Orthogonal :—circles : 4170, 4182—4; of in- and circum-circles of a triangle, [Q.18](#).
- circle and conic : [E.7](#).
- coefficient system : [A.2.61](#).
- coordinates : [C.60](#); curvilinear, [JP.26](#).
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- \* projection : 1087.
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- substitution : [J.67](#) : [M.13](#) : [Z.24](#).
- surfaces : [C.29](#), spheres 36, 54, 59, 72, 79, 87, [ths17](#) and [21](#) : [J.84](#) : [JP.17](#) : [L.43.44.46.47.63](#) : [N.51](#) : [P.73<sub>2</sub>](#) : [Pr.21](#) : [Q.19](#).
- cubic eq. for : [C.76<sub>2</sub>](#).
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- systems : [C.67.75<sub>1</sub>](#) : [M.7](#); condition, [J.83](#); quadric, [J.76](#); parallel, [M.24](#).
- triple : [A.55—58](#) : [An.63.77.85](#) : [C.alg58.67](#); cyclic, [G.21.22](#) : [J.84](#); quartic, [82](#) : [Z.23](#).
- trajectories : [L.45](#) : [Me.80](#) : [Z.17](#).
- of circles : [Me.85](#).
- of circular sections of an ellipsoid : [L.47](#).
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- of a moveable sphere : [C.42](#).
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- Orthomorphic projection of an ellipsoid on a sphere : [A.13](#).
- Orthomorphosis of a circle into a parabola : [Q.20](#).
- Orthoptic :—lines of a conic : [A.57](#).
- loci of : [LM.13](#) : [Pr.37](#); of 3 tangs. to a quadric, [E.40](#).
- surface of a quadric : [J.50](#).
- Orthotomic circles : [Me.64.66](#) : [Q.2](#).
- \* Osculating :—circle : 5724 : [L.39](#).
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- \* cone : 5727 : angle of, 5752.
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- paraboloid : [JP.15](#) : [N.82](#) : of a quadric, L.38<sub>2</sub>.
- \* plane : 5721, eq 5733 : and radii of curv. at a multiple point of a gauche curve : [An.71](#) : [C.68](#).  
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- sphere : — [Mém.20](#) : [N.70](#) : of curve of intersection of two surfaces, cn : [C.83](#).  
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- surfaces : [C.79<sub>2</sub>](#), degree of 98 : [L.41,80](#) : of quadrics, [N.60](#).
- Oval of Cassini : see "Cassinian oval."
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- \*Pantograph : 5423 : [Mém.31](#) : [TE.13](#).
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- \*Parabola : geo. 1220—44 : anal. 4200—39 : eq [CM.2](#) : [N.42<sub>2</sub>,54,70](#) : geo [CM.4](#) : [Me.71](#) : cn 1249 : [thsN.60,63,71,76,80<sub>2</sub>](#).  
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- \* quadrature of : 6127 : [An.55](#).
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- Parallel curves : [J.55,ths32](#) : [LM.3](#) : Q.11 : [Z.5](#).  
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- \* of ellipse : 4960 : [A.39](#) : [An.60](#) : [N.44<sub>2</sub>](#) : [Q.12](#).
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- Parallelogram with sides through four given points : [A.39](#).
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- \*Parallelopipeds : on conjugate diameters : 5648.  
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- Partial differences : question in analysis : [J.16](#).
- \*Partial differential equations (P.D.E.) : 3380—3445 : [C.3<sub>2</sub>,11,16,78,95,96](#) : [thsCD.3](#) : [J.58,80,prs26](#) : [JP.7,10,11](#) : [L.36,80,83](#) : [M.11](#) : [Z.6,8,18](#).
- \*P.D.E., first order : 3399—3410 : [A.33](#), tr50 : [An.55,69](#) : [C.14<sub>2</sub>,53<sub>2</sub>,54<sub>2</sub>](#) : [CD.7](#) : [CM.1<sub>2</sub>](#) : [J.2,17](#) : tr [JP.22](#) : [L.75](#) : [M.9,11,th20](#) : [Z.22](#).
- \* complete primitive connected with any solution : 3405.
- \* derivation of the general primitive and singular solution from the complete primitive : 3401.
- \* derivation of a singular solution from the differential equation : 3403.  
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- \* Charpit's method : 3399.
- Jacobi's first method : [C.79,82](#) : and ap. to Pfaff's pr, [J.59](#).
- Jacobi-Hamilton method : [ML.3](#).
- Lie's method : [M.6,8](#).
- Weiler's method : [M.9](#).
- \* law of reciprocity : 3446.  
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P.D.E., first order—(continued):

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- \* singular solution: 3401—3: [J.66](#).
- systems of: [A.56](#): [M.11,17](#).
- theorem of Jacobi: [C.45](#).
- 3 variables: [J.64](#).
- \*  $n$  variables: 3409: [A.22](#): [J.60](#): LM. 10,11.
- with constant coefficients: [M.61,5](#).
- integration by calc. of variations: [C.14](#).
- $z = px + qy + F(p, q)$ : [Z.5](#).
- $x^a y^b z^c p^m q^n = A$ : [CM.1](#).
- $(1 + P_1 + \dots + P_n) \chi d_x, d_y^n Z = Q$ : [Z.13](#).
- \* P.D.E., first order, linear: 3381—95: reduction to, [C.15](#): [J.81](#): [Me.78](#).
- \*  $Pz_x + Qz_y = R$ : 3383: extension to  $n$  variables, 3384.
- $L(px + qy - z) - Mp - Nq + R = 0$ : [C.83](#).
- $a(yu_z - zu_y) + b(zu_x - xu_z) + c(xu_y - yu_x) = 1$ : [Q.8](#).
- \*  $Pt_x + Qt_y + \dots + Rt_z = S$ : 3387.
- \*  $z_x = \frac{y}{\sqrt{(y^2 - x^2)}}$ ;  $z_{xy} = x^2 + y^2$ : 3390—1.
- \*  $(x - a)z_x + (y - b)z_y = c - z$ : 3392.
- \*  $x^2 + y^2 + z^2 = 2ax$ : 3393.
- \* simultaneous: 3396—7; ex. 3398: [J.81](#).
- $u = v_y$  and  $u_y = -v_x$ : [J.70](#).
- \* P.D.E., second order: 3420—45: [A.33](#): [C.54,70,78,98](#): JP.tr22: [L.72](#): [M.15](#): [Me.76,77](#): [M.61,3](#): [N.83](#): [P.46](#): transf. of, [C.97](#).
- in two independent variables: trA. 54; trNo.81: [C.92](#); transf. of, [97](#): [M.24](#).
- in 4 and 5 variables: [Mém.13](#).
- \*  $Rr + Ss + Tt = V$ , Monge's method: 3423: [CM.3](#): [N.76](#): [Q.6](#).
- \*  $Rr + Ss + Tt + U(rt - s^2) = V$ : 3424, 3434—40: [J.61](#).
- \*  $Rr + Ss + Tt + Pp + Qq + Zz = U$ : 3442.
- $r + t = 0$ : [A.2](#): CM.1: [J.59,73,74](#): [L.43](#).
- $r + t + k^2z = 0$ : [M.1](#).
- \*  $r - a^2t = 0$ : 3433.
- \*  $r - a^2t = \phi(x, y)$ , &c.: 3565.
- $z(r - a^2t) = 2np$ : [E.13](#).
- $r - a^2y^{2b}t = 0$ : [E.27,28](#).
- $r = q^{2m}t$ : [C.98](#).
- $r = q$ : [J.72](#).

P.D.E., second order—(continued):

- $d_x(p \sin x) + t + n(n+1)z \sin^2 x = 0$ : [L.46](#).
- $r(1+q^2) = t(1+p^2)$ : [J.58](#).
- $q^2r - A^2t + \frac{A^2}{\eta} \eta_y q = 0$ : [E.22](#).
- $d_x(px) + d_y(qx) = 0$ : [Z.28](#).
- construction of explicitly integrable equations of the form  $s = z\lambda(x, y)$ : [JP.28](#).
- $s = \frac{2f'(x)F'(y)}{\{f'(x) + F'(y)\}^2} z$ : C.81.
- $a^2d_{xy} \log \lambda \pm \lambda = 0$ : [L.53](#).
- $s + Pp + Qq + Z = 0$ : [Me.76](#).
- \*  $s + ap + bq + abz = V$ : 3444.
- $(ax + by + c)s + a\lambda q + b\mu p = 0$ : [A.33](#).
- $(x+y)^2 s + a(x+y)p + b(x+y)q + cz = 0$ : [A.33,38](#).
- $z^2(zs - pq)^2 + q = F(y)$ : [A.70](#).
- $rt - s^2$ : [GeoQ.2](#).
- \*  $P = (rt - s^2)^n Q$ , Poisson's eq.: 3441.
- $(1+r)t + (1+t)r - 2pq = 0$ : [An.53](#).
- \*  $q(1+q)r + p(1+p)t - (p+q+2pq)s = 0$ : 3432.
- $4s^2 + (r-t)^2 = 4k^2$ : approx. integrn., [C.74](#).
- $As + Bq + \psi(r, p, q, x, y, z) = 0$ : [C.93](#).
- $(\log z)_{xy} + az = 0$ : [C.36](#).
- $u_{2t} = R_r \frac{u}{r}$  where  $t = \int \frac{r dr}{\sqrt{(2Rr^2 + A^2)}}$ : [L.38](#).
- \*  $u_{2x} + u_{2y} + u_{2z} = 0$  (see also "Spherical harmonics"): 3551, 3626, [J.36](#): Mo. 78.
- \*  $u_x + u_y + u_z = xyz$ : 3552.
- \*  $au_x + bu_y + cu_z = xyz$ , &c.: 3554.
- \*  $xu_{2x} + au_x - q^2xu = 0$ : 3618.
- \*  $a^2(u_{2x} + u_{2y} + u_{2z}) = u_{2t}$ : 3629: [C.7](#): [LM.7](#).
- integration by change of variables: [C.74](#).
- P.D.E., third order, two independent variables: [LM.8](#): [N.83](#).
- P.D.E., fourth order:  $\Delta \Delta u = 0$ : [C.69](#).
- P.D.E., any order: [No.73](#): [C.80,89](#): [M.11,13](#).
- $x^{\frac{n}{2}} z_{nx} = z_{ny}$ : [Z.7](#).
- two independent variables: [C.75](#): [CP.8](#).



P.D.E., any order—(continued):

any number of functions and independent variables: [C.80](#).

and ap. to physics: [JP.13](#).

of cylinders: [Me.77](#).

and elliptic functions: [J.36](#); hyper-ell., [J.99](#).

integration by definite integrals: [An.59](#): [C.94](#): [L.54](#).

of dynamics: [C.5](#): [J.47](#).

Hamiltonian: [M.23](#): [Z.11](#).

of heat: [L.48](#); of sound, [L.38](#).

integrated in series: [C.15,16](#).

of Laplace: [G.23](#).

linear: [An.77](#): [C.13,90](#): [CD.9<sub>2</sub>](#): [CM.2](#): [J.69](#): [JP.12<sub>2</sub>](#): [L.39](#).

of orthogonal systems of surfaces: [Ac.4](#): [C.77](#).

with periodical coefficients: [C.29<sub>3</sub>](#).

of physics: [L.72,47](#).

P.D.E., simultaneous: [C.92,th78](#): [LM.9](#): [M.23](#): [Z.20](#).

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P.D.E., system of: [C.13,74,81](#).

$\alpha^m z_{mt} = x^{2m} z_{mx}$ : [A.30,31](#), by  $z = e^{at}f(x)$ .

$z_{nx} = x^m z_{(m+n)y} + F_1(y) + xF_2(y) + \dots + x^{m-1}F_m(y)$ : [A.51](#).

$Az_{xt} + (d_{2x} + d_{2y} + \dots)^n z = 0$ : [C.94](#).

$dz = Hdx + Kdy + Ldp + Mdq + Ndr + \&c.$ : [J.14](#).

Partial differentials of  $\frac{x}{x^2+y^2}$ : [J.11](#).

\*Partial fractions: [235](#), 1915: [A.30,66](#): [C.46,49,78<sub>3</sub>](#): [CM.1](#): [G.2](#): [J.1,5,9,10,11,22,32,50<sub>2</sub>](#): [JP.3](#): [L.46](#): [Mém.9](#): [N.45<sub>2</sub>,64,69](#): [Q.5](#).

Partition of numbers (see also "Numbers" and "Indeterminate eqs."): [AJ.2,5,6](#): [An.57,59](#): [At.65](#): [C.80,86,90,91](#): [CP.8](#): [J.13,61,85](#): [M.14](#): [Man.55](#): [Me.78,79](#): [Mém.13](#): [geo.ap.20,44](#): [Mél.1](#): [N.69,85](#): [P.50,56,58](#): [Pr.7,8](#): [Q.1<sub>2</sub>,2,7,15](#): [Z.20,24](#).

by Arbogast's derivatives: [L.82](#).

of complex numbers in Jacobi's th: [C.96](#).

by elliptic and hyper-elliptic functions: [J.13](#).

formula of verification: [Pr.24](#).

into 2 squares: [An.50,52,54](#): [C.87](#): [J.49](#): [LM.8,9](#): [N.54,78,alg65](#): odd squares, [Q.19](#).

into 3 squares: [J.40](#): [L.59,60](#).

into 4 squares: [C.99](#): [L.68](#): [Pr.9](#): [Q.1](#). 4 odd, or 2 even and 2 odd: [Q.19,20](#).

into 5 squares: [C.97,98](#): [J.35](#).

Partition of numbers—(continued):

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into  $p$  squares: [C.39,90](#): [L.61](#): [N.54](#).

and an integer: [L.57](#).

into the product of two sums of sqs.: [L.57](#).

into parts, the sum of any two to be a sq.: [Mém.9](#).

into 2 cubes: [L.70](#).

into sum or difference of 2 cubes: [AJ.2](#).

into 4 cubes: [N.79](#).

into maximum  $n$ th powers: [C.95](#).

into 10 triangular numbers: [C.62](#).

formation of numbers out of cubes: [J.14](#).

2 squares whose sum is a sq.: [E.20](#): [N.50](#).

3 squares, the sum of every two being a sq.: [E.17](#).

4 squares, the sum of every two being a sq.: [E.16](#).

3 nos., the sum or diff. of two to be a sq.: [Mém.18](#).

2 sums of 8 sqs. into 8 sqs.: [Me.78](#).

a sum of 4 sqs. into the product of 2 sums of 4 sqs.: [TL.21](#).

$n$  nos. whose sum is a sq. and sum of sqs. a biquadrate: [E.18,22,24](#).

a quadric into a sum of squares: [N.80,81](#).

$n^3 - m^3$  into  $3p^2 + q^2$ : [N.49](#).

a square into a sum of cubes: [N.67](#).

a cube into a sum of cubes: [E.22,23](#).

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$n^{12} - 9m^{12}$  or its double into 2 cubes: [N.81](#).

3 nos. whose sum is a cube, sum of sqs. a cube, and sum of cubes a sq.: [E.26](#).

5 biquadrates whose sum is a square [E.20](#).

of  $n$  into 1, 2, 3, &c. different numbers: [E.34](#).

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- \*Pascal's theorem: 4781: [AJ.2](#): [CD.3](#),  
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mum surface: [L.57](#).  
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  - \* three points lying on, cond.: 4036, t.c 4615.
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  - \* squares of differences: [541](#): ap, [N.50](#): [E.40](#).  
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  - \* incommensurable: [506](#) (see "Sturm's th.")  
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- Roots of an equation—(continued):  
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  - \* squares of differences: [541](#): [CM.1](#):  
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  - \* Scales of notation: [342](#): J.1: [L.48,5ry](#),  
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  - \* Sections of the cone: 1150.
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  - \* Self-conjugate triangle: 4755, 4967:  
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- Self-conjugate triangle—(continued):  
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- \* Self-reciprocal triangle: 1020.

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- \* Series: (see also "Summation" and  
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[64](#): [N.59,th62,70](#): [Q.3](#): [Z.15,16](#),  
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Useful summations:

- \*  $x + \frac{x^2}{2} + \frac{x^3}{3} + \dots = \log \frac{1}{1-x}$ : [156](#).  
 \*  $x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \log(1+x)$ : [155](#).  
 \*  $x + \frac{x^3}{3} + \frac{x^5}{5} + \dots = \frac{1}{2} \log \frac{1+x}{1-x}$ : [157](#).  
 \*  $x - \frac{x^3}{3} + \frac{x^5}{5} - \dots = \tan^{-1} x$ : [791](#).  
 \*  $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = e^x - 1$ : [150](#).  
 $x - \frac{x^2}{2!} + \frac{x^3}{3!} - \dots = 1 - \frac{1}{e^x}$ :  
 $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots = \frac{e^x - e^{-x}}{2}$ :  
 \*  $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sin x$ : [764](#).  
 $\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots = \frac{e^x + e^{-x}}{2}$ :  
 \*  $\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \dots = 1 - \cos x$ : [765](#).  
 \*  $1^p + 2^p + \dots + n^p$ : 2939: [A.65](#): [Me.78](#).  
 \*  $p = 1, 2, 3$ , or 4: 276: [A.64](#): [E.34](#).  
 $1^p - 2^p + 3^p - \dots$  &  $1^p - 3^p + 5^p - \dots$ :  
[J.7](#).  
 $\sum n^a x^n$ : [A.27](#).  
 $\sum (a+n)^b x^n$ : [N.56](#).  
 $\sum (a_n + b_n x^c) x^{k-n}$ : [Z.15](#).  
 $1 - \frac{n-3}{2} + \frac{(n-4)(n-5)}{2 \cdot 3} - \dots$ : [J.20](#).

Series—(continued):

- $a \pm nb + C(n, 2) c \pm \&c.$ : [J.31](#).  
 \*  $n^r - n(n-1)^r + C(n, 2)(n-2)^r - \dots$ :  
[285](#);  $r = n$ , [CM.1](#).  
 $1 + \frac{a}{b} x + \frac{a(a+1)}{b(b+1)} x^2 + \dots$ : [J.37](#); with  
 $x = 1$ , [J.2](#).  
 $\sum \frac{1}{(a+nd)^k}$ : [N.59](#).  
 deductions from  
 $\left( \sum \frac{a_2^{(n)} x^n}{a^{(n)} n!} \right) e^{-x} = \sum \frac{x^{2n}}{(a+1)_2^{(n)} 2^n n!}$ :  
[LM.9](#).  
 \*  $1 \pm \frac{1}{2^n} + \frac{1}{3^n} \pm \dots$  and  $1 \pm \frac{1}{3^n} + \frac{1}{5^n} \pm \dots$ :  
 2940—4: [A.41](#): [LM.8](#): [Q.7](#): with  
 $n = 1, 2, \dots, 8$ ; 2945: [E.32,39](#): [G.10](#):  
[N.79](#): [Z.3](#). Note that by (2391),  
 $1 - \frac{1}{3^2} + \frac{1}{5^2} - \&c. = - \int_0^1 \frac{\log x}{1+x^2} dx$ .  
 $\sum \frac{x^n}{n^2}$ : [Mém.11](#): with  $x = 1$ , [J.5](#).  
 $\sum \frac{n^x}{n!}$ : [A.61](#).  $\sum \frac{n^x}{a^n}$ , [A.41](#).  
 $x - \frac{x^7}{7!} + \frac{x^{13}}{13!} - \dots$ : [E.44](#); with  $x = 1$ ,  
[J.5](#).  
 $\sum \frac{(n-1)x^n}{n! a^{n-1}}$ : [A.50](#).  
 $\sum (-1)^n \frac{x^n}{(a+2n)(1-x)^n}$ : [Q.6](#).  
 $\sum \frac{1}{x^{(n)}}$ : [E.36](#).  
 \*  $\sum a_n x^{(n)}$ : 2709.  
 $\sum (-1)^n \frac{x^{(n)}}{n+1!}$ : [A.26](#).  
 $\sum \frac{1 \cdot 3 \dots 2n-1}{n! 2^n (2n+1)}$ : [L.60](#).  
 $1 + \frac{(q^s-1)(q^s-1)}{(q-1)(q^r-1)} x +$   
 $\frac{(q^s-1)(q^{s+1}-1)(q^s-1)(q^{s+1}-1)}{(q-1)(q^2-1)(q^r-1)(q^{r+1}-1)} x^2$   
 $+ \dots$ : [J.32,70](#).  
 $\sum \left\{ \left( \frac{n+1}{1} \right) \left( \frac{s-3}{0} \right) + \dots \right\}$ : [G.11](#).  
 $\sum \frac{x}{an^m + a_1 n^{m-1} + \dots + a_m}$ : [A.35](#).



Series—(continued):

- $\sum K_n \frac{x^{na+\beta}}{na+\beta}$ ,  $\beta$  a pos. integer  $< a$ ,  
 $K_n$  = the general term of some  
 recurring series: [C.86](#).
- $\sum \frac{a}{n^{1+\alpha}}$ : [A.34](#).
- $\sum_{n=1}^{\infty} \frac{q^{n-1}}{n^2} E\left(\frac{n^2}{q}\right)$ : [C.50](#).
- \*  $\sum \left( \frac{\sin}{\cos} 2n+1\phi \right) / n^k$ : [J.54](#); with  $k=2$ ,  
 2960—1, [J.8](#).
- $\sum \sin^3(2n+1)\phi / (2n+1)^4$ : [E.39](#).
- $\sum \left( \frac{\sin}{\cos} n\phi \right)^{2 \text{ or } 4} / n^4$  and  
 $\sum \left( \frac{\sin}{\cos} n\phi \right)^3 / n^3$ : [L.73](#).
- \*  $\sum \left( a^n \frac{\sin}{\cos} n\phi \right) / n$ : 2922—3.
- $\sum \frac{\sin}{\cos} n^2 \frac{2r\pi}{p}$ : [L.40](#).
- \*  $\sum \frac{n \sin n\theta}{a^2+n^2}$ : 2962: [M.5](#).
- $\sum A_n \cos^n \theta \frac{\cos}{\sin} n\theta$ : [Z.1](#).
- \*  $\sum \frac{\sin}{\cos} (a+n\beta)$ : 800: [Q.3](#);
- \*  $\sum c^n \frac{\sin}{\cos} (a+n\beta)$ , 783;
- \*  $\sum \frac{c^n}{n!} \frac{\sin}{\cos} (a+n\beta)$ , 788.
- $\sum f(n) \frac{\sin}{\cos} n\theta$ : [J.42](#): [L.52](#).
- $\sum H(m, n) x^n \frac{\sin}{\cos} n\phi$ : [J.41](#).
- $\sum \frac{1}{2^n} \tan \frac{\phi}{2^n}$ : [A.44](#).
- $\sum f(nx)$ : [L.51](#).  $\sum A_n \phi(n) x^n$ , [J.25, 28](#).
- $\sum_1^m f\left(x + \frac{n}{m}\right)$ : [A.22](#).
- from  $\int x(1-x)^n dx$ : [A.47](#).
- from  $\int_0^{\frac{\pi}{2}} \cos^2 x \log \frac{1+\sin x}{1-\sin x} dx$ : [G.10](#).
- of Abel: [C.93](#): [N.85](#).
- application: [thA.48](#); to arith, [G.7](#).
- in a.p.: see "Arithmetical progression."
- analogous series: [N.69](#).
- a.p. and g.p. combined: [A.9](#).
- with Bernoulli's nos.: [An.53](#); and binomial coefficients, [A.23](#).

Series—(continued):

- binomial (see "Binomial theorem"):
- analogous series: [E.35](#): [J.32](#): [N.82](#); with inverse binomial coefficients, [Me.80](#).
- coefficients independently determined: [A.18](#).
- whose coefficients are the sums of divisors of the exponents, sq. of this series: [Me.85](#).
- combination: [A.26](#).
- \* convergent: 239: [A.2, 6, 8, 14, 26, 41, 67, 69](#); [No.44](#): [C.10, 11, 28, 40, 43](#); [J.2, 3, 11, 13, 16, 22, 42, 45, 76](#): [L.39—42](#): [M.10, 20—22](#): [Me.64](#): [N.45, 46, 67, 69, 70](#); [P.87](#): [Z.10, 11](#).
- and of [d.1](#) with a periodic factor: [L.53](#).
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- of symbolic functions into isotropic means: [C.43](#).
- of  $\tan^{-1} \sqrt{\frac{a-c}{c+x}}$  + symmet.  $y, z$ : [CD.9](#).
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- reduction to intransitive: [C.21](#).
- \* Transversals: 967—74: [A.13,18,27,30,56](#); [CD.5](#); [CM.1](#); [J.8](#); [G.22](#); [Me.t.c68,75](#); [N.43,48](#); [TE.t.c24](#).
- orthogonal: [AJ.3](#).
- of plane alg. curves: [Z.19](#).
- of two points: [A.66](#).
- parallel: [A.13,57](#).
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- Trees, analytical: [AJ.4](#).
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- \* Triangle: [700](#); [A.17,19,22,29,33,36,43,46,61](#); [G.21](#); [J.50](#); [M.geol7](#); [Me.q.c62](#); [N.42,43](#).
- \* angles of: [677](#); 738—45: [A.65](#); [P.28](#); division, [A.51,58](#); sum, [geo960](#); [C.69,70](#); difference, [Q.8](#).
- \* area: [707,4036—41](#); [A.45,57](#); [CM.2](#); [Mém.13](#).
- \* bisectors: — of angles: [709](#), [742](#), [932](#), 4628,—30.
- \* of sides: [738](#), 922i, [951](#), 4631: [Mém.pr10,13](#).
- \* central line: [957](#), 4644.
- and circle, ths.: [A.9,40,47,60](#); [Q.7,8,10](#).
- and 3 concentric circles: [Me.85](#).
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- \* of reference in t.c: 4006.
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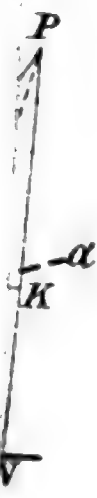


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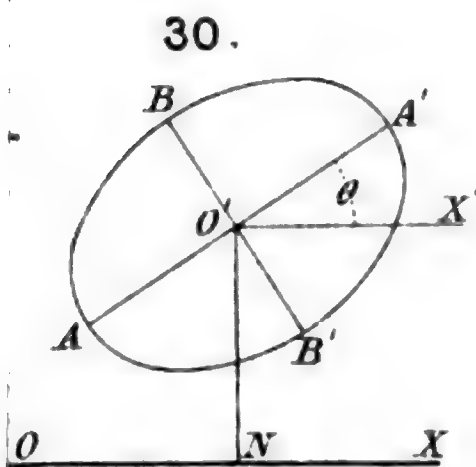
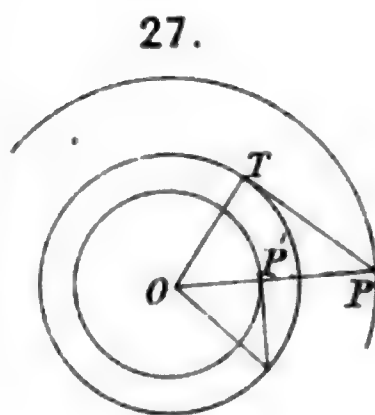
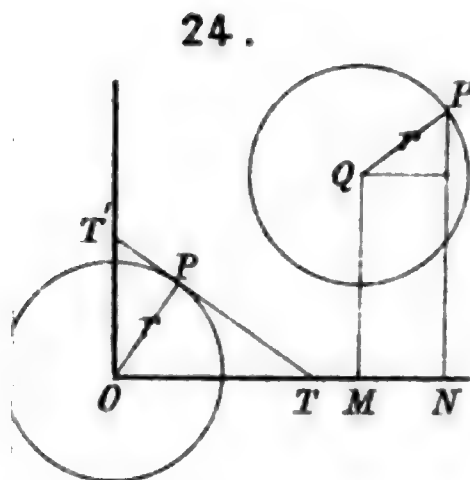
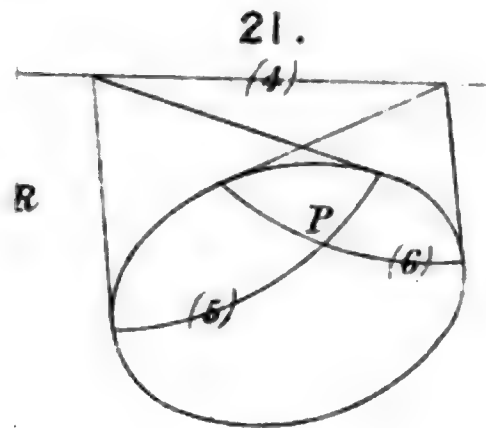
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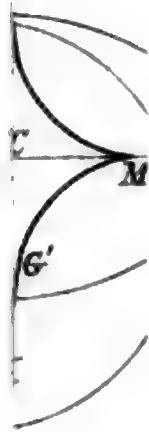




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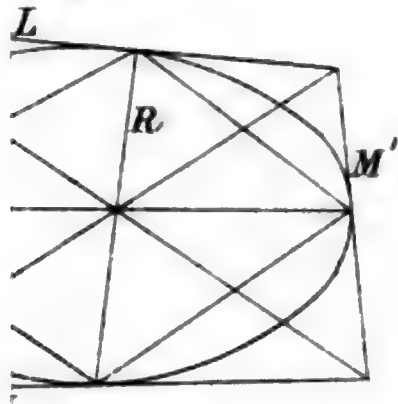




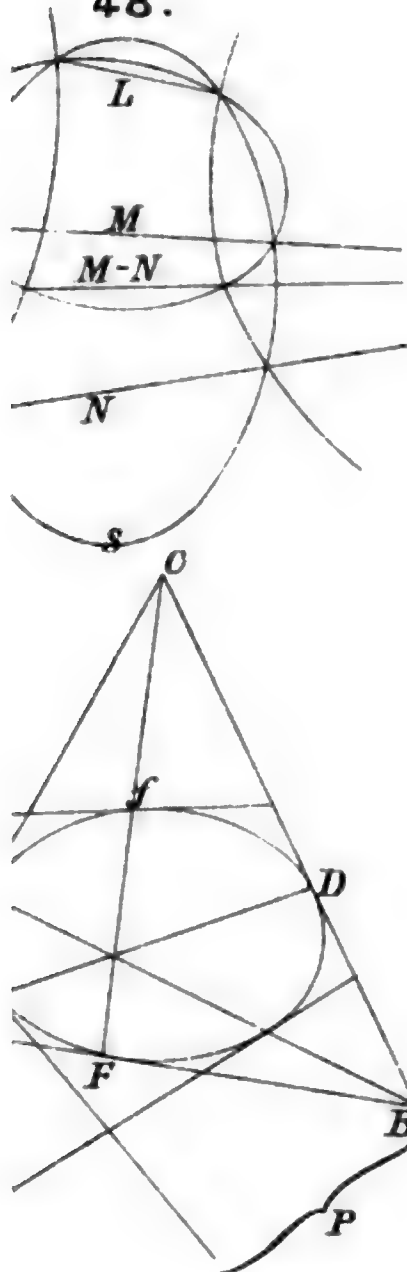


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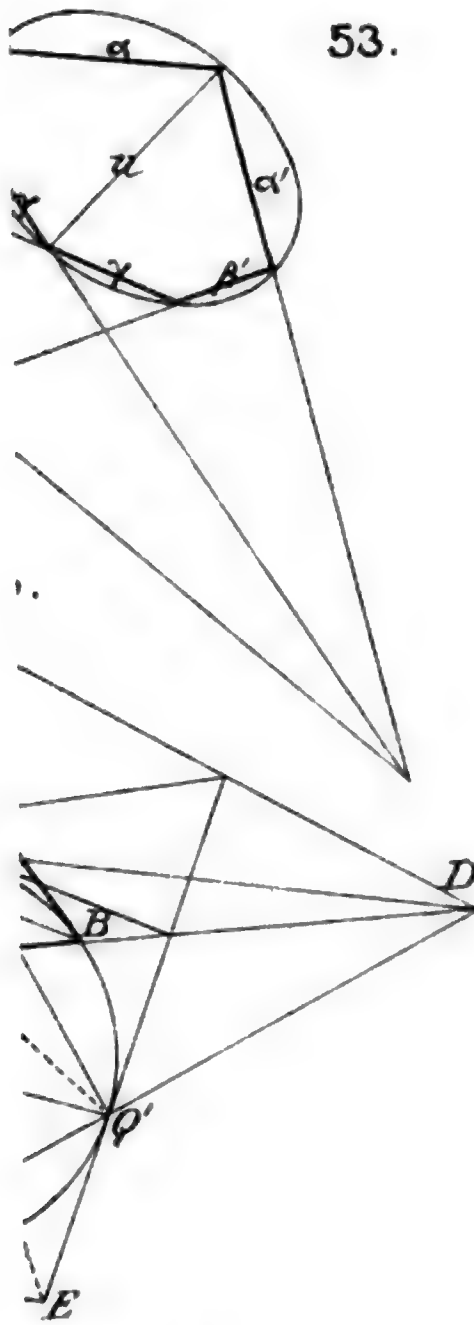


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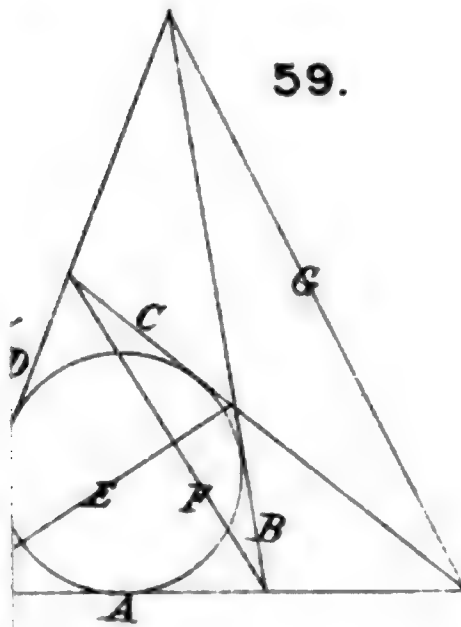




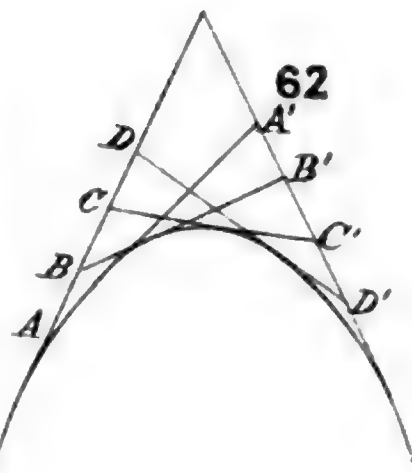
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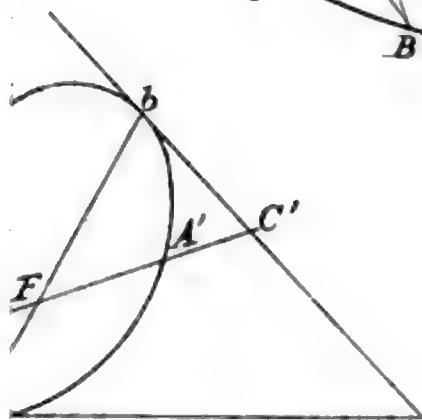
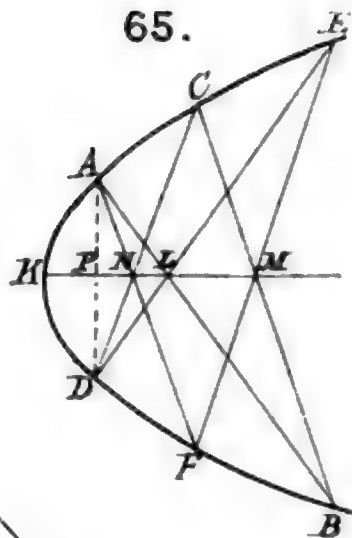
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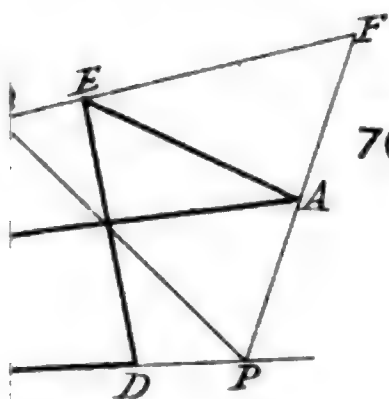
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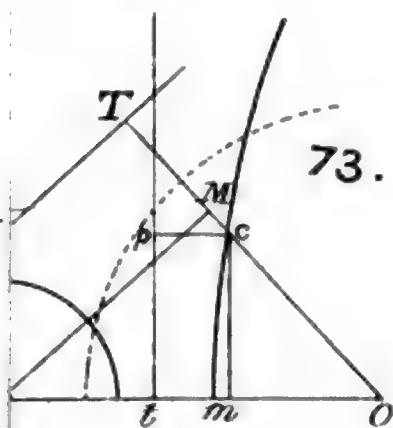
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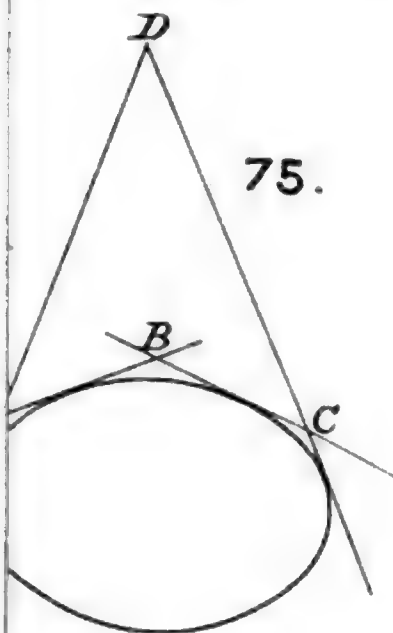
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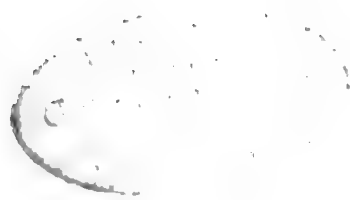


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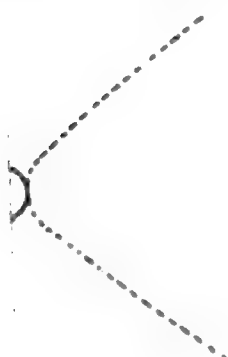
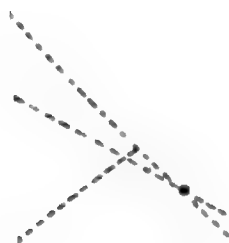


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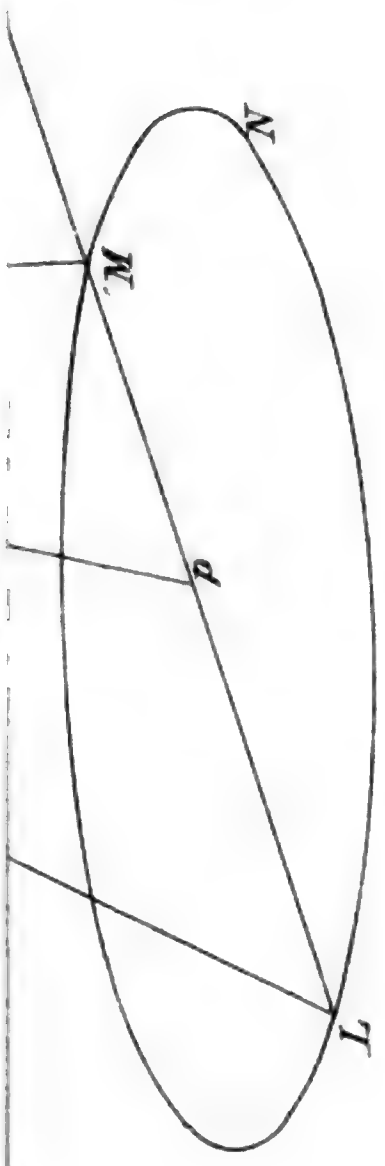
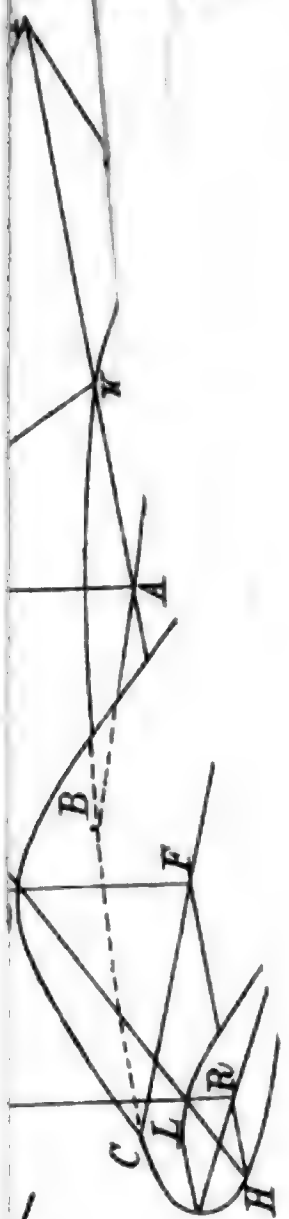


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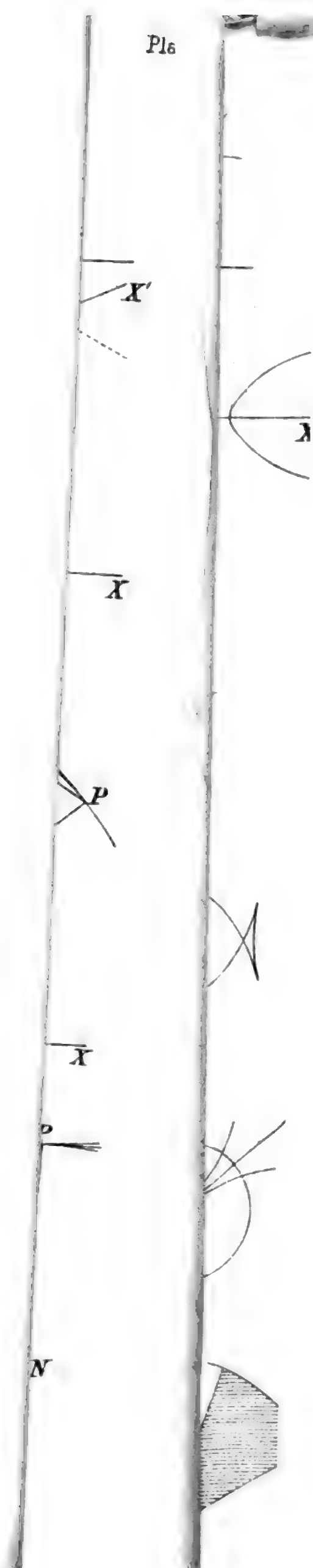
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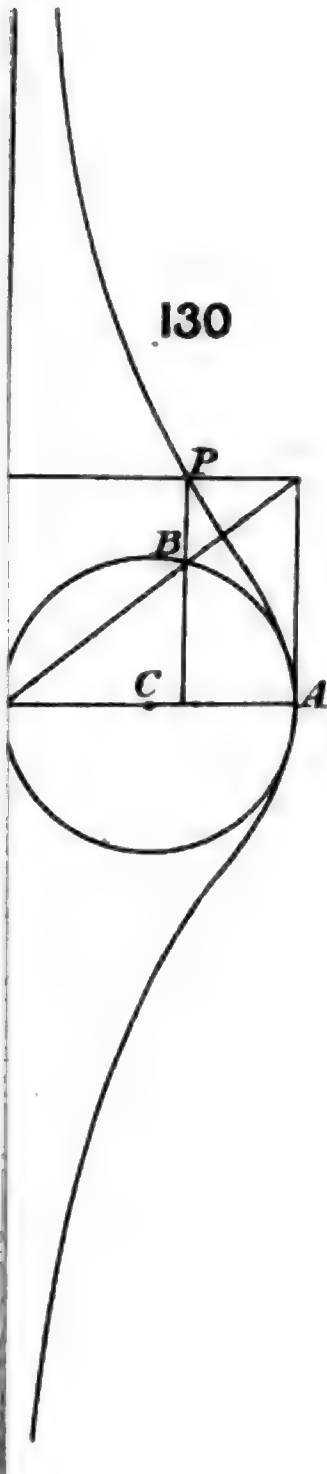
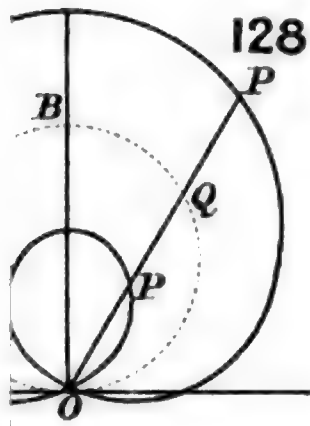
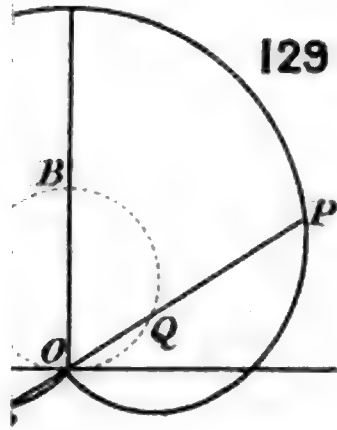




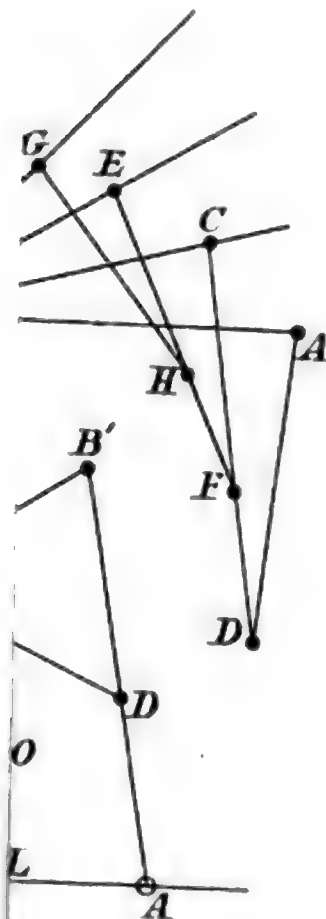
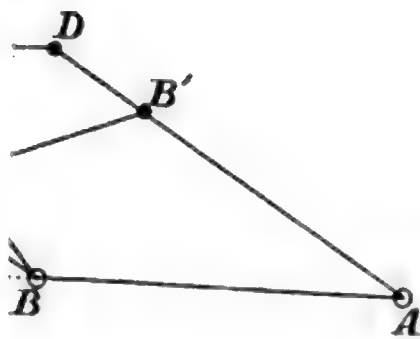
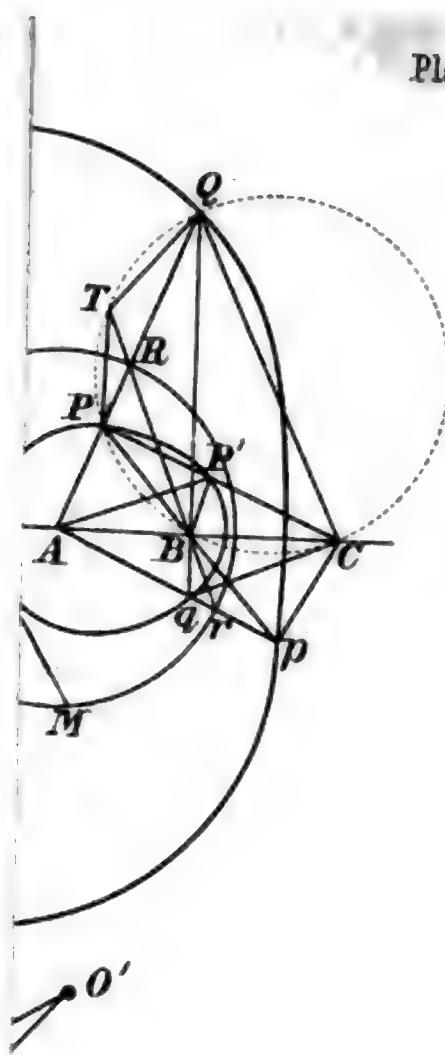








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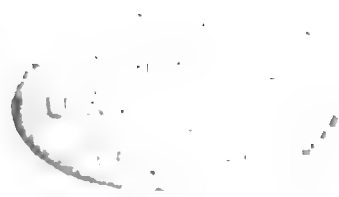
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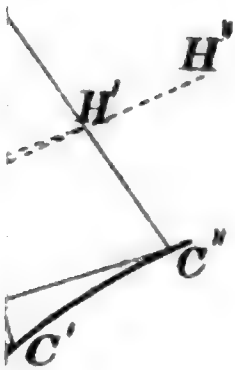
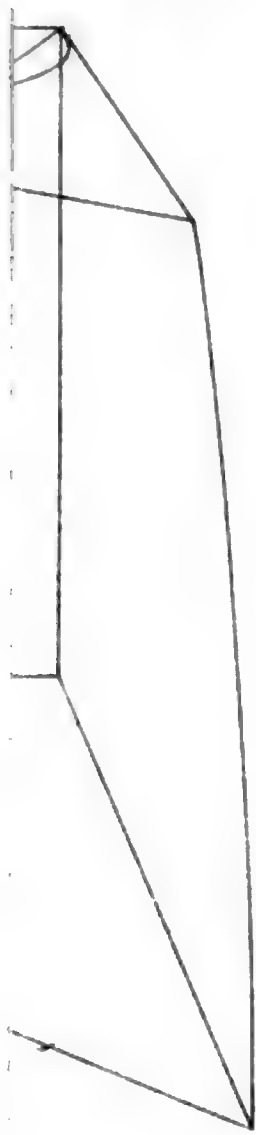




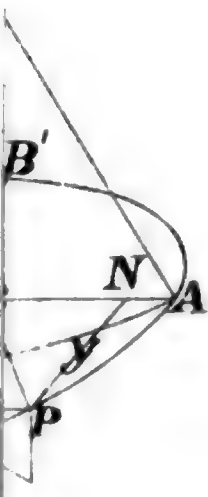




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